Solution Set 2: Linear Transformations

p.52, #13. For W_1, a basis is given by

$$\{(0,1,0,0,0),(0,0,0,0,1),(1,0,1,0,0),(1,0,0,1,0)\}.$$

For W_2, a basis is given by

$$\{(0,1,1,1,0),(1,0,0,0,-1)\}.$$

In the first case, the dimension is 4, and in the second case, it is 2.

Non-book Problem. We use Lagrange interpolation. The four lagrange polynomials are

$$f_0(x) = \frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} = \frac{x^3-6x^2+11x-6}{-6},$$

$$f_1(x) = \frac{(x)(x-2)(x-3)}{(1)(-1)(-2)} = \frac{x^3-5x^2+6x}{2},$$

$$f_2(x) = \frac{(x)(x-1)(x-3)}{(2)(1)(-1)} = \frac{x^3-4x^2+3x}{-2},$$

$$f_3(x) = \frac{(x)(x-1)(x-2)}{(3)(2)(1)} = \frac{x^3-3x^2+2x}{6}.$$

To find the appropriate polynomial, we take the sum

$$0f_0+1f_1+2f_2+4f_3 = \frac{x^3-5x^2+6x}{2} - (x^3-4x^2+3x) + 2\frac{x^3-3x^2+2x}{3},$$

or $\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{4}{3}x$.

p.69, #3. T is linear because $T(a(x_1,y_1)+(x_2,y_2)) = T(ax_1+x_2, ay_1+y_2) = (ax_1+x_2+ay_1+y_2, 0, 2ax_1+2x_2-ay_1-y_2) = (ax_1+ay_1, 0, 2ax_1-ay_1)+(x_2+y_2, 2x_2-y_2) = aT(x_1,y_1)+T(x_2,y_2)$. A basis for the kernel is $\{(0,0)\}$. A basis for the image (which can be generated by the image of a basis), is $\{(1,0,2),(1,0,-1)\}$. So the nullity is 0 and the rank is 2. Since the dimension of V is 2, this satisfies the dimension theorem. Lastly, T is one-to-one, as its nullity is zero, but it is not onto, as its rank is not 3.

p.69, #9. a) $T(2(1,0)) = T(2,0) = (1,0)$, but $2T(1,0) = (2,0)$. These are not the same, so T is not linear.
b) $T(2(1,0)) = T(2,0) = (2,4)$, but $2T(1,0) = (2,2)$. These are not the same, so T is not linear.
c) $T(2(\pi/2,0)) = T(\pi,0) = (0,0)$, but $2T(\pi/2,0) = (2,0)$. These are not the same, so T is not linear.
d) $T((1,0)+(-1,0)) = T(0,0) = (0,0)$, but $T(1,0)+T(-1,0) = (1,0)+(1,0) = (2,0)$. These are not the same, so T is not linear.
e) $T(2(1,0)) = T(2,0) = (3,0)$, but $2T(1,0) = (4,0)$. These are not the same, so T is not linear.

p.70, #11. Since $(1,1)$ and $(2,3)$ form a basis of $\mathbb{R}^2$ (they are lin. indep., so since $\mathbb{R}^2$ is

2-dimensional they form a basis), we know that there exists a unique linear transformation that takes them to any given vectors, in this case, $(1, 0, 2)$ and $(1, -1, 4)$. Since $(8, 11) = 2(1, 1) + 2(2, 3)$, we have $T(8, 11) = 2T(1, 1) + 3T(2, 3) = 2(1, 0, 2) + 3(1, -1, 4) = (5, -3, 16)$.

p.72, #30. We want to show that $\ker T_W = \ker T \cap W$. Well, if $w \in \ker T_W$, then by definition $w \in W$, and also, $w \in \ker T$ as $T(w) = 0$. Conversely, if $w \in \ker T \cap W$, then $T_W(w) = T(w) = 0$, so $w \in \ker T_W$. So $\ker T_W = \ker T \cap W$, as anything in one side is in the other side.

Now, we show that im $T_W = T(W)$. Well, if $x \in$ im T_W, then there exists $w \in W$ such that $T_W(w) = x$. But then $T(w) = X$, so $x \in T(W)$. Conversely, if $x \in T(W)$, then there exists $w \in W$ such that $T(w) = x$. But then $T_W(w) = x$, so $x \in$ im T_W.

Extra Credit, #86. For the first part, given f injective (that is, 1-1), we want to define $g(b)$, for any $b \in$ im f. Well, say that $b = f(a)$. We know there is only one a for which this works, because f is injective. So we define $g(b) = a$ in this case. We see that $g \circ f(a) = a$, so $g \circ f$ is the identity on A.

In the second case, given f surjective, we want to define $g(b)$, for any $b \in B$. Well, since f is surjective, we know that there exists an a such that $f(a) = b$. Then let $g(b) = a$ in this case. We see that $f \circ g(b) = b$ for any $b \in B$, and thus $f \circ g$ is the identity on B.