Midterm Solutions

1. a) Is a subspace.

b) Not a subspace because it doesn't contain $(0,0,0)$.

c) Is a subspace.

d) Not a subspace because it doesn't contain $(0,0,0)$.

e) Not a subspace because it contains $(1,1,1)$ but not $2(1,1,1)=(2,2,2)$.

f) Is a subspace.

g) Not a subspace because it doesn't contain $(0,0,0)$.

h) Is a subspace (just the point $(0,0,0)$!).

i) Not a subspace because it doesn't contain $(0,0,0)$.

j) Is a subspace.

2. a) Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $M\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}M=\begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$. So the set in question is those M such that $-b=c, a=d, -d=-a$, and $c=-b$. In other words, all matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. A basis for this set is

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

It has dimension 2.

b) Given a 3rd degree real polynomial $f(x)=a+bx+cx^2+dx^3$, $f(0)=a$ and $f''(0)=2c$. So the set in question is those polynomials such that $a=2c$, or polynomials of the form $2c+bx+cx^2+dx^3$. A basis for this set is $\{x, x^2+2, x^3\}$.

3. A and B give rise in the usual way to linear transformations $L_A: F^n \to F^m$ and $L_B: F^k \to F^n$. Since $AB=0$, we know that $L_{AB}=0$, so $L_AL_B=0$. This means that $L_A(L_B(v))=0$ for any $v\in F^k$, and thus that the image of L_B lies inside the kernel of L_A. Since $\text{im } L_B \subseteq \ker L_A$, we must have $\dim(\text{im } L_B)\le \dim(\ker L_A)$, or $\text{rank } L_B \le \text{nullity } L_A$. By the dimension formula, $\text{nullity } L_A = n - \text{rank } L_A$, so we have $\text{rank } L_B \le n - \text{rank } L_A$, or $\text{rank } L_A + \text{rank } L_B \le n$. Finally, since $\text{rank } A = \text{rank } L_A$ and $\text{rank } B = \text{rank } L_B$, we conclude that $\text{rank } A = \text{rank } B \le n$.

4. a) We have that

$$
\begin{aligned}
T_A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \\
T_A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -c & a-d \\ 0 & c \end{pmatrix} \\
T_A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} b & 0 \\ d-a & -b \end{pmatrix} \\
T_A\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}
\end{aligned}
$$

So with respect to the basis β, the matrix of T_A is

$$
\begin{pmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{pmatrix}.
$$

b) If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the matrix $[T_A]_\beta^\beta$ becomes

$$
\begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
$$

This means that T_A sends the matrix $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ to $T_A(B) = \begin{pmatrix} g & h-e \\ 0 & -g \end{pmatrix}$. The rank of this transformation is 2, a basis for the image being given by

$$
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.
$$

The nullity of this transformation is also 2, a basis for the kernel being given by

$$
\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
$$

c) Saying that A commutes with every 2×2 matrix B is equivalent to saying that $AB - BA = 0$ for all 2×2 matrices B, which is equivalent to saying that T_A is the zero transformation. From part A, we see that this is true if and only if $b = c = 0$ and $a = d$, i.e., if and only if A is of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ (such a matrix is called a *scalar matrix*, because it is a scalar times the identity).

5. We form the augmented matrix of the system, which is

$$
\left(\begin{array}{ccccc|c} 3 & 2 & 1 & -1 & -1 & 6 \\ 1 & -1 & -1 & -1 & 2 & 12 \\ -1 & 2 & 3 & 1 & -1 & 18 \end{array}\right).
$$

Row reducing to echelon form, we obtain

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -2/3 & 1 & 12 \\ 0 & 1 & 0 & 2/3 & -3 & -30 \\ 0 & 0 & 1 & -1/3 & 2 & 30 \end{array}\right).$$

This represents the equivalent system:

$$\begin{aligned} x_1 + 0x_2 + 0x_3 - 2/3x_4 + x_5 &= 12 \\ 0x_1 + x_2 + 0x_3 + 2/3x_4 - 3x_5 &= -30 \\ 0x_1 + 0x_2 + x_3 - 1/3x_4 + 2x_5 &= 30 \end{aligned}$$

Letting $t = x_4$ and $u = x_5$, we obtain $x_1 = 12 + 2/3t - u$, $x_2 = -30 - 2/3t + 3u$ and $x_3 = 30 + 1/3t - 2u$. In other words the solutions are parameterized by,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 12 \\ -30 \\ 30 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

Since there are 2 parameters, the solution space has dimension 2.

6. We showed in class that the rank of BC must be no greater than the rank of B and no greater than the rank of C. Since B is 3×1 and C is 1×3, they each can have rank at most 1. So BC can have rank at most 1.

Conversely, say A has rank 1. Then it has only one linearly independent column. Assume, without loss of generality, that the first column of A is nonzero (we can do this because we know A is not the zero matrix, and it doesn't change the proof to assume that the first column is nonzero, as opposed to one of the other columns). Then the other columns must be mutliples of the first column, so A looks like

$$\begin{pmatrix} x & c_1x & c_2x \\ y & c_1y & c_2y \\ z & c_1z & c_2z \end{pmatrix}.$$

But this means we can write

$$A = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} 1 & c_1 & c_2 \end{pmatrix},$$

so we set $B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} 1 & c_1 & c_2 \end{pmatrix}$ to get $A = BC$.

7. a) To prove that S has dimension 3, we show that the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 4 & 1 \\ 3 & 10 & 11 & 4 \\ 4 & 13 & 18 & 7 \end{pmatrix},$$

whose columns are e_1, e_3, e_3, e_4, has rank 3. Row reducing, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which has rank 3, as it is in echelon form and has 3 nonzero rows. Since row reduction does not change the rank of a matrix, the original matrix also has rank 3. This means that the columns span a space of dimension 3, so S has dimension 3.

b) Since columns 1, 2, and 3 are pivotal, and thus linearly independent, we know that they were linearly independent the whole way through. So e_1, e_2, e_3 are linearly independent, and thus form a basis for the 3 dimensional space S.

c) No, we cannot do this. For instance, $e_1 - e_3 + 2e_4 = 0$, so e_1, e_3, and e_4 are not linearly independent. So they do not form a basis for S.

8. The statement is true. Let A be the coefficient matrix, so that the system is of the form $Ax = b$, with A an $m \times n$ matrix of rank m. Since $L_A : F^n \to F^m$ has rank m, it is onto. This means that, for any $b \in F^m$, there exists $x \in F^n$ such that $L_A(x) = b$, or in other words, $Ax = b$. This x is a solution to the system.