

Solution Set 0: Techniques of Proof—Induction and Contradiction

Solution 1: For the case $n = 0$, this just says $0 = 0$. For the induction step, assume that

$$\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

for a given n . Then write

$$\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^n i^2 + (n+1)^2.$$

Using the induction hypothesis, we have that the right hand side is equal to

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} = \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}, \end{aligned}$$

which is what we wanted to prove.

Solution 2: For the case $k = 0$, this just says that 0 is divisible by 8, which is true. For the induction step, assume that $5^k - 4k - 1$ is divisible by 8, for a given k . We want to prove that $5^{k+1} - 4(k+1) - 1$ is divisible by 8. Well,

$$5^{k+1} - 4(k+1) - 1 = (5^k - 4k - 1) + (5^{k+1} - 5^k - 4) = (5^k - 4k - 1) + (4 \cdot 5^k - 4).$$

By the induction hypothesis, $5^k - 4k - 1$ is divisible by 8, so it suffices to show that $4 \cdot 5^k - 4$ is divisible by 8. But 5^k is an odd number, so $4 \cdot 5^k$ has a remainder of 4 when divided by 8. Thus, $4 \cdot 5^k - 4$ is divisible by 8.

Solution 3: For the cases $n = 0$ and $n = 1$, there is clearly only one way for the bunny to hop up the steps, so this proves the base case, as $F_0 = F_1 = 1$. Now, say the number of ways to hop up an n -step staircase is F_n for all numbers up to a given n . Then, for an $n + 1$ step staircase, We know that the bunny's last hop is either 1 step or 2 steps. By the induction hypothesis, the number of ways to hop up the $n + 1$ step staircase with the last hop being 1 step is F_n and the number of ways to hop up with the last hop being 2 steps is F_{n-1} . (WHY?) Since, for any way of hopping up the $n + 1$ step staircase involves a last hop of either 1 or 2 steps, the total number of ways is $F_n + F_{n-1} = F_{n+1}$, which is what we wanted to prove.

Solution 4: Given a differentiable, real-valued function defined on \mathbb{R} , if its derivative is uniformly zero, then it is a constant (this is a true statement, which can be proved using the mean value theorem).

Solution 5: We can write this statement in the form, "All real square matrices with nonzero determinant are invertible over the real numbers." The negation is, "There exists a real square matrix with nonzero determinant, but which is not invertible over the real numbers." The original statement is true, while the negation is false.

Solution 6: One answer is, “If a matrix is not invertible over the real numbers, then it is not a real square matrix with nonzero determinant.” Another answer is “If a real square matrix is not invertible over the real numbers, then its determinant is zero.”

Solution 7: For a contradiction, assume that there exists a way to stuff the n pigeons into k holes with no hole containing more than p pigeons (this is the negation of the consequent). Since we have k holes, with no more than p pigeons in each hole, we must have no more than pk pigeons. But we have n pigeons, and we have assumed $n \geq pk + 1$. This is a contradiction, so it is impossible to stuff the n pigeons into k holes with no hole containing more than p pigeons, so long as $n \geq pk + 1$. Thus any way of stuffing the n pigeons into the k holes must result in at least one hole with more than p pigeons.