Lecture 3

Smooth divisors, local large deviations and CLT's

The derivation of asymptotic formulae from multivariate generating functions is in some ways parallel to what we have just seen in the univariate case. As you might guess, it is more complicated. Cauchy's integral works in any dimension, but the business of complex contour integration becomes quite geometric once the dimension is high enough for there to be nontrivial geometry, that being (complex) dimension two. I have devoted a considerable portion of my efforts this decade to creating an infrastructure for deriving asymptotics from multivariate generating functions. Some tokens of this may be seen in the mvGF website http://www.cs.auckland.ac.nz/ mcw/Research/mvGF/asymultseq/ created by my collaborate Mark Wilson, and in my collection of Father's Day "Asymptotics of a multivariate generating function" dinner plates. I hope you will find, as a result of these lectures, that the multivariate theory is not mystifying and in fact forms a cogent framework that complements all the work that has been done to derive multivariate generating functions for various combinatorial applications.

The number of possible asymptotic behaviors of $\{a_\mathbf{r}\}$ is great, but a basic estimate holds in many cases:

$$a_\mathbf{r} \sim C(\hat{\mathbf{r}})|\mathbf{r}|^{(1-d)/2}\mathbf{Z}(\hat{\mathbf{r}})^{-\mathbf{r}} \tag{3.1}$$

where $\hat{\mathbf{r}}$ is the unit vector parallel to $\mathbf{r}$. In order to motivate the technicalities of the derivation, which are substantial, I will devote one lecture to the statement and consequences of the first main theorem, which proves the basic estimate. The next lecture will contain the proof of this theorem and some of its generalizations.

3.1 The smooth point theorem

Critical points and minimality

Our immediate goal is to explain (3.1) and to formulate this estimate into a precise theorem. Let F be a meromorphic function, written as P/Q where P and Q are entire functions. In the generalizations below we will examine further the different Laurent expansions and their domains o convergence, but for now it suffices to consider a particular Laurent expansion

$$F(\mathbf{Z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$$

and its domain of convergence; this domain, $\mathcal{D}$, is **pseudo-convex**, meaning that it is the union of tori $\exp(\mathbf{x} + i\mathbb{R}^d)$ as $\mathbf{x}$ varies over some convex set, B; see, e.g., [Hör90]. We begin with the minimizing point $\mathbf{x}_*(\hat{\mathbf{r}})$ which leads to the specification of the point $Z(\hat{\mathbf{r}})$ in (3.1).

Given a real unit vector $\mathbf{r}$, let $h = h_{\mathbf{r}}$ denote the $\mathbb{R}$-linear function $\mathbf{x} \mapsto -\mathbf{r} \cdot \mathbf{x}$. If $\mathbf{Z}$ is a smooth point of $\mathcal{V}$, we say that it is a critical point for h if $dh|_{\mathcal{V}}(\mathbf{Z}) = 0$. There is a notion of criticality for non-smooth points, which requires the introduction of a stratification of $\mathcal{V}$; this notion is useful for theorems beyond the most basic, but for now we do not need it because smoothness is a hypothesis of the basic theorem.

Let $c(\mathbf{r}) := \inf_{\mathbf{x} \in B} h_{\mathbf{r}}(\mathbf{x})$ be the infimum value of h on B; if $c(\mathbf{r}) > -\infty$ then there is a point $\mathbf{x} \in \partial B$ with $h(\mathbf{x}) = c$. If B is strictly convex then the point $\mathbf{x}(\mathbf{r})$ is unique. In the case of a unique minimizing point, we denote this point by $\mathbf{x}_*(\mathbf{r})$. A linear function cannot be minimized on the interior of a set, so F fails to be analytic in a neighborhood of some $\exp(\mathbf{x}_* + i\mathbf{y})$, and meromorphicity of F implies that $F(\exp(\mathbf{x}_* + i\mathbf{y})) = 0$, or in other words,

$$\mathrm{ReLog}^{-1}(\mathbf{x}_*) \cap \mathcal{V} \neq \emptyset\,.$$

The points of $\mathrm{ReLog}^{-1}(\mathbf{x}_*) \cap \mathcal{V}$ are called **minimal** points. A further classification introduced in [PW02] is these points are called **strictly minimal**, **finitely minimal** or **torally minimal** according to whether the cardinality of $\mathrm{ReLog}^{-1}(\mathbf{x}_*) \cap \mathcal{V}$ is one, a finite number greater than one, or infinity. The more general definition of critical point makes this classification irrelvant, but the simplest statement of the basic theorem is the one from [PW02], in which strict minimality is assumed. Observe that $\mathbf{x}_*$ depends on $\mathbf{r}$ only through its direction, $\hat{\mathbf{r}}$. We define

$$\mathbf{Z}(\mathbf{r}) = \mathbf{Z}(\hat{\mathbf{r}}) = \mathbf{x}_*(\hat{\mathbf{r}}) \tag{3.2}$$

whenever strict minimality holds, that is, $\mathrm{ReLog}^{-1}(\mathbf{x}_*(\hat{\mathbf{r}})) \cap \mathcal{V}$ has cardinality one.

A first theorem

Any point where that the gradient of Q does not vanish is a smooth point. If in particular, $\frac{\partial Q}{\partial z_d}$ does not vanish a point at $\mathbf{Z}$, then in a neighborhood of $\mathbf{Z}$, the projection to the first $d-1$ coordinates is a diffeomorphism. This will still be true after the logarithm is applied, coordinatewise, hence there is a smooth function $g : \mathbb{C}^{d-1} \to \mathbb{C}$ such that $f(\mathbf{0}) = 0$ and

$$Q(\exp(\mathbf{z}+\mathbf{u})) = 0 \Leftrightarrow u_d = g(u_1, \ldots, u_{d-1}) \tag{3.3}$$

for all $\mathbf{u}$ is a neighborhood of the origin in $\mathbb{C}^{d-1}$. For any complex valued function W and any point $\mathbf{x}$, let

$$\mathcal{H}(W, \mathbf{x}) := \left(\frac{\partial^2 W}{\partial x_i \, \partial x_j}(\mathbf{x}) \right)_{1 \le i,j \le d-1}$$

denote the Hessian matrix of W at the point $\mathbf{x}$. The function g, defined above, depends on $\mathbf{Z}$, so the Hessian matrix $\mathcal{H}(g, \mathbf{0})$ depends on $\mathbf{Z}$ as well; denote by $\mathcal{H}(\hat{\mathbf{r}})$ the quantity $\mathcal{H}(g, \mathbf{0})$ in the case that $\mathbf{Z} = \mathbf{Z}(\hat{\mathbf{r}})$. The following result is stated and proved in the case of ordinary power series as Theorem 3.5 of [PW02].

Theorem 3.1. *Let $F = P/Q = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$ on some domain $\mathcal{D} = \exp(B + i\mathbb{R}^d)$. Then*

$$a_{\mathbf{r}} \sim (2\pi |r_d|)^{(1-d)/2} \frac{P}{z_d (\partial Q / \partial z_d)} (\mathbf{Z}(\hat{\mathbf{r}})) \, |-\mathcal{H}(\hat{\mathbf{r}})|^{-1/2} \mathbf{Z}(\hat{\mathbf{r}})^{-\mathbf{r}} \tag{3.4}$$

as $\mathbf{r} \to \infty$, uniformly as $\hat{\mathbf{r}}$ varies over a compact set K on which $\mathbf{Z}(\hat{\mathbf{r}})$ is a strictly minimal smooth point such that P and $(\partial Q/\partial z_d)(\mathbf{Z})$ and $|\mathcal{H}(\hat{\mathbf{r}})|$ are all nonvanishing. $\square$

Extensions

Some easy generalizations are as follows. It follows for free from the proof that the hypothesis of mermorphicity may be weakened to meromorphicity in any domain $\exp(B' + i\mathbb{R}^d)$ where B' is an open set containing B as well as each $\mathbf{x}_*(\hat{\mathbf{r}})$ for $\hat{\mathbf{r}} \in K$, the compact set in the conclusion. Another extension with virtually no cost is proved as [PW02, Corollary 3.7].

Corollary 3.2. *If the conditions of Theorem 3.1 hold except that strict minimality is replaced by finite minimality, then*

$$a_{\mathbf{r}} \sim (2\pi |r_d|)^{(1-d)/2} \sum_{\mathbf{Z} \in W(\hat{\mathbf{r}})} \frac{P}{z_d (\partial Q / \partial z_d)} (\mathbf{Z}) \, |-\mathcal{H}(\mathbf{Z})|^{-1/2} \mathbf{Z}^{-\mathbf{r}} \tag{3.5}$$

where $W(\hat{\mathbf{r}}) := \mathrm{ReLog}^{-1}(\mathbf{x}_*(\hat{\mathbf{r}})) \cap \mathcal{V}$.

Extensions to general Laurent series and torally minimal points requires the discussion of amoebas and critical points of stratified spaces, which I will include in the next lecture along with one of the two proofs of Theorem 3.1.

3.2 Local large deviations

The minimizing point $\mathbf{x}_*$ looks like, and is, a dominating point from a large deviation computation. In this section, I will show how the classical local large devation estimate for partial sums of IID lattice variables follows rather directly from Theorem 3.1.

Tilted distributions, tilted covariance and rate function

Let ν be a probability measure on $\mathbb{Z}^d$ and assume sub-exponential tails, that is, $\nu(\mathbf{r}) = o(e^{-\epsilon|\mathbf{r}|})$ for some $\epsilon > 0$. Assume also that ν is not supported on a proper sub-lattice of $\mathbb{Z}^d$. Let ℓ be a real vector and define the tilted distribution ν_ℓ by

$$\nu_\ell(\mathbf{r}) = \frac{\nu(\mathbf{r})e^{\ell\cdot\mathbf{r}}}{\sum_{\mathbf{r}'} \nu(\mathbf{r}')e^{\ell\cdot\mathbf{r}'}} \,.$$

Let $\overline{\nu_\ell}$ denote the mean of the distribution ν_ℓ and let Γ_ℓ denote its covariance matrix; it follows from the nondegeneracy assumption that the determinant $|\Gamma_\ell|$ is strictly positive. We will see shortly that the means $\overline{\nu_\ell}$ of the tilted distributions are distinct. Let Λ denote the set of means of the tilted distributions. For $\mathbf{m} \in \Lambda$, define

$$\begin{aligned}
\ell(\mathbf{m}) &:= \ell \text{ such that } \overline{\nu_\ell} = \mathbf{m}\,; \\
\beta(\mathbf{m}) &:= -\mathbf{m}\cdot\ell(\mathbf{m})\,; \\
\Gamma(\mathbf{m}) &:= \Gamma_{\ell(\mathbf{m})}\,.
\end{aligned}$$

Theorem 3.3 (local LD for lattice disributions with small tails)**.** *The n-step transition probabilities $p_n(\cdot)$ satisfy*

$$p_n(\mathbf{r}) \sim (2\pi n)^{-d/2}\,|\Gamma(\mathbf{r}/n)|^{-1/2}\; e^{n\beta(\mathbf{r}/n)} \tag{3.6}$$

as $n \to \infty$, uniformly as $\mathbf{r}/n$ ranges over any compact subset of Λ.

We now derive this from Theorem 3.1.

Generating functions

The probability generating function

$$\Phi(\mathbf{Z}) := \sum_{\mathbf{r}\in\mathbf{Z}^d} \nu(\mathbf{r})\mathbf{Z}^{\mathbf{r}}$$

is an entire function, by the sub-exponential tail assumption. Letting $\mathbf{L} := \exp(\ell)$, the coordinatewise exponential, and $\mathbf{LZ}$ denote the coordinatewise product $(L_1Z_1, \ldots, L_dZ_d)$, the probability generating functions for the tilted distributions are given by

$$\Phi_\ell(\mathbf{Z}) = \frac{1}{\Phi(\mathbf{L})}\Phi(\mathbf{LZ}) \tag{3.7}$$

Let $\phi(\mathbf{z}) := \log \Phi(\exp(\mathbf{z}))$ denote the logarithmic generating function. Changing variables, we find the simple relation

$$\phi_\ell(\mathbf{z}) = \phi(\ell + \mathbf{z}) - \phi(\ell)\,. \tag{3.8}$$

Lemma 3.4.

$$\overline{\nu_\ell} \;=\; \nabla\phi(\ell) \tag{3.9}$$

$$\Gamma_\ell \;=\; \mathcal{H}(\phi, \ell)\,. \tag{3.10}$$

The vectors $\nabla\phi(\ell)$ are distinct for distinct ℓ. The function $\ell \mapsto \phi(\ell) - \mathbf{r}\cdot\ell$ has a unique minimum at ℓ if $\nabla\phi(\ell) = \mathbf{r}$ and is not bounded from below if there is no ℓ with $\nabla\phi(\ell) = \mathbf{r}$.

Proof: When $\ell = \mathbf{0}$, it is well known (and easy to verify directly) that $\overline{\nu} = \nabla\Phi(1, \ldots, 1)$. Changing variables then gives $\overline{\nu} = \nabla\phi(\mathbf{0})$. The statement from general ℓ follows from (3.8). For the covariance matrix, again it is well known and easy to verify that the untilted covariance matrix is given by $\Gamma = \mathcal{H}(\Phi, (1, \ldots, 1)$. Again, a change of variables gives $\Gamma = \mathcal{H}(\phi, \mathbf{0})$ and again the result for general ℓ follows from (3.8). The final statement of the lemma is a consequence of the fact that all the matrices Γ_ℓ are positive definite. $\square$

The spacetime generating function is the rational function in $d+1$ variables defined by

$$F(\mathbf{Z}, Y) := \sum_{\mathbf{r}\in\mathbb{Z}^d}\sum_{n\geq 0} p_n(\mathbf{0}, \mathbf{r})\mathbf{Z}^{\mathbf{r}}Y^n$$

where $p_n(\cdot, \cdot)$ are the n-step transition probabilities and we have named the $d+1$ coordinates $Z_1, \ldots, Z_d, Y$. Using the pseudo-norm $|(\mathbf{r}, n)| := n$, we have $\widehat{(\mathbf{r}, n)} = (\mathbf{r}/n, 1)$, which we abbreivate as $\hat{\mathbf{r}} = \mathbf{r}/n$. The convolution identity for p_n implies that

$$F(\mathbf{Z}, Y) = \frac{1}{1 - Y\Phi(\mathbf{Z})}\,. \tag{3.11}$$

Lemma 3.5. *The open domain of convergence $\mathcal{D}$ of F is* $\exp(B + i\mathbb{R}^d)$ *where*

$$B := \{(\ell, y) : y < -\phi(\ell)\}\,.$$

Proof: Fix any $\ell \in \mathbb{R}^d$. Then $Y\Phi(\mathbf{L}) = 1$, whence F has a pole at $(Y, \mathbf{L})$ and $(y, \ell) \notin B$. On the other hand, if $y + \phi(\ell) < 0$, then $Y\Phi(\mathbf{L}) < 1$. But $\Phi(\mathbf{Z})$ is a generating function with nonnegative coefficients, so the series $\sum_n (Y\Phi(\mathbf{L}))^n$ is summable, and it follows that the generating function F is absolutely summable in a neighborhood of $(\mathbf{L}, Y)$. $\square$

Proof of Theorem 3.3

We wish to apply Theorem 3.1 with $P = 1$ and $Q = 1 - Y\Phi(\mathbf{Z})$ (and d replaced by $d + 1$). Fix $\hat{\mathbf{r}} \in \Lambda$. By definition, $\mathbf{x}_*$ is the unique minimizer of $-(\hat{\mathbf{r}}, 1) \cdot (\mathbf{x}, y)$ as $\mathbf{x}$ and y vary with $y \leq -\phi(\mathbf{x})$. In other words, it is where the minimum of $\phi(\mathbf{x}) - \hat{\mathbf{r}} \cdot \mathbf{x}$ occurs, which we know from Lemma 3.4 to be at $\ell(\hat{\mathbf{r}})$.

Next, I claim that $(\mathbf{L}(\hat{\mathbf{r}}), 1/\Phi(\mathbf{L}(\hat{\mathbf{r}})))$ is a strictly minimal smooth point of $\mathcal{V}$. Clearly this is a smooth point of $\mathcal{V}$. To see strict minimality, note that for any other $(\mathbf{Z}, Y)$ with $|Z_j| = L_j$ and $|Y| = 1/\Phi(\mathbf{L})$, the aperiodicity assumption (ν is not supported on a proper sublattice) implies that the series $F(\mathbf{Z}, Y)$ have moduli summing to less than one, unless $\mathbf{Z} = \mathbf{L}$, in which case the sum is not one unless $Y = 1/\Phi(\mathbf{L})$. This proves that $(\mathbf{L}, 1/\Phi(\mathbf{L}))$ is a strictly minimal point.

We have already seen that $\partial Q/\partial Y$ and $|\mathcal{H}(\hat{\mathbf{r}})|$ are nonvanishing for all $\hat{\mathbf{r}} \in \Lambda$, so the hypotheses of Theorem 3.1 are verified and it remains only to interpret the conclusion. Easily,

$$\mathbf{Z}(\hat{\mathbf{r}})^{-\mathbf{r}} = \exp(-\mathbf{r} \cdot \mathbf{x}_*(\hat{\mathbf{r}})) = -n\hat{\mathbf{r}} \cdot \ell(\hat{\mathbf{r}}) = n\beta(\hat{\mathbf{r}})$$

by the definition of β. Together with the facts that $r_d = n$ and $Y(\partial Q/\partial Y) = Y\Phi(\mathbf{Z}) = 1$ on $\mathcal{V}$ and that d is now $d + 1$, this gives

$$a_{\mathbf{r}} \sim (2\pi n)^{-d} |-\mathcal{H}(\mathbf{r}/n)|^{-1/2} e^{n\beta(\mathbf{r}/n)}\,.$$

The proof is finished by observing that the parametrization in the definition of $\mathcal{H}$ is just $y = -\phi(\mathbf{x})$, whence $-\mathcal{H}(\mathbf{r}/n)$ is the Hessian matrix of ϕ at $\mathbf{r}/n$, which is equal to $|\Gamma(\mathbf{r}/n)|$ by (3.10). $\square$

3.3 Local CLT

Let Γ be a positive definite $d \times d$ matrix. The discrete Gaussian approximation $\mathcal{N}_n$ with mean $\mathbf{m}$ and covariance Γ is defined by

$$\mathcal{N}_n(\mathbf{r}) := (2\pi n)^{-d/2}|\Gamma|^{-1/2} \exp\left[\frac{1}{2}n^{-1}(\mathbf{r} - n\mathbf{m})^T\Gamma^{-1}(\mathbf{r} - n\mathbf{m})\right] . \tag{3.12}$$

The following result is true assuming only a third moment (see [LL08, Theorem 2.3.11]), though we are concerned here only with the sub-exponential case.

Theorem 3.6 (Local CLT)**.** *Let $\{\mathbf{X}_n\}$ be IID random variables in $\mathbb{Z}^d$ with mean $\mathbf{m}$, covariance Γ, sub-exponential tails, and not supported on any proper sublattice. Then the partial sums $\{\mathbf{S}_n\}$ satisfy*

$$\mathbb{P}(\mathbf{S}_n = \mathbf{r}) \sim \mathcal{N}_n(\mathbf{r}) \tag{3.13}$$

if $\mathbf{r} - n\mathbf{m} = o(n^{2/3})$.

Often, one speaks of the "central limit regime" where $\mathbf{r} - n\mathbf{m} = O(\sqrt{n})$ and the "large deviation regime" where $\mathbf{r} - n\mathbf{m} = \Theta(n)$ as two distinct regimes separated by some murky middle ground. In fact, in the case of small tails, the central limit result is a corollary of the large deviation result, the latter providing a seamless transition between (3.13) and the region of exponential decay. While it is hardly rocket science, it is worth seeing a derivation of the Local CLT from the Local LD result. This is really a fact about convex duals, as shown by the following fact whose proof may be found in [Roc66].

Lemma 3.7. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be convex and of class C^2 and let β be the convex dual, defined by*

$$\beta(\mathbf{r}) := \inf_{\ell} \phi(\ell) - \mathbf{r} \cdot \ell \, .$$

Then the quadratic term in ϕ at $\ell(\mathbf{r})$ is related to the quadratic term in β at $\mathbf{r}$ by inversion of the representing matrix:

$$\mathcal{H}(\phi, \ell(\mathbf{r})) = \mathcal{H}(\beta, \mathbf{r})^{-1} \, .$$

In particular, when ϕ is the logarithmic generating function and β the rate function for a small-tailed distribution, then the expression (3.10) *for $\mathcal{H}(\phi, \ell)$ implies*

$$\beta(\mathbf{r} + \mathbf{u}) = \beta(\mathbf{r}) + \mathbf{u} \cdot \nabla\beta(\mathbf{r}) + \frac{1}{2}\mathbf{u}^T\Gamma_\ell^{-1}\mathbf{u} \, . \tag{3.14}$$

$\square$

Proof of Theorem 3.6: As long as $\mathbf{r} - n\mathbf{m} = o(n)$, we have $\mathbf{r}/n \to \mathbf{m}$ so the constant $|\Gamma(\mathbf{r}/n)|^{-1/2}$ converges to $|\Gamma|^{-1/2}$. The rate function $\beta(\hat{\mathbf{r}})$ is smooth and achieves its maximum of zero at $\hat{\mathbf{r}} = \mathbf{m}$. The quantities $\ell(\mathbf{m})$, $\beta(\mathbf{m})$ and $\nabla\beta(\mathbf{m})$ all vanish, so (3.14) becomes

$$\beta(\hat{\mathbf{r}} + \mathbf{u}) = \frac{1}{2}\mathbf{u}^T\Gamma^{-1}\mathbf{u} + O(|\mathbf{u}|^3)\,.$$

Passing the factor of n^{-1} out of the argument then yields

$$\begin{aligned} n\beta\left(\frac{\mathbf{r}}{n}\right) &= n\left[\left(\frac{\mathbf{r}}{n} - \mathbf{m}\right)^T \Gamma^{-1}\left(\frac{\mathbf{r}}{n} - \mathbf{m}\right) + O\left(\left|\frac{\mathbf{r}}{n} - \mathbf{m}\right|^3\right)\right] \\ &= n^{-1}(\mathbf{r} - n\mathbf{m})^T\Gamma^{-1}(\mathbf{r} - n\mathbf{m}) + O\left(n^{-2}|\mathbf{r} - n\mathbf{m}|^3\right)\,. \end{aligned}$$

When $|\mathbf{r} - n\mathbf{m}| = o(n^{2/3})$, the last term goes to zero, and (3.6) and (3.12) imply (3.13). $\square$

3.4 Smooth point applications

Limiting probabilities of the form given by the basic theorem are called **Gaussian** or **Ornstein-Zernike**. Whenever asymptotics take this form, there is a central limit result. One such result is formulated in [PW08, Theorem 3.29].

Theorem 3.8. *Suppose the d-variate generating function $F = \sum a_{\mathbf{r}}\mathbf{Z}^r$ has nonnegative coefficients and let μ_k denote the "k^{th} slice" probability measure on $\mathbb{Z}^{d-1}$ defined by*

$$\mu_k(r_1, \ldots, r_{d-1}) := \frac{1}{\sum_{r_d = k} a_{\mathbf{r}}}\, a_{r_1,\ldots,r_{d-1},k}\,.$$

Define $G(x) := F(1, \ldots, 1, x)$. Suppose the G has a unique singularity of minimal modulus which is a simple pole at $X_0 = e^{x_0}$. If the map $\hat{\mathbf{r}} \mapsto \mathbf{Z}(\hat{\mathbf{r}})$ has nonsingular Hessian matrix Γ at $\hat{\mathbf{r}} := \nabla(F \circ \exp)(1, \ldots, 1, x_0)$ then

$$\sup_{\mathbf{r}} k^{(d-1)/2}|\mu_k(\mathbf{r}) - \mathcal{N}(\mathbf{r})| \to 0$$

where $\mathcal{N}$ is the discrete Gaussian approximation with mean $\hat{\mathbf{r}}$ and covariance Γ^{-1}. Letting $|\mathbf{r}|$ denote the L^1 norm $\sum_{j=1}^d r_j$, a similar result holds with μ_k replaced by the diagonal slice measure $\nu_k(\mathbf{r}) := a_{\mathbf{r}}/\sum_{|\mathbf{r}|=k} a_{\mathbf{r}}$ and $(1, \ldots, 1, x)$ replaced by $(x, \ldots, x)$.

This shows at once the utility and the limitations of the basic theorem. On the one hand, deducing Gaussian behavior from the form of the generating function is a powerful technique. In [CC86], for instance, a Gaussian limit law for directed percolation paths is derived by showing little more than meromorphicity of the generating function in a domain $\exp(B' + i\mathbb{R}^d)$ where B' extends just beyond B. A number of applications of the basic smooth point theorem are worked in detail in the survey paper [PW08], one of which will be reproduced in the remainder of this lecture. On the other hand, there are plenty of situations in which we know the limit behavior not to be Gaussian, so there must be a number of other types of generating functions for which quite different limit results hold. I will be able to touch on these only briefly, though more may be found in [PW04] and [BP08].

Application: horizontally convex polyominoes

A **horizontally convex polyomino** (HCP) is a union of cells $[a, a+1] \times [b, b+1]$ in the two-dimensional integer lattice such that the interior of the figure is connected and every row is connected. Formally, if $S \subseteq \mathbb{Z}^2$ and $P = \bigcup_{(a,b)\in S}[a, a+1] \times [b, b+1]$ then P is an HCP if and only if the following three conditions hold: $B := \{b : \exists a, (a, b) \in S\}$ is an interval; the set $A_b := \{a : (a, b) \in S\}$ is an interval for each $b \in \mathbb{Z}$; and whenever $b, b + 1 \in B$, the sets A_b and A_{b+1} intersect.

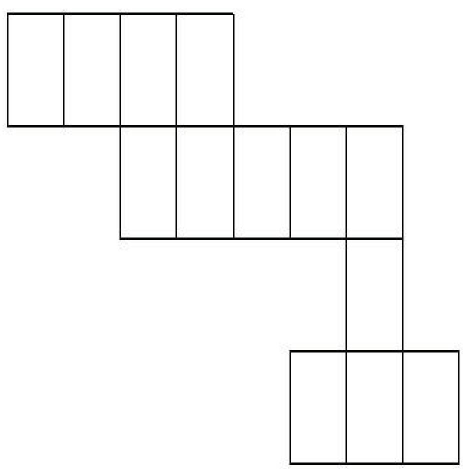

Figure 3.1: An HCP with 13 cells and 4 rows.

Let a_n be the number of HCP's with n cells, counting two as the same if they are translates of one another. Pólya [Pól69] proved that

$$\sum_n a_n x^n = \frac{x(1-x)^3}{1 - 5x + 7x^2 - 4x^3} . \tag{3.15}$$

Further discussion of the origins of this formula and its accompanying recursion may be found in [Odl95] and [Sta97]. The proof in [Wil94, pages 150–153] shows in fact that

$$F(x,y) = \sum_{n,k} a_{nk} x^n y^k = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x-x^2+x^3+x^2y)}\,, \tag{3.16}$$

where a_{nk} is the number of HCP's with n cells and k rows. Let us find an asymptotic formula for a_{rs}.

All the coefficients of $F(x,y)$ are nonnegative, which implies that the least positive real zero on any ray is a minimal point. We may verify without too much trouble that these points are strictly minimal and smooth.

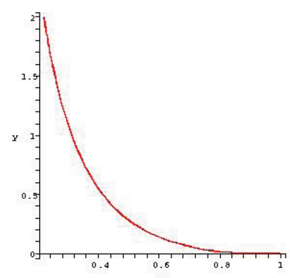

Figure 3.2: Minimal points of $\mathcal{V}$ in the positive real quadrant.

For bivariate generating functions $P(x,y)/Q(x,y)$, an explicit formula

$$\frac{P}{z_d \partial Q/\partial z_d}(\mathbf{Z}(\hat{\mathbf{r}}))\, \mathcal{H}(\hat{\mathbf{r}})^{-1/2} = P(x,y)\sqrt{\frac{-y\partial Q/\partial y}{K}}$$

is given in [PW02, Theorem 3.1], where (using subscripts for parial derivatives)

$$K(x,y) = -x^2 Q_x^2 y Q_y - x Q_x y^2 Q_y^2 - x^2 y^2 \left((Q_{xx} Q_y^2 + Q_{yy} Q_x^2 - 2 Q_{xy} Q_x Q_y\right)\,. \tag{3.17}$$

Using Maple's Groebner package, we find that

```
Basis([Q,diff(Q,x),diff(Q,y)] , tdeg(y,x));
```

returns the trivial basis [1], indicating no singularities, and

```
Basis([Q,K] , plex(y,x));
```

returns a basis whose first entry is the elimination polynomial

$$(5x^4 + 28x^3 + 62x^2 + 28x + 5)(x-1)^{12}$$

whose only real zero is at $x = 1$. Therefore, as long as $\mathbf{x}_* \neq (1,0)$, the Hessian does not vanish and the hypotheses of Theorem 3.1 are satisfied.

We deduce that the asymptotics for a_{rs} are uniform as s/r varies over a compact subset of the interval $(0,1)$ and are given by

$$a_{rs} \sim C(\hat{\mathbf{r}})\, r^{-1/2} x(r)^{-r} y(r)^{-s}$$

where we may use Maple to determine x, y and C as explicit functions of $L := s/r$, giving asymptotics for the number of HCP's with a given ratio of height to area. To determine x, we compute

```
Basis([Q,L*x*diff(Q,x) - y*diff(Q,y)] , plex(y,x));
```

returning the elimination polynomial

$$(L+1)x^4 + 4(L+1)^2x^3 + 10(L^2 + L - 1)x^2 + 4(2L-1)^2x + (L-1)(2L-1)\,.$$

The least positive real solution to this for a given L is $x(L)$. The value of $y(L)$ is a polynomial in x and L, which may be recovered from the second Gröbner basis element. We may plug these values in numerically to compute $K(x(L), y(L))$. Alternatively, we may use computer algebra again.

```
Basis([Q,L*x*diff(Q,x) - y*diff(Q,y),w-K] , plex(y,x,w));
```

produces a univariate elimination polynomial for w, satisfied when w takes on the value $K(x,y)$ for (x,y) constrained to satisfy $Q = 0$ and $yQ_y - LXQ_x = 0$. If you try it, you will find that K is quartic over $\mathbb{Q}[L]$, the coefficients of the annihilator of K being polynomials of degree 25 in L.