Solution Set 7: Inner Product Spaces

Proof of 6.1b. We have $\langle x, cy\rangle = \overline{\langle cy, x\rangle} = \overline{c}\overline{\langle y, x\rangle} = \overline{c}\langle x, y\rangle$.

Proof of 6.2a. $\|cx\| = \sqrt{\langle cx, cx\rangle} = \sqrt{c\overline{c}\langle x, x\rangle} = |c|\langle x, x\rangle$.

p.322, #4. $\langle A, A\rangle = \text{tr } A^*A = \text{tr } \begin{pmatrix} 1 & 3 \\ 2-i & -i \end{pmatrix} \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} = \text{tr } \begin{pmatrix} 10 & 2+4i \\ 2-4i & 6 \end{pmatrix} = 16.$ So $\|A\| = 4$. Likewise, $\langle B, B\rangle = \text{tr } B^*B = \text{tr } \begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix} = \text{tr } \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = 4$. So $\|B\| = 2$. $\langle A, B\rangle = \text{tr } B^*A = \text{tr } \begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} = \text{tr } \begin{pmatrix} 1-4i & 3-i \\ 3i & -1 \end{pmatrix} = -4i$.

p.322, #8. a) Not positive: $\langle (1,2), (1,2)\rangle = 1 - 4 = -3$.

b) Not positive: $\langle -I, -I\rangle = -2$.

c) Not positive: $\langle 1, 1\rangle = 0$.

p.323, #10. $\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$. Since x and y are orthogonal, the middle two terms are zero. So we are left with $\langle x, x\rangle + \langle y, y\rangle$, which is equal to $\|x\|^2 + \|y\|^2$. This is the pythagorean theorem in $\mathbb{R}^2$, when we let x and y represent the legs of a right triangle (which are orthogonal) and $x + y$ is the hypotenuse.

p. 335, #2a. Set $w_1 = (1, 1, 1), w_2 = (0, 1, 1), w_3 = (0, 0, 1)$. Using the Gram-Schmidt algorithm, we set $v_1 = w_1/\|w_1\| = \frac{1}{\sqrt{3}}(1, 1, 1)$. Now we calculate v_2. To get an orthogonal vector, we calculate $w_2 - \langle w_2, v_1\rangle v_1 = (0, 1, 1) - \frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}(1, 1, 1) = (-2/3, 1/3, 1/3)$. Normalizing gives $v_2 = \frac{1}{\sqrt{6}}(-2, 1, 1)$.

Lastly, we find v_3. To get an orthogonal vector, we calculate $w_3 - \langle w_3, v_1\rangle v_1 - \langle w_3, v_2\rangle v_2$. Plugging in for w_3, v_1, and v_2, we get $(0, 0, 1) - \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}(1, 1, 1) - \frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}(-2, 1, 1) = (0, 0, 1) - (1/3, 1/3, 1/3) - (-1/3, 1/6, 1/6) = (0, -1/2, 1/2)$. Normalizing gives $v_3 = \frac{1}{\sqrt{2}}(0, -1, 1)$. So our orthonormal basis is $\frac{1}{\sqrt{3}}(1, 1, 1)$, $\frac{1}{\sqrt{6}}(-2, 1, 1)$, and $\frac{1}{\sqrt{2}}(0, -1, 1)$.

p.335, #4. We seek $(x, y, z) \in \mathbb{C}^3$ such that $(x, y, z) \cdot (1, 0, i) = (x, y, z) \cdot (1, 2, 1) = 0$. These two conditions yield the equations $x - iz = 0$ and $x + 2y + z = 0$. We set $x = iz$ (from the first condition) and substitute into the second condition to obtain $iz + 2y + z = 0$. So $y = \frac{(-1-i)}{2}z$. So a generator for $S^\perp$ is $(i, \frac{-1-i}{2}, 1)$.

p.337, #17c. We first find an orthonormal basis for W. W is two dimensional, so we need only find two linearly independent vectors. Let's start with the basis $\{1, x\}$. We apply Gram-Schmidt. 1 is already normalized, so we don't need to do anything with it. So set $v_1 = 1$. Then calculate $x - \langle x, 1\rangle \cdot 1$. This is $x - 1/2$. We also find out that $\|x - 1/2\|^2 = \langle x - 1/2, x - 1/2\rangle = \int_0^1 (x^2 - x + 1/4)dx = 1/3 - 1/2 + 1/4 = 1/12$. So $\|x - 1/2\| = \frac{1}{\sqrt{12}}$. Normalizing, we get $v_2 = \sqrt{12}(x - 1/2)$.

Now, we find the orthogonal projection using the formula from class—it is equal to $\langle h(x), v_1\rangle v_1 + \langle h(x), v_2\rangle v_2$, which is $\int_0^1 h(x)dx + \sqrt{12}(x-1/2)\int_0^1(\sqrt{12}(x-1/2)(h(x)))dx = 4+3/2-2/3+12(2+1-1/2-2-3/4+1/3)(x-1/2) = x+13/3$.

p.337, #18. We calculate the distance from $4+3x-2x^2$ to $x+13/3$, which is the closest point, because of the theorem from class. This distance is $\|-1/3+2x-2x^2\|$. Well, $\|-1/3+2x-2x^2\|^2 = \int_0^1(-1/3+2x-2x^2)^2dx = 4/5+4/3+1/9-2/3+4/9-2 = 1/45$. So $\|-1/3+2x-2x^2\| = 1/\sqrt{45}$.

Extra Credit: p.325, #23. a) This is true, because the length of a vector is always positive.

b) $d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1|\|y-x\| = \|y-x\| = d(y,x)$.

c) $d(x,y) = \|x-y\|$. Since $(x-z)+(z-y) = x-y$, the triangle inequality gives that $\|x-z\|+\|z-y\| \geq \|x-y\|$. So $d(x,y) \leq d(x,z)+d(z,y)$.

d) $d(x,x) = \|x-x\| = \|0\| = 0$.

e) $d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$. This has to be positive if $x-y \neq 0$, by the rules of inner products. So it is positive if $x \neq y$.