Solution Set 4: Elementary Matrices and Row Reduction

p.142, #2. A gets transformed into B by $C_2 - 2C_1$. B goes to C by $R_2 - R_1$. To go to I_3, the following sequence is one that works:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{pmatrix}$$
$$\xrightarrow{\frac{-1}{2}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix}$$
$$\xrightarrow{R_3 + 3R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_2 - R_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_1 - 3R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

p.142, #7. We could prove this directly by multiplication, but instead we will use a slicker method. Since E is an $n \times n$ suqare matrix, L_E takes vectors of length n to vectors of length n. In particular, if E represents R_{ij}, then $(L_E)v$ is the same as v, except the ith and jth entries are switched. Since applying L_E twice switches them back again, we have that $(L_E \circ L_E)(v) = v$, and thus that $L_E \circ L_E = Id$. But $L_E \circ L_E = L_{E^2}$. So $L_{E^2} = Id$ and thus E^2 is the identity (by Theorem 2.15 (b) and (f)).

p.155, #2d. Since the second row is twice the first, there is only one linearly independent row. So the rank is 1.

p.155, #2g. Looking at the columns, we see that three of them are the same, and the zero column can never be part of a linearly independent set. So there is only one linearly independent column, and the rank is 1.

p.156, #3. If A is the zero matrix, then the rank is zero because there are not linearly independent columns (as the zero vector is never part of a linearly independent set). Conversely, if the rank of A is zero, it means than A has no linearly independent columns. This means that no column can be nonzero, i.e., A is the zero matrix.

p. 156, #6e. Since T is a linear transformation between two isomorphic vector spaces (both isomorphic to $\mathbb{R}^3$), it is invertible if and only if it is injective. To prove that T is injective, note that $T(f) = (0, 0, 0)$ if and only if $f(-1) = f(0) = f(1) = 0$, i.e., if and only if $-1, 0, 1$ are all roots of f. But f has degree no greater than 2, so it can only have 3 roots if it is the zero polynomial. Thus, the kernel of T is zero, and T is injective. So T is invertible.

To find the inverse of T, we write T as a matrix with respect to the standard bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$. Since $1 \mapsto (1,1,1), x \mapsto (-1,0,1)$, and $x^2 \mapsto (1,0,1)$, the matrix is

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The inverse of this matrix is

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix},$$

so $L_{A^{-1}} : \mathbb{R}^3 \to P_2(\mathbb{R})$ is the inverse transformation T^{-1}.

Note: You can also find the inverse by using Lagrange interpolation to find polynomial preimages via T for the points $(1,0,0), (0,1,0)$, and $(0,0,1)$. Then T^{-1} on these vectors will be defined by taking them to their repsective preimages.

p.157, #14. a) Say $w \in \text{im } (T+U)$. Then $w = (T+U)(v) = T(v) + U(v)$ for some v. But this means that $w \in \text{im } T + \text{im } U$, because $T(v) \in \text{im } T$ and $U(v) \in \text{im } U$. So $\text{im } (T+U) \subseteq \text{im } T + \text{im } U$.

b) First note that $\dim(\text{im } T + \text{im } U) \leq \dim(\text{im } T) + \dim(\text{im } U) = \text{rank } T + \text{rank } U$ (because a basis for im T combined with a basis for im U certainly span im $T+$im U). Since $\text{im } (T+U) \subseteq \text{im } T + \text{im } U$, we have that $\dim(\text{im } (T+U)) \leq \dim(\text{im } T + \text{im } U) \leq \dim(\text{im } T) + \dim(\text{im } U)$. But this says $\text{rank } (T+U) \leq \text{rank } T + \text{rank } U$.

c) This follows from part (b) by substituting L_A for T and L_B for U and noting that $L_{A+B} = L_A + L_B$. Specifically, $\text{rank } (A+B) = \text{rank } L_{A+B} = \text{rank } L_A + L_B \leq \text{rank } L_A + \text{rank } L_B = \text{rank } A + \text{rank } B$.

p.157, #15. For clarity's sake, let's say A has p columns and B has q columns. Then, for $j \leq p$ (the left side of the line), we have that $M(A|B)_{ij} = \sum_{k=1}^{n} M_{ik}A_{kj}$ and for $j > p$ (the right side of the line), $M(A|B)_{ij} = \sum_{k=1}^{n} M_{ik}B_{k(j-p)}$. But these are just the respective entries of MA and MB, specifically, MA_{ij} and $MB_{i(j-p)}$. Now, the ijth entry of $(MA|MB)$ is MA_{ij} if $j \leq p$ and $MB_{i(j-p)}$ if $j > p$. So we see that the entries of $M(A|B)$ and $(MA|MB)$ match. So $M(A|B) = (MA|MB)$.

p.158, #21. Consider $L_A : \mathbb{R}^n \to \mathbb{R}^m$. I claim that finding B such that $AB = I_m$ is the same as finding $T : \mathbb{R}^m \to \mathbb{R}^n$ linear such that $L_A T = Id_{\mathbb{R}^m}$. The claim is true because, if β is the standard basis for $\mathbb{R}^n$ and γ is the standard basis for $\mathbb{R}^m$, we can take $B = [T]_\gamma^\beta$, resulting in $L_B = T$. So $Id_{\mathbb{R}^m} = L_A T = L_A L_B = L_{AB}$, and thus $AB = I_m$.

To find the desired T, let $(e_1, \ldots, e_m)$ be the standard basis for $\mathbb{R}^m$. Then define $T(e_i) = v_i$, for some v_i such that $L_A(v_i) = e_i$. Expand by linearity to define T on all of $\mathbb{R}^m$. We know that such a v_i exists, because L_A has rank m, and is thus onto. Then $L_A T(e_i) = e_i$ for all i. Since $L_A T$ is the identity on a basis, it is the identity transformation on $\mathbb{R}^m$.