Solution Set 5: Diagonalization and Change of Basis

p.209, #3. The matrix on the left hand side is gotten from the matrix on the right hand side by applyting the elementary row operations $2R_1, 3R_2, 7R_3$, and then $R_2 + 5/7R_3$. These multiply the determinant by 2, 3, 7, and 1 respectively, for a total multiplication of 42. So $k = 42$.

p.210, #21. The matrix

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

row reduces, using only elementary row operations of type 3, to

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

The determinant of this matrix is 95, as it is upper triangular, so we just multiply the diagonal entries. Since we used only row operations of type 3, which do not change the determinant, 95 is also the determinant of the original matrix.

p.210, #26. Since we can transform $-A$ to A by multiplying each of the n columns by -1, we have that the determinant of $-A$ is equal to $(-1)^n$ times the determinant of A. So they will be the same if and only if n is even or if $\det A = 0$.

p.217, #9. If M is nilpotent, then there exists some k such that $M^k = 0$. So $\det M^k = 0$. But $\det M^k = (\det M)^k = 0$. So since $(\det M)^k = 0, \det M = 0$.

p.108, #3d. We seek the matrix $[I_{P_2(\mathbb{R})}]_{\beta'}^{\beta}$, which, as we said in class, is equal to the matrix $([I_{P_2(\mathbb{R})}]_{\beta}^{\gamma})^{-1}[I_{P_2(\mathbb{R})}]_{\beta'}^{\gamma}$, where γ is the standard basis $(1, x, x^2)$. This matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix},$$

which comes out to

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$

p.109, #9. Since A and B are similar, write $A = Q^{-1}BQ$. Then $\operatorname{tr} A = \operatorname{tr} Q^{-1}BQ = \operatorname{tr} (Q^{-1}B)Q = \operatorname{tr} Q(Q^{-1}B) = \operatorname{tr} B$, where for the second to last equality, we use exercise 12 of section 2.3 (which was on the previous homework).

p.247, #2d. We know that $[L_A]_\beta = Q^{-1}AQ$ for $Q = [Id_{\mathbb{R}^3}]_\beta^{std}$, so

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}.$$

So

$$[L_A]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix},$$

which comes out to

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$

Extra Credit. Clearly the statement is true for all 1×1 matrices. So assume it is true for all $n \times n$ matrices. We want to prove it for all $(n+1) \times (n+1)$ matrices. For cleaner notation, let $A = [a_{ij}]$ be an $(n+1) \times (n+1)$ matrix, and denote by A_{ij} not the ijth entry of A, but rather the $n \times n$ matrix obtained by eliminating the ith row and jth column of A. Also, let $A^T = [a_{ij}^T]$ and denote by A_{ij}^T the $n \times n$ matrix obtained by eliminating the ith row and jth column of A^T. Now, calculating the determinant of A by expansion along the first row, we get

$$\det A = \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det A_{1i}.$$

By inspection, one can see that $(A_{1i})^T = A_{i1}^T$, or $A_{1i} = (A_{i1}^T)^T$. Also, $a_{1i} = a_{i1}^T$. So our formula for $\det A$ can be rewritten as

$$\det A = \sum_{i=1}^{n} (-1)^{1+i} a_{i1}^T \det((A_{i1}^T)^T).$$

But by the induction hypothesis, $\det(A_{i1}^T)^T = \det(A_{i1}^T)$, so we can write

$$\det A = \sum_{i=1}^{n} (-1)^{1+i} a_{i1}^T \det(A_{i1}^T).$$

But this is just the formula for the determinant of A^T evaluated by expansion along the first column. So $\det A = \det A^T$.