Chapter 3

Univariate asymptotics

Throughout this chapter, $f(z) = \sum_{n=0}^{\infty} a^n z^n$ will be a univariate generating function for the sequence $\{a_n\}$.

3.1 Rational functions: an explicit formula

For rational functions in one variable, estimation is not needed because there is an explicit formula for a_n. Some special cases form the basis for this. The first is obvious and the second is easy to check by induction on k.

$$\text{If } f(z) = \frac{1}{1 - z/r} \qquad \text{then } a_n = r^{-n} \tag{3.1}$$

$$\text{If } f(z) = \tfrac{1}{(1-z/r)^k} \qquad \text{then } a_n = \tbinom{n+k-1}{k-1} r^{-n} \tag{3.2}$$

Now let $f = p(z)/q(z)$ be any rational function, where p and q are relatively prime and we may assume $q(0) = 1$.

Case 1: distinct roots.

Suppose the roots $r_1, \ldots, r_t$ of q are distinct. Then

$$q(z) = \prod_{j=1}^{t} \left(1 - \frac{z}{r_j}\right) .$$

Let $q_j(z) := q(z)/(1 - z/r_j)$. The ideal generated by all the q_j is all of $\mathbb{C}[z]$, hence any polynomial p may be written as $\sum p_j q_j$ for some polynomials p_j. This proves the **partial fraction** expansion

$$f \;=\; \frac{p}{q} \;=\; \sum_{j=1}^{t} \frac{p_j \, q_j}{q} \;=\; \sum_{j=1}^{t} \frac{p_j}{1 - z/r_j} \, .$$

This may be written in the canonical form

$$f(z) = p_0(z) + \sum_{j=1}^{t} \frac{c_j}{1 - z/r_j}$$

where $\{c_j\}$ are constants that will shortly be evaluated as

$$c_j = \frac{p(r_j)}{q'(r_j)}. \tag{3.3}$$

By (3.1), for $n > \deg(p_0)$ we have $a_n = \sum_{j=1}^{t} c_j r_j^{-n}$. The leading term of this approximation is

$$a_n \sim \frac{p(r_*)}{q'(r_*)} r_*^{-n}$$

where r_* is the root of minimum modulus. If there are several roots of minimal modulus, $r_1, \ldots, r_\nu$, then the leading term is

$$a_n \sim \sum_{j=1}^{\nu} \frac{p(r_j)}{q'(r_j)} r_j^{-n} = |r_j|^{-n} \left(\sum_{j=1}^{\nu} \frac{p}{q'}(r_j)\omega_j^n \right)$$

where $\omega_j = r_j/|r_j|$ is on the unit circle. In the special case where the minimum modulus poles of f are on the unit circle, then if $\{a_n\}$ are real, ω_j are necessarily roots of unity and one obtains the general eventually periodic real sequence $\{a_n\}$ in this way.

When there is just one root of minimum modulus, the second term is exponentially smaller than the first:

$$a_n = c_* r_*^{-n} \left(1 + O\left(\left| \frac{r_*}{r_\dagger} \right|^n \right) \right)$$

where $r_\dagger$ is the next smallest root, and there is a polynomial correction if $r_\dagger$ is a multiple root. While exponentially good estimates are the best one normally hopes for, there can be a problem if you don't know how close $|r_\dagger|$ is to r_* or even which root has the least modulus. To see how to make these determinations automatically, consult [GS96].

Case 2: repeated roots.

Let the root r_j have multiplicity m_j and let $q_j(z)$ now denote $q(z)/(1-z/r_j)^{m_j}$. The same argument as shows there is a partial fraction expansion, one canonical form of which is

$$f(z) = p_0(z) + \sum_{j=1}^{t} \frac{p_j(z)}{(1 - z/r_j)^{m_j}}$$

with p_0 a polynomial and p_j polynomials of degree at most $m_j - 1$ and not vanishing at r_j. One can further break down $p_j/(1 - z/r_j)^{m_j}$ as a sum $\sum_{i=0}^{m_j-1} c_{ji}/(1 - z/r_j)^i$. By (3.2),

$$a_n = \sum_{j=1}^{t} \sum_{i=0}^{m_j} c_{ji} \binom{n+i-1}{i-1} r_j^{-n}.$$

The binomial coefficients, viewed as functions of n, are polynomials of degree $i-1$. The leading term(s) in this sum are the ones that minimize $|r_j|$ and among those, maximize i. If there is only one maximum multiplicity root of minimum modulus, let the root be denoted r_* and the multiplicity i_*. The leading term becomes

$$a_n \sim c_* \, r_*^{-n} \binom{n + i_* - 1}{i_* - 1}, \tag{3.4}$$

where it will be shown shortly that

$$c_* = \frac{p(r_*)}{q_*(r_*)} \,. \tag{3.5}$$

Meromorphic functions of one variable

Let us compare the formal power series solution, which is complete but specialized, to an analytic solution. First note that the radius of convergence, R, of the power series for f is equal to the minimum modulus of a singularity for f, which we have denoted r_*. For any power series with radius of convergence R, a preliminary estimate is obtained by integrating $z^{-n-1}f(z)$ over a circle of radius $R - \epsilon$:

$$\frac{1}{n} \log a_n \leq -\log(R - \epsilon) \,.$$

If f is integrable on the circle of radius R this improves to

$$a_n = O(R^{-n}) \,.$$

Since there is a singularity of modulus R, we know the series does not converge for $|z| > R$, so $\log a_n \geq n(-\log R - \epsilon)$ infinitely often. This establishes

$$\limsup \frac{1}{n} \log |a_n| = -\log R \,. \tag{3.6}$$

Thus we get the correct exponential rate, at least for the lim sup, with no work at all.

Next, use Cauchy's integral formula to write

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dz}{z^{n+1}} f(z)$$

where $\mathbb{C}$ is any contour enclosing the origin and contained in the domain Of convergence of f. Let $\mathbb{C}$ be a circle of radius $r < r_*$, and let $\mathbb{C}'$ be a circle of radius $R > r_*$. Assume that r_* is the only root of q of minimum modulus and that the moduli of other roots are greater than R. We then have, by the residue theorem,

$$\int_{\mathcal{C}} \frac{dz}{z^{n+1}} f(z) - \int_{\mathcal{C}'} \frac{dz}{z^{n+1}} f(z) = -2\pi i \, \mathrm{Res}\,(z^{-n-1} f(z); r_*). \tag{3.7}$$

It is conceivable the reader may have gotten this far but never seen residues, in which case the text [Con78] is recommended and it can meanwhile be taken on faith that the residue at a simple pole and pole of order $k > 1$ respectively are defined by

$$\begin{aligned}\mathrm{Res}\,(g;r) &= \lim_{z\to r}(z-r)g\,, \\ \mathrm{Res}\,(g;r) &= \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}((z-r)^k g)\,.\end{aligned}$$

If $f = p/q$ has a simple pole at r_* then the residue is just $r_*^{-n-1}p(r_*)/q'(r_*)$. The integral over $\mathcal{C}'$ is bounded by $2\pi R^{-n-1}\sup_{|z|=R}|f(z)|$, and is therefore exponentially smaller than the residue. Thus the leading term asymptotic for a_n is

$$a_n = -r_*^{-n}\frac{p(r_*)}{q'(r_*)} + O(R^{-n}).$$

In fact we may send $\mathcal{C}'$ to infinity, thus picking up all the terms. This will be a sum of terms $-r_i^{-n-1}p(r_i)/q'(r_i)$. This makes good on the promise to prove (3.3), showing also that even when one can work everything out algebraically, an analytic approach may still add something.

If there is more than one root of minimum modulus, one may simply sum the contributions. If the root r_j appears with multiplicity m_j, then the residue at r_j comes out to be

$$\binom{n+m_j-1}{m_j-1} r_j^{-n}\frac{p(r_j)}{q_j(r_j)} + O\left(n^{d-2}r_j^{-n}\right)\,.$$

The lower order terms are easy to compute though the expressions are longer due to iterating the derivative.

Aside from providing a shortcut to the constants, the analytic approach has the advantage of generality. The partial fractions approach required that $f = p/q$ be a rational function. The residue computation gives an answer whenever f is **meromorphic** in a disk of radius greater than $|r_*|$, meaning that it is the quotient of analytic functions. In other words, if $f(z) = p(z)/q(z)$ with p and q analytic on a disk of radius R and q vanishing at some point a inside the disk, then the previous estimates are still valid: $a_n = a^{-n}p(a)/q'(a) + O(R-\epsilon)^{-n}$ if a is a simple pole and $a_n = cn^{m-1}a^{-n} + O(n^{m-2}a^{-n})$ if q has a root of multiplicity $m > 1$ at a.

3.2 Saddle point methods

One of the crowning achievements of elementary complex analysis is development of techniques to evaluate integrals by deforming the contour of integration. Much of this can be grouped together as "saddle point methods", aimed at discovering the best deformation. In several variables, topology comes into play, and this forms the content of Chapter 9, which is the heart of these notes. To prepare for this, and because it is useful in itself, a tutorial in saddle point integration is given in this section.

When the modulus of an integrand falls steeply on either side of its maximum, most of the contribution to the integral comes from a small interval about the maximum. If that were so, then multiplying the integrand by the length of the interval where the modulus is near its maximum (or doing something slightly more fancy) would give an easy estimate. Most contours, however, do not have this property. To see this, note first that this estimate cannot hold if the contour can be deformed so as to decrease the maximum modulus of the integrand, since then the integral would be less than the claimed estimate. Let γ be a contour and denote the logarithm of the integrand by I. At a point z_0 where the modulus of the integrand is maximized, $\mathrm{Re}\{I'\}$ vanishes along γ. Generically, $\mathrm{Im}\{I'(z_0)\}$ will not vanish along γ. By the Cauchy-Riemann equations, $\mathrm{Re}\{I'\}$ in the direction perpendicular to the contour is equal to the $\mathrm{Im}\{I'\}$ along γ. When this does not vanish, γ may be locally perturbed, fixing the endpoints but pushing the center in the direction of increasing $\mathrm{Re}\{\phi\}$, thereby decreasing the maximum modulus $|e^I|$ of the integrand on the contour.

In other words, generically, the original contour does not pass through such a point. The univariate saddle point method consists of the following steps.

(i) locate the zeros of I' (a discrete set of points)

(ii) see whether the contour can be deformed so as to obtain minimize $\mathrm{Re}\{I\}$ at such a point

(iii) estimate the integral via a Taylor series development of the integrand

In Chapter 5 we will see that for families parametrized by λ of integrals

$$\int A(z)\exp(-\lambda\phi(z))$$

(into which form the Cauchy integral may be put), one may often get away with approximating the critical point $z_0(\lambda)$ the critical point z_0 for ϕ, ignoring A and removing the dependence of z_0 on λ. This approximation is often good enough to provide a complete asymptotic approximate expansion of the integral.

Here, we consider cases where this does not work but where we can deal directly with $z_0(\lambda)$. For the second step above not to fail, either f must be entire or the saddle point (where I_n' vanishes) must be in the interior of the domain of convergence of f. This is not asking too much, and the method is widely applicable. I am not sure when Cauchy's formula was first combined with contour integration methods to estimate power series coefficients. One seminal paper from 1956 [Hay56] defines a broad class of functions, called **admissible functions** for which the saddle point method can be shown to work and the Gaussian approximation mechanized. The title of Hayman's paper refers to the fact that when one takes $f(z) = e^z$, one recovers Stirling's approximation to $n!$.

Examples of Hayman's method

The next few examples apply Hayman's methods to univariate generating functions derived in the Chapter 2. They all rely on the estimate

$$\int_\gamma A(z)\exp(-\lambda\phi(z)) \sim A(z_0)\sqrt{\frac{2\pi}{\phi''(z_0)}}\exp(-\lambda\phi(z_0))\,, \tag{3.8}$$

which holds when A and ϕ are smooth and $\mathrm{Re}\,\{\phi\}$ is minimized in the interior of γ at a point z_0 where ϕ'' does not vanish. (In the notation, A is for "amplitude" and ϕ is for "phase".) This is generalized and proved in Theorem 5.1 of Chapter 5. However, to show that it is elementary, a direct verification is provided for the first example.

Example 3.1 (ordered-set partitions: an isolated essential singularity). Evaluate (2.14) at $y = 1$ to obtain the exponential generating function

$$f(z) = \exp\left(\frac{z}{1-z}\right)$$

for the number a_n of partitions of $[n]$ into ordered sets. By Cauchy's formula (1.2),

$$\frac{a_n}{n!} = \frac{1}{2\pi i}\int z^{-n-1}\exp\left(\frac{z}{1-z}\right)\,dz\,.$$

We will show that

$$a_n \sim n!\sqrt{\frac{1}{4\pi e}}n^{-3/4}\exp(2\sqrt{n})\,.$$

To estimate this, we find the critical point z_n. Denoting the logarithm of the integrand by $I_n(z) := -(n+1)\log z + \dfrac{z}{1-z}$ and compute the derivative

$$I_n' = \frac{-n-1}{z} + \frac{1}{(1-z)^2}\,.$$

This vanishes at a value $1-\beta_n$ where $\beta_n = n^{-1/2} + O(n^{-1})$. The only singularities of the integrand are at $z = 0, 1$, so we may expand the contour to a circle passing through $1-\beta_n$. It is a little more convenient to work in the Riemann sphere (there is no singularity at infinity), so that we may deform the contour into the line $1-\beta_n+ix$. Denote this contour by γ.

The hope now is that the integral is well approximated by integrating the degree-two Taylor approximation of I_n. Specifically, we hope that (cf. (3.8) with $\lambda = n+1$ and $\phi = -I_n(1-\beta_n)$)

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\exp(I_n(1-\beta_n+it))\,(i\,dt) \tag{3.9}$$

$$\approx \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[I_n(1-\beta_n) + \frac{1}{2}I_n''(1-\beta_n)(it)^2]\,dt \tag{3.10}$$

$$= \sqrt{\frac{1}{2\pi}I_n''(1-\beta_n)}\exp(I_n(1-\beta_n)). \tag{3.11}$$

This hope is easily verified as follows. We compute

$$\begin{aligned} I_n''(1-\beta_n) &= \frac{n+1}{(1-\beta_n)^2} + \frac{2}{\beta_n^3} \\ &= (2+o(1))n^{3/2} \end{aligned}$$

This tells us that the main contribution to (3.9) should come from the region where $|t|$ is not much larger than $n^{-3/4}$. Accordingly, we pick a cutoff a little greater than that, say $L = 2n^{-3/4}\log n$, and break the integrals (3.9) and (3.10) into two parts, $|t| \leq L$ and $|t| > L$. Up to the cutoff the two integrals are close, and past the cutoff they are both small.

The contribution to (3.10) when $|t| > L$ is a Gaussian tail and is $o(\exp(I_n(1-\beta_n) - \log^2 n))$. The contribution to (3.9) when $|t| > L$ may be bounded in two parts. When $|t| < n^{-1/2}$, use the fact that $|z| \geq 1$ on the line of integration to see that the modulus of the integrand is at most $M(t)\exp(I_n(1-\beta_n))$ where

$$|M(t)| = \exp\left[\mathrm{Re}\left\{\frac{1}{\beta_n + it} - \frac{1}{\beta_n}\right\}\right].$$

We then compute

$$\begin{aligned}
\mathrm{Re}\left\{\frac{1}{\beta_n + it} - \frac{1}{\beta_n}\right\} &= \frac{\beta_n}{\beta_n^2 + t^2} - \frac{1}{\beta_n} \\
&= \frac{\beta_n^{-1}}{1 + \beta_n^{-2}t^2} - \beta_n^{-1} \\
&= \frac{-\beta_n^{-3}t^2}{1 + \beta_n^{-2}t^2} \\
&\leq \frac{-1 + o(1)}{2} n^{3/2}t^2
\end{aligned}$$

because $\beta_n \sim n^{-1/2}$. Thus again we have part of a Gaussian tail and again the contribution is $o(\exp(I_n(1-\beta_n)) - \log^2 n)$. When $|t| > n^{-1/2}$ we need to use the z^{-n-1} term as well:

$$\begin{aligned}
\frac{|\exp(I_n(1-\beta_n+it))|}{\exp(I_n(1-\beta_n))} &\leq \frac{|1-\beta_n|^n}{|1-\beta_n+it|^n}\exp\left(\mathrm{Re}\left\{\frac{1}{\beta_n+it} - \frac{1}{\beta_n}\right\}\right) \\
&\leq (1+t^2)^{-n/2}\exp\left(\mathrm{Re}\left\{\frac{1}{\beta_n + in^{-1/2}} - \frac{1}{\beta_n}\right\}\right).
\end{aligned}$$

Here, the bound of $(1+t^2)^{-n/2}$ on the z^{-n-1} term follows from the fact that $1 - \beta_n < 1$ and that $|x/(x+it)|$ is increasing in $x \geq 0$. Integrating from $t = n^{-1/2}$ to ∞ and using our previous computation of the real part of the above difference shows that the contribution is at most the integrand at $t = 0$ times a factor of $\exp(-(\frac{1}{2} + o(1))\sqrt{n})$.

When $|t| \leq L$, we use the Taylor approximation

$$\left|I_n(1-\beta_n+it) - I_n(1-\beta_n) + \frac{1}{2}t^2 I_n''(1-\beta_n)\right| \leq \frac{1}{6}t^3 \sup_{|s|\leq L}|I_n'''(s)|.$$

The RHS is bounded by $(1+o(1))t^3n^2$, and hence by $8n^{-1/4}\log^3 n$. Since the integrand of (3.10) is everywhere positive, this implies that the difference between the integrals (3.9) and (3.10) on $|t| \leq L$ is at most $\exp(n^{-1/4}\log^3 n) - 1$ times the integral (3.10), as desired.

Having established that (3.11) is the leading term, we finally compute it. Using the formula for $I_n''(z)$ and the formula

$$\beta_n = n^{-1/2} - \frac{1}{2}n^{-1} + O(n^{-3/2})$$

we get

$$\begin{aligned}
& \sqrt{\frac{1}{2\pi I_n''(1-\beta_n)}} \exp(I_n(1-\beta_n)) \\
= \quad & (1+o(1))\sqrt{\frac{1}{4\pi n^{3/2}}} \exp\left(-(n+1)\log(1-\beta_n) - 1 + \frac{1}{\beta_n}\right) \\
= \quad & (1+o(1))\sqrt{\frac{1}{4\pi n^{3/2}}} \exp\left(-(n+1)(-n^{-1/2} + O(n^{-3/2})) - 1 + n^{1/2} + \frac{1}{2} + O(n^{-1/2})\right) \\
= \quad & (1+o(1))\sqrt{\frac{1}{4\pi e}} n^{-3/4} \exp(2\sqrt{n}).
\end{aligned}$$

Note that computing the full asymptotic development is almost as easy. The cutoff is calibrated so that the remainder after k terms of the Taylor expansion is always small on $|t| \leq L$, and essentially the same computation suffices to derive an asymptotic series.

Example 3.2 (involutions: an entire function). Let $f(z) = \exp(z + z^2/2)$ be the exponential generating function for the number a_n of involutions in the permutations group S_n, as in Example 2.37. This is an entire function, so we apply Hayman's method. Denote

$$I_n(z) = \log(f(z)z^{-n-1}) = z + \frac{z^2}{2} - (n+1)\log z\,.$$

Setting the derivative equal to zero gives the quadratic $z^2 + z - (n+1) = 0$. The roots are $-\frac{1}{2} \pm \sqrt{n + \frac{5}{4}}$. The coefficients a_n are positive, whereas $\exp(I_n(z))$ alternates in sign near the negative root so a_n cannot be approximated by the integrand near the negative root and we therefore try taking $z_0 = \sqrt{n + \frac{5}{4}} - \frac{1}{2}$.

Let γ be the circle centered at the origin through z_0. It is easy to verify that real part of I_n on γ is maximized at z_0. The estimate

$$[z^n]f = \frac{1}{2\pi i}\int_\gamma \exp(I_n(z_n))\,dz \sim \exp(z_0)\sqrt{\frac{1}{2\pi\, I_n''(z_0)}}$$

is justified the same way as in the previous example. Using the approximations

$$\begin{aligned}
z_0 &= n^{1/2} - \frac{1}{2} + \frac{5}{8}n^{-1/2} + O(n^{-3/2}) \\
\frac{z_0^2}{2} &= \frac{1}{2}n - \frac{1}{2}n^{1/2} + \frac{3}{4} + o(1) \\
\log(z_0) &= \frac{1}{2}\log n - \frac{1}{2}n^{-1/2} + \frac{1}{2}n^{-1} + O(n^{-3/2}) \\
I_n(z_0) &= -\frac{1}{2}n\log n + \frac{1}{2}n + n^{1/2} - \frac{1}{2}\log n - \frac{1}{4} + o(1) \\
I_n''(z_0) &= 2 + o(1)
\end{aligned}$$

and using Stirling's formula to approximate $n!$, we find that

$$\begin{aligned}
a_n &= n!\,[z^n]f(z) \\
&\sim \exp(n\log n - n)\sqrt{2\pi n}\exp(I_n(z_0))\sqrt{\frac{1}{2\pi\, I_n''(z_0)}} \\
&\sim \exp(n\log n - n)\; n^{1/2}\;\exp(-\frac{1}{2}n\log n + \frac{1}{2}n^{1/2} - \frac{1}{2}\log n - \frac{1}{2}))\; 2^{-1/2} \\
&= \exp\left(\frac{1}{2}n\log n - \frac{1}{2}n + n^{1/2} - \frac{1}{2}\log 2 - \frac{1}{4} + o(1)\right).
\end{aligned}$$

Evidently, a_n is near $\sqrt{n!}$. Is there a reason that the number of involutions should be roughly the square root of the number of permutations? Pulling out the $\sqrt{n!}$ yields the slightly more transparent

$$a_n \sim \sqrt{n!}e^{\sqrt{n}}(8\pi e n)^{-1/4}.$$

3.3 Circle methods

When f has a branch singularity, for example a logarithm or non-integral power, there is often no way to manoeuvre the contour through a saddle. As we have seen when deriving the crude estimate (3.6), pushing the contour to the boundary of the disk of convergence will give some improvement. This leads to **circle methods**, such as Darboux' Theorem. A preliminary estimate well known to harmonic analysts is

Lemma 3.3. *Suppose a complex-valued function f on the circle γ of radius R is k times continuously differentiable for some integer $k \geq 0$. Then*

$$\int_\gamma z^{-n-1}f(z)\,dz = O(n^{-k}R^{-n})$$

as $n \to \infty$.

PROOF: Replacing f by $f(z/R)$ we may assume without loss of generality that $R = 1$. Integrating by parts,

$$\int_\gamma z^{-n} f(z)\, dz = \int_\gamma \frac{1}{n-1} z^{1-n} f'(z)\, dz$$

with the $\int_\gamma \frac{z^{1-n}}{1-n} f(z)\, dz$ term dropping out because γ has no boundary. By induction on k.

$$\int_\gamma z^{-n} f(z)\, dz = \frac{1}{k!\binom{n-1}{k}} \int_\gamma z^{k-n} f^{(k)}(z)\, dz\,.$$

Since $f^{(k)}$ is continuous it is bounded on the unit circle, so this last integral is bounded independently of n and the lemma follows from $\binom{n-1}{k} \sim n^k$. □

The following version of Darboux' Theorem may be found in [Hen88, Theorem 11.10b].

Theorem 3.4 (Darboux). *Suppose that $F(z) = (1-z/R)^\alpha \psi(z)$ for some $R > 0$, some $\alpha \notin \mathbb{Z}^+$, and some function ψ with radius of convergence greater than R. Denote the coefficients of ψ expanded about R by $\psi(z) = \sum_{n=0}^\infty b_n (R-z)^n$. Then the coefficients $\{a_n\}$ of F have asymptotic expansion*

$$a_n \sim R^{-n} \sum_{k=0}^\infty c_k n^{-\alpha-1-k}\,.$$

The coefficients c_k are given by explicit linear combinations of $b_0, \ldots, b_k$ and the leading term is

$$a_n \sim \frac{\psi(R)}{\Gamma(-\alpha)} n^{-\alpha-1} R^{-n}\,.$$

PROOF: Again, assume without loss of generality that $R = 1$. Begin by recalling some elementary facts about the power series $(1-z)^\alpha$. It coefficients are the formal binomial coefficients $(-1)^n \binom{\alpha}{n}$ defined by

$$\binom{x}{n} := \prod_{j=1}^n \frac{x-j+1}{j}\,.$$

As a function of n, these coefficients are asymptotically

$$(-1)^n \binom{\alpha}{n} \sim \frac{1}{\Gamma(-\alpha)} n^{-\alpha-1}\,. \tag{3.12}$$

Furthermore, an asymptotic series for $\binom{\alpha}{n}$ in terms of decreasing powers n^{-x-1-k} is known. Therefore, a triangular linear map converts an asymptotic series $a_n \sim \sum_{k=0}^\infty c'_k (-1)^n \binom{\alpha+k}{n}$ into a series $a_n \sim \sum_{k=0}^\infty c_k n^{-\alpha-1-k}$, with $c_0 = c'_0/\Gamma(-\alpha)$.

To find the series in c'_k, let m be an integer greater than $\Re\{-\alpha\}^+$ and let ψ_m be the m^{th} remainder term in the Taylor series for ψ:

$$(1-z)^m \psi_m(z) = \psi(z) - \sum_{k=0}^{m-1} b_k (1-z)^k\,.$$

Multiplying by $(1-z)^\alpha$ yields

$$f(z) - \sum_{k=0}^{m-1} b_k (1-z)^{\alpha+k} = (1-z)^{\alpha+m}\psi_m(z)$$

on the open unit disk, and hence as formal power series. Taking the z^n-coefficient on both sides yields

$$a_n - \sum_{k=0}^{m-1} b_k (-1)^n \binom{\alpha+k}{n} = [z^n](1-z)^{\alpha+m}\psi_m(z)\,. \tag{3.13}$$

This proves the desired expansion, provided that the right-hand side of (3.13) is $o(n^{-\alpha-m})$.

By assumption, $\alpha + m \geq 0$, so the function $(1-z)^{\alpha+m}\psi_m$ is $\lfloor \alpha + m \rfloor$ times continuously differentiable on the unit circle, which implies that the right-hand side of (3.13) is $O(n^{-\lfloor \alpha+m \rfloor})$. This is not quite small enough, but replacing m by $m+1$ adds a term known to be $O(n^{-\alpha-m-1})$ to the left-hand side of (3.13) while reducing the right-hand side to $O(n^{-\lfloor \alpha+m+1 \rfloor})$, which is good enough. □

Example 3.5 (2-regular graphs: an algebraic singularity). Let $f(z) = e^{-z/2-z^2/4}/\sqrt{1-z}$ be the exponential generating function for 2-regular graphs that was derived in Example 2.38. Apply Darboux' Theorem with $R = 1$, $\alpha = -1/2$ and $\psi = \exp(-z/2 - z^2/4)$. Then $\psi(0) = e^{-3/4}$, $\Gamma(-\alpha) = \sqrt{\pi}$ and the number a_n of 2-regular graphs on n labelled vertices is estimated by

$$a_n \sim n! \frac{e^{-3/4}}{\sqrt{\pi n}}\,.$$

3.4 Transfer theorems

A closer look at the proof of Darboux' Theorem shows that one can do better. Analyticity of $f/(R-z)^\alpha$ beyond the disk of radius R was used only to provide a development of f in decreasing powers of $R-z$. Also, a sharper estimate than Lemma 3.3 will allow us to obtain a sufficiently good estimate on the remainder term without replacing m by $m+1$, which is crucial if the hypotheses are weakened to a finite asymptotic expansion.

There are various results along these lines, my favorite among which are the **transfer theorems** of Flajolet and Odlyzko [FO90]. Their idea was that the estimates could be improved to $a_n = O(n^{-\alpha-1})$ for the coefficients of *any* power series $f(z)$ that is $O(1-z)^\alpha$. They reduced the necessary domain of analyticity of f as far as possible (to a neighborhood of the unit disk in the slit plane) and while they were at it they generalized the scope beyond powers to other branch singularities.

To state the main theorem of [FO90], we define the class **alg-log** to be the class of functions that are a product of a power of $R-z$, a power of $\log(1/(R-z))$ and a power of $\log\log(1/(R-z))$. Analogously to (3.12), one begins with a description of asymptotics for all functions in the class **alg-log**. The reader is referred to [FO90] for the proof of the following lemma.

Proposition 3.6 ([FO90, Theorem 3B]). *Let α, γ and δ be any complex numbers other than nonnegative integers and let*

$$f(z) = (1-z)^\alpha \left(\frac{1}{z}\log\frac{1}{1-z}\right)^\gamma \left(\frac{1}{z}\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right)^\delta .$$

Then the Taylor coefficients $\{a_n\}$ of f satisfy

$$a_n \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}(\log n)^\gamma (\log\log n)^\delta .$$

□

Remark 3.7. When α, γ or δ is a nonnegative integer, different formulæ hold. For example, for the case $\alpha \in \mathbb{Z}^+, \gamma \notin \mathbb{Z}^+, \delta = 0$, the estimate

$$a_n \sim C n^{-\alpha-1}(\log n)^{\gamma-1} \tag{3.14}$$

is known: the coincidence of α with a nonnegative integer causes an extra log in the denominator.

Given a positive real R and an $\epsilon \in (0, \pi/2)$, the so-called **Camembert-shaped region**,

$$\{z : |z| < R+\epsilon, z \neq R, |\arg(z-R)| \geq \pi/2 - \epsilon\},$$

denoted $\Delta(R,\epsilon)$, is shown in figure 3.1.

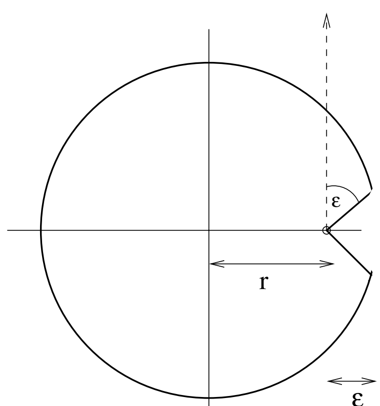

Figure 3.1: a Camembert-shaped region

Theorem 3.8 (Transfer Theorem). *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be analytic in a Camembert-shaped region $\Delta(R,\epsilon)$. If $g(z) = \sum_{n=0}^\infty b_n z^n \in$ alg-log, then the following hold.*

(i)

$$f(z) = O(g(z)) \Rightarrow a_n = O(b_n) ;$$

(ii)

$$f(z) = o(g(z)) \Rightarrow a_n = o(b_n)\,;$$

(iii)

$$f(z) \sim g(z) \Rightarrow a_n \sim b_n$$

In particular, when $f(z) \sim C(r-z)^\alpha$, this result subsumes Theorem 3.4.

So as not to devote too much space to computation, I will only prove this theorem with the class alg-log replaced by the class of powers $(1-z)^\alpha$; this result still greatly improves Darboux' Theorem.

PROOF FOR $g(z) = (1-z)^\alpha$: Assume without loss of generality that $R = 1$. Note that for $g = (1-z)^\alpha$, the n^{th} coefficient is of order $n^{-\alpha-1}$. Next, note that the assumption that $f(z) = O((1-z)^\alpha)$ near $z = 1$ implies (using only continuity, not analyticity) that for some K, $|f(z)| \leq K|1-z|^\alpha$ everywhere on $\Delta(R, \epsilon)$. The contour of integration in Cauchy's formula will be a contour γ constructed as the union of four pieces. Let γ_1 be the circular arc parameterized by $1 + n^{-1}e^{it}$ for $\xi \leq t \leq 2\pi - \xi$. Let γ_2 be the line segment between $1 + n^{-1}e^{i\xi}$ and the number β of modulus $1 + \eta$ and $\arg(\beta - 1) = \xi$. Let γ_3 be the arc on the circle of radius $1 + \eta$ running between β and $\overline{\beta}$ the long way, and let γ_4 be the conjugate of γ_2. We will bound the absolute value of the integral on each segment separately, so we need not worry about the orientations. The value of η is chosen so as to be less than $R - 1$ and $\xi < \pi/2$ is chosen as large as is necessary to make γ contained in the Camembert region.

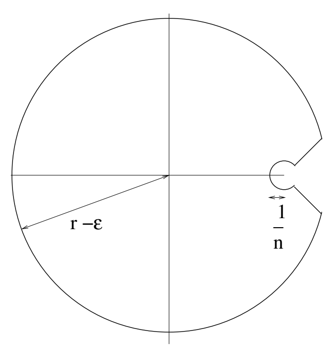

Figure 3.2: the contour γ

On γ_1, the modulus of f is at most $Kn^{-\alpha}$, the modulus of z^{-n-1} is at most $(1-n^{-1})^{-n-1} \leq 2e$, and the integral of $|dz|$ is at most $2\pi n^{-1}$, leading to a contribution of size at most $6Kn^{-\alpha-1}$. On γ_3 the z^{-n-1} factor in the integral reduces the modulus to at most $C(\eta)(1+\eta)^{-n}$ which is of course $O(n^{-N})$ for any N. Since the order of the n^{th} coefficient of g is $n^{-\alpha-1}$, we are in good shape so far.

By symmetry, we need now only do the computation for γ_2. Set $\omega = e^{i\xi}$ and parametrize the integral as $z = 1 + (\omega/n)t$ for $t = 1$ to En for a constant, $E = |\beta - 1|$. We have $|f(z)| \leq K|z-1|^\alpha =$

$K(t/n)^\alpha$ and

$$|z^{-n-1}| = \left|1 + \frac{\omega t}{n}\right|^{-n-1}$$

so

$$\begin{aligned}\int_{\gamma_2} |f(z)||z^{-n-1}||dz| &\leq \int_1^{En} K\left(\frac{t}{n}\right)^\alpha \left|1 + \frac{\omega t}{n}\right|^{-n-1} \frac{dt}{n} \\ &\leq Kn^{-\alpha-1} \int_1^\infty t^\alpha \left|1 + \frac{\omega t}{n}\right|^{-n-1} dt. \qquad (3.15)\end{aligned}$$

We need to see that the integral in (3.15) is bounded above for sufficiently large n. The bound $|1 + \omega t/n| \geq 1 + \mathrm{Re}\,\{\omega t/n\} = 1 + (t/n)\cos(\xi)$ implies an upper bound of

$$J_n := \int_1^\infty t^\alpha \left(1 + \frac{t\cos(\xi)}{n}\right)^{-n} dt$$

for the integral in (3.15). The integrand is monotone decreasing in n, and clearly finite for $n > 1+\alpha^+$, so the decreasing limit is

$$J := \lim_{n\to\infty} J_n = \int_1^\infty t^\alpha e^{-t\cos(\xi)}\, dt$$

which is finite. We have now bounded all four integrals by multiples of $n^{-\alpha-1}$, so the proof of the first assertion is complete.

The proof of the second assertion is contained in here as well. When $|f| \leq Kg$ then the integral over γ_1 is bounded above by $6Kn^{-\alpha-1}$, the integral over γ_3 is $o(n^{-\alpha-1})$, and the integral over γ_2 is bounded by JK. Furthermore, the contributions to each of these four integrals from parts of γ at distance greater than δ from 1 are $o(n^{-\alpha-1})$ for any fixed $\epsilon > 0$. If $f(z) = o(g(z))$ at $z = 1$, then for any $\epsilon > 0$ there is a δ such that $|f(z)| \leq \epsilon|g(\mathbf{z})|$ when $|1 - z| \leq \delta$. It follows that $a_n \leq (2J + 6 + o(1))\epsilon n^{-\alpha-1}$. This is true for every $\epsilon > 0$, whence $a_n = o(n^{-\alpha-1})$.

Finally, the third assertion is an immediate consequence of the first two assertions. ☐

Example 3.9 (Catalan numbers). Let $a_n := \frac{1}{n+1}\binom{2n}{n}$ be the n^{th} Catalan number. The generating function for these was shown in Example 2.13 to be given by

$$f(z) := \sum_{n=0}^\infty a_n z^n = \frac{1 - \sqrt{1-4z}}{2z} = \frac{1 - 2\sqrt{\frac{1}{4} - z}}{2z}\,.$$

There is an algebraic singularity at $r = 1/4$, near which the asymptotic expansion for f begins

$$f(z) = 2 - 4\sqrt{\frac{1}{4} - z} + 8(\frac{1}{4} - z) - 16(\frac{1}{4} - z)^{3/2} + O(\frac{1}{4} - z)^2\,.$$

Note that $f/\sqrt{1/4 - z}$ is not analytic in any disk of radius $1/4 + \epsilon$, since both integral and half-integral powers appear in f, but f is analytic in a Camembert-shaped region. Theorem 3.8 thus gives (note that the integral powers of $(1-z)$ do not contribute):

$$\begin{aligned} a_n &\sim \left(\frac{1}{4}\right)^{1/2-n} n^{-3/2} \frac{-4}{\Gamma(-1/2)} + \left(\frac{1}{4}\right)^{3/2-n} n^{-5/2} \frac{-16}{\Gamma(-3/2)} + O(n^{-7/2}) \\ &= 4^n n^{-3/2} \frac{(-4)(\frac{1}{4})^{1/2}}{\Gamma(-1/2)} + 4^n n^{-5/2} \frac{(-16)(\frac{1}{4})^{3/2}}{\Gamma(-3/2)} + O(n^{-7/2}) \\ &= 4^n \left(\frac{n^{-3/2}}{\sqrt{\pi}} - n^{-5/2} \frac{3}{2\sqrt{\pi}} + O(n^{-7/2}) \right). \end{aligned}$$

Example 3.10 (branching random walk: logarithmic singularity). For an example including a logarithmic term, recall from Example 2.12 the implicit equation

$$\phi(z) = [(1-p)z + p\phi(\phi(z))]^2 .$$

This characterizes the probability generating function for the number, X, of particles to reach the origin in a binary branching nearest-neighbor random walk with absorption at the origin. It is shown in [Ald05] that there is a critical value, satisfying $16p(1-p) = 1$, such that for all greater p X is sometimes infinite, while for lesser p, X is never infinite. At the critical value, X is always finite, and it is of interest to know the likelihood of large values of X.

In order to apply the transfer theorem, we will show that

$$\phi(z) = 1 - \frac{1-z}{4p} - (c + O(1)) \frac{1-z}{\log(1/(1-z))}, \tag{3.16}$$

where $c = \log(1/(4p))/(4p)$. It follows from this and (3.14) that

$$a_n \sim cn^{-2} (\log n)^{-2},$$

so that Z has a first moment but not a "$1 + \log$" moment.

To show (3.16), fix $0 < z_0 < 1$ and let $z_n = \phi^{(-n)}(z_0)$ so that $z_n \uparrow 1$. The recursion for ϕ gives

$$z_n = ((1-p)z_{n+1} + pz_{n-1})^2.$$

Changing variables to $y_n = 1 - z_n$ gives

$$\begin{aligned} y_n &= 1 - ((1-p)(1-y_{n+1}) + p(1-y_{n-1}))^2 \\ &= 1 - (1 - ((1-p)y_{n+1} + py_{n-1}))^2. \end{aligned}$$

Solving for y_{n+1} gives

$$y_{n+1} = \frac{1 - \sqrt{1-y_n} - py_{n-1}}{1-p}.$$

Setting $x_n = y_n/(4p)^n$ and using $16p(1-p) = 1$ gives

$$x_{n+1} = 2x_n - x_{n-1} + O(y_n)^2.$$

Verifying first that y_n is small, we then have $x_n \sim An + B$, whence $y_n \sim (4p)^n(An + B)$. We may write this as

$$y_{n+1} = 4py_n + (1 + o(1))\frac{y_{n+1}}{n+1} = 4py_n + (1 + o(1))\frac{y_{n+1}}{\log y_{n+1}/\log(4p)}.$$

Let $z = 1 - y_{n+1}$ so $\phi(z) = 1 - y_n$. We then have

$$1 - \phi(z) = \frac{1-z}{4p} - (1 + o(1))\frac{1-z}{4p}\frac{\log(4p)}{\log(1-z)},$$

proving (3.16).

Notes

One of the earliest and most well known uses of generating function analysis to obtain asymptotics was Hardy and Ramanujan's derivation of asymptotics for the number of partitions of an integer [HR17a]. The original argument used a tauberian theorem and the behavior of the generating function $f(s)$ as $s \uparrow 1$ through real values, though later followup work (see, e.g., [HR17b]) used circle methods. I have not found it possible to trace the use of branchpoint methods or smooth saddle point methods, though the seminal paper of Hayman in 1956 [Hay56] was perhaps the earliest influential work in this area.

The exposition in this chapter does not follow any one source though it owes a debt to Chapter 11 of [Hen91] and to the beautiful paper of Flajolet and Odlyzko [FO90]. A good reference book for this material is Flajolet and Sedgewick's text [FS08].

Exercises

Exercise 3.1. Use the exponential generating function $f(z) = \exp(e^z - 1)$ for number a_n of partitions of $[n]$ (Example 2.36) to derive the estimate

$$a_n = (\log n + O(1))^n .$$

Exercise 3.2. Find the approximate value of the least positive zero of the generating function f from Exercise 2.5. You can do this with or without rigorous bounds. Prove that the least positive zero has the least modulus of any singularity of f (Hint: use the fact that the coefficients a_n of f are positive) and use this to estimate the limsup logarithmic growth rate $\limsup_{n\to\infty} n^{-1} \log a_n$. Prove that this limsup is equal to the liminf, so the limit exists.