

Solution Set 1: Vector Spaces

p.14, #12. It is easy to see that addition is commutative and associative, scalar multiplication is associative and distributes over addition (in the scalar and the vector space), that $0 \in V$, and that $1 \cdot f = f$ for any f . What needs checking is that all elements f have additive inverses, that if $f, g \in V$, then $f + g \in V$, and that if $f \in V$ and $a \in \mathbb{R}$, then $af \in V$. To prove the first assertion, given f an even function, we see that $-f$ is an even function because $(-f)(x) = -1 \cdot f(x) = -1 \cdot f(-x) = (-f)(-x)$. To prove the second assertion, we note that $(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x)$. To prove the third assertion, we note that $(af)(x) = a \cdot f(x) = a \cdot f(-x) = (af)(-x)$.

p. 15, #19. V is not a vector space under these operations, as multiplication does not distribute over addition of scalars. For example, $(2 + 3)(1, 1) = (5, 1/5)$, but $2(1, 1) + 3(1, 1) = (5, 5/6)$.

p. 21, #19. First, assume that $W_1 \subseteq W_2$ or that $W_2 \subseteq W_1$. In the first case, we have that $W_1 \cup W_2 = W_2$, which is a subspace of V , and in the second case, we have that $W_1 \cup W_2 = W_1$, which is also a subspace of V . So we have proved one direction.

Now, assume that $W_1 \cup W_2$ is a subspace of V . For a contradiction, assume that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then there exists an element w that is in W_2 but not W_1 , and an element w' that is in W_1 but not W_2 . Then $w + w'$ is not in W_1 , because if it were, $w + w' - w' = w$ would be in W_1 , which is false. Likewise, $w + w'$ is not in W_2 , because if it were, $w + w' - w = w'$ would be in W_2 , which it is not. So $w + w'$ is not in either W_1 or W_2 , so it is not in their union. This shows that $W_1 \cup W_2$ is not a vector space, and is thus not a subspace of V . This is a contradiction, so in fact we have that either $W_1 \subseteq W_2$ or that $W_2 \subseteq W_1$.

p.21, #20. We prove this by induction on n . For the case $n = 1$, the statement just says that if $w_1 \in W$, then $a_1 w_1$ is in W for any a_1 in the field of scalars, which is true by the definition of a subspace. Now, assume that the statement is true for a given n , i.e., that if w_1, \dots, w_n are elements of W , then $a_1 w_1 + \dots + a_n w_n \in W$ for any scalars a_1, \dots, a_n . Then, given vectors $w_1, \dots, w_{n+1} \in W$, and scalars a_1, \dots, a_{n+1} , we have that

$$a_1 w_1 + \dots + a_{n+1} w_{n+1} = (a_1 w_1 + \dots + a_n w_n) + a_{n+1} w_{n+1},$$

which is in W because both $a_1 w_1 + \dots + a_n w_n$ and $a_{n+1} w_{n+1}$ are, the first by the induction hypothesis and the second by the definition of a vector space.

p.33, #8. The symmetric 2 by 2 matrices are all of the form $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$. We can write such a matrix as $aM_1 + bM_2 + cM_3$, so the span of $\{M_1, M_2, M_3\}$ contains all of the symmetric 2 by 2 matrices. Conversely, any matrix of the form $aM_1 + bM_2 + cM_3$ is equal to $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$, which is symmetric, so the span of $\{M_1, M_2, M_3\}$ is contained in the set of 2 by 2 symmetric matrices. Since the span both contains and is contained in the set of 2 by 2 symmetric matrices, it is equal to the set of 2 by 2 symmetric matrices.

p. 39, #14. Assume M has n columns. Call the columns v_i . They are elements of F^n . Assume, for a contradiction, that the columns are linearly dependent. Then we can write

$$a_1 v_1 + \dots + a_n v_n = 0,$$

with at least one a_i not equal to zero. Let i be the highest index such that $a_i \neq 0$. Then $a_1v_1 + \cdots + a_nv_n = a_iv_1 + \cdots + a_iv_i$. Now, since the diagonal entries are non-zero, the i th component on v_i is not zero. But for all $j < i$, the i th component of v_j is zero, because M is upper triangular. So the i th component of $a_1v_1 + \cdots + a_nv_n = a_iv_1 + \cdots + a_iv_i$ is the same as the i th component of a_iv_i , which is not zero, as a_i is not zero and the i th component of v_i is not zero. But this means that $a_1v_1 + \cdots + a_nv_n$ cannot be zero, as it has a nonzero component. This contradicts our assumption. So the columns must be linearly independent.

p. 51, #2b. The three vectors do not form a basis for \mathbb{R}^3 , as they are not linearly independent. In particular, $3(2, -4, 1) + 4(0, 3, -1) - 1(6, 0, -1) = 0$.

p.53, #19. a) Given V a vector space of dimension n and S a spanning subset, use the following algorithm: Pick a nonzero element s_1 of S (such an element exists, as S spans V). If $n = 1$, we can stop here and we have a basis, as the one element is linearly independent. Otherwise, we pick a new element s_2 , which is not in the span of what we have picked so far (again, such an element exists, as S spans V). Then s_1 and s_2 are linearly independent. If $n = 2$, we have our basis, as the two elements are linearly independent. If not, pick a nonzero element s_3 of S , which is not in the span of s_1 and s_2 . Repeating this process until we pick n elements will give our desired basis.

b) We know that S contains a basis for V , and that such a basis has n elements. So S contains at least n elements.

p.54, #30. a) Let W_1 be the xy -plane, which has dimension 2, and let W_2 be the x -axis, which has dimension 1. Then $W_1 \cap W_2 = W_2$, which has dimension 1.

b) Let W_1 be the xy -plane, which has dimension 2, and let W_2 be the z -axis, which has dimension 1. Then $W_1 \cap W_2 = \mathbb{R}^3$, which has dimension 3.

c) Let W_1 be the xy -plane, which has dimension 2, and let W_2 be the xz -plane, which has dimension 2. Then $W_1 \cap W_2$ is the x -axis, which has dimension 1. But $W_1 + W_2$ is \mathbb{R}^3 , which has dimension 3.

Extra Credit: p.53, #27 a) Since W_1 and W_2 are finite dimensional, their intersection $W_1 \cap W_2$ is as well, so we can take a basis $\{u_1, \dots, u_k\}$. Also, since $W_1 \cap W_2 \subset W_1$, we can extend this basis to a basis of W_1 (and likewise for W_2). So let $\{u_1, \dots, u_k, v_1, \dots, v_m\}$ be a basis for W_1 and let $\{u_1, \dots, u_k, w_1, \dots, w_p\}$ be a basis for W_2 . We claim that

$$B = \{u_1, \dots, u_k, v_1, \dots, v_m, w_1, \dots, w_p\}$$

is a basis for $W_1 + W_2$. Assuming this claim is true, it shows that $W_1 + W_2$ is finite dimensional, and that

$$\dim W_1 + W_2 = k + m + p = (k + m) + (k + p) - k = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

To prove the claim, we note that B spans $W_1 + W_2$, because B spans W_1 and W_2 (convince yourself of this!). To show that B is linearly independent, assume for a contradiction that we

can write

$$a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m + c_1w_1 + \cdots + c_pw_p = 0,$$

with the a_i, b_i, c_i scalars, not all of which are zero. Well, we have

$$a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m = -(c_1w_1 + \cdots + c_pw_p),$$

and since the left hand side is in W_1 , so is the right hand side. But the right hand side is in W_2 by our setup, so it is in $W_1 \cap W_2$. This means that it is in the span of $\{u_1, \dots, u_k\}$. But since $\{u_1, \dots, u_k, w_1, \dots, w_p\}$ is linearly independent, we must have $c_1 = c_2 = \cdots = c_p = 0$. By using the same argument applied to the equality

$$a_1u_1 + \cdots + a_ku_k + c_1w_1 + \cdots + c_pw_p = -(b_1v_1 + \cdots + b_mv_m),$$

we obtain that $b_1 = b_2 = \cdots = b_m = 0$. But then, since that u_i are a basis of $W_1 \cap W_2$, we must have $a_1 = a_2 = \cdots = a_k = 0$. This is a contradiction, so B is linearly independent. Thus B is a basis.

b) Given that $V = W_1 + W_2$, we know that $V = W_1 \oplus W_2$ iff $W_1 \cap W_2 = \{0\}$, or equivalently, that $\dim W_1 \cap W_2 = 0$. But by part (a), we have

$$\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(V).$$

So $\dim(W_1 \cap W_2) = 0$ precisely when $\dim V = \dim(W_1) + \dim(W_2)$.

Extra Credit: practice problem #38. S is indeed a vector subspace of \mathbb{R}^3 . If we write the system as

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 52 & 37 \\ 31 & 1287 & 389 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then we see that if (x_1, y_1, z_1) and (x_2, y_2, z_2) are solutions, then

$$\begin{aligned} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 52 & 37 \\ 31 & 1287 & 389 \end{pmatrix} \begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ az_1 + bz_2 \end{pmatrix} = \\ & a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 52 & 37 \\ 31 & 1287 & 389 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 & 1 \\ 1 & 52 & 37 \\ 31 & 1287 & 389 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \\ & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus $(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \in S$, so S is a vector subspace. The exact same argument works for any homogeneous system.

If we have a nonhomogeneous system, then the set of solutions is NOT a vector subspace of \mathbb{R}^3 , as if (x_1, y_1, z_1) and (x_2, y_2, z_2) are solutions, you can verify that $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ are solutions to the associated homogeneous system, and thus not solutions to the nonhomogeneous system.