Chapter 4

Integration on manifolds

The steps outlined at the end of Section 1.3 require several fundamental results concerning differential forms and their integrals. Up to this point, the exposition has been largely self-contained. However, to develop these results from scratch would require inclusion of semester-long graduate-level courses in differentiable manifolds and algebraic topology. This would be wasteful, and yet not all of these results are familiar to many working combinatorialists, and certainly most graduate students would benefit from a synopsis of these. To quote them without providing any explanation would render these lecture notes inaccessible to much of their intended audience. Instead, in this chapter and the next, I aim to give enough background so that the relevant results may be stated precisely and their proofs may be located in the literature. In other words, a reader who is only vaguely aware of such results should be able, with the aid of his or her library, to make sense of the statements and verify that they are known to be true.

In this chapter, we tackle the first two of the five steps listed at the end of Section 1.3. These two steps are the replacement of $a_{\mathbf{r}}$ by an integral which is a variant of (1.4):

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\mathcal{C}} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z} \tag{1.4'}$$

where $\mathcal{C}$ is any cycle homologous to T in the domain $\mathcal{M}$ of holomorphy of the integrand. To make sense of this, we must define differential forms and their integrals, state the multivariate Cauchy formula, construct the singular homology of the domain $\mathcal{M}$, and connect the singular homology of $\mathcal{M}$ to the integration of exact forms over cycles in $\mathcal{M}$. An outline for the review of differential topology in this chapter will be as follows.

- **Section 4.1:** Real differential forms and their integrals over chains are defined, leading to Stokes' Theorem (Theorem 4.1).
- **Section 4.2:** Complex forms are defined and Cauchy's integral formula (Theorem 4.4) is stated.

- **Section 4.3:** Singular homology is constructed, the long exact homology sequence is derived (Theorem 4.7) and the relation between homology and integration is stated (Theorem 4.13). This section also contains discussions of the homology of pairs and the Künneth product formula.

4.1 Differential forms in $\mathbb{R}^n$

Manifolds

The notion of a manifold is undoubtedly familiar, but there are several different formalizations. The manifolds relevant to us will be submanifdolds of euclidean space, so I will use definitions specific to $\mathbb{R}^n$. We define a d-manifold to be a subset $\mathcal{M}$ of $\mathbb{R}^n$ such that every point in $\mathcal{M}$ has a neighborhood in $\mathbb{R}^n$ whose intersection $\mathcal{N}$ with $\mathcal{M}$ is diffeomorphic to $\mathbb{R}_d$ (or equivalently, to the open unit ball in $\mathbb{R}^d$. Here, diffeomorhpic means there is a map $\phi : \mathcal{N} \to \mathbb{R}^d$ such that both ϕ and ϕ^{-1} are smooth (that is, members of the class C^∞ of infinitely differentiable functions).

The exterior algebra on $\mathbb{R}^d$

To elucidate the discussion of differential forms, let us first distinguish this from the other common notion of integration. A smooth map $\gamma : [0,1] \to \mathbb{R}^n$ may be thought of as a curve, embedded or immersed in Euclidean space. There are two somewhat different notions of integration on γ. First, there is a natural measure on the range of γ, namely one-dimensional Hausdorff measure $\mathbf{m}$. The integral $\int_\gamma f\, d\mathbf{m}$ is then defined for any measurable $f : \mathbb{R}^n \to \mathbb{R}$ as a Lebesgue integral. This is not the kind of integration we will be interested in, so there is no need to proceed further in defining Hausdorff measure and Lebesgue integration.

The notion we are interested is more akin to a line integral from physics: a vector-like integrand $f\,dx + g\,dy$ is integrated along the curve $\{(\gamma_1(t), \gamma_2(t)) : 0 \le t \le 1\}$ by taking the inner product of the vector (f, g) with the tangent vector $d\gamma := (\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt})$ to obtain

$$\int_\gamma f\,dx + g\,dy := \int_0^1 \left[f(\gamma(t))\frac{d\gamma_1}{dt} + g(\gamma(t))\frac{d\gamma_2}{dt}) \right] dt.$$

This is an oriented notion of integration, in the sense that parametrizing the curve γ in the reverse direction would change the sign on the dot product and hence the whole integral. One may observe however, using the chain rule, that a different parametrization γ in the same orientation produces the same integral. Based on this concept, we now define general real differential forms and their integrals.

Let $\mathcal{M}$ be a real d-manifold in $\mathbb{R}^n$. A collection of k-dimensional vector spaces $\{V_\mathbf{p} : \mathbf{p} \in \mathcal{M}\}$, one for each point of $\mathcal{M}$, is a **bundle** over $\mathcal{M}$ if it is locally coordinatized by $\mathcal{M} \times \mathbb{R}^k$. A (smooth)

section of a bundle is a smooth map $\mathbf{p} \mapsto u(\mathbf{p})$ with $u(\mathbf{p}) \in V_{\mathbf{p}}$. The bundle with $V_{\mathbf{p}} = T_{\mathbf{p}}(\mathcal{M})$, the tangent space to $\mathcal{M}$ at $\mathbf{p}$, is called the **tangent bundle**, denoted by $T(\mathcal{M})$. Let $T^*(\mathcal{M})$ denote the **cotangent bundle** defined by letting $V_{\mathbf{p}} = T_{\mathbf{p}}(M)^*$ be the dual space to $T_{\mathbf{p}}(\mathcal{M})$. To see why we need this, note that the form $f\,dx + g\,dy$ from the line integral example was an element of the cotangent bundle. For any k-dimensional real vector space V and any $p \leq k$, an **alternating p-linear function** on V is a linear map from p-tuples of element of V to $\mathbb{R}$, which is anti-symmetric in each pair of arguments. These form a vector space, $\Lambda_p(V)$. Let $\Lambda(V)$ denote the direct sum over p of $\Lambda_p(V)$. The following facts about $\Lambda(V)$ for any k-dimensional vector space, V, may be found in [War83, Chapter 2].

(i) $\Lambda(V)$ is a graded vector space and the dimension of $\Lambda_p(V)$ is $\binom{k}{p}$.

(ii) An associative anti-symmetric product may be defined taking p elements of $\Lambda_1(V)$ into $\Lambda p(V)$, as follows:
$$\alpha_1 \wedge \cdots \wedge \alpha_p(v_1, \ldots, v_p) = \det\left(\alpha_i(v_j)\right) .$$

(iii) Any vector space basis for $\Lambda_1(V)$ generates $\Lambda(V)$ as a ring.

(iv) Let $\{v_1, \ldots, v_k\}$ be any basis for V. For any $\Phi := \{i_1 < \cdots < i_p\}$ of integers from 1 to k, let $v_\Phi^* := v_{i_1}^* \wedge \cdots \wedge v_{i_p}^*$. A basis for $\Lambda_p(V)$ is given by $\{v_\Phi^* : \Phi \text{ a subset of size } p\}$.

The bundle for which $V_p = \Lambda(T_{\mathbf{p}}(\mathcal{M}))$ is called the **exterior algebra bundle**, $\Lambda_k(\mathcal{M})$, and it is graded by the decomposition into the direct sum of $\Lambda_p(\mathcal{M})$. A section of $\Lambda_p(\mathcal{M})$ is called a **differential k-form**. The set of these is denoted $E^k(\mathcal{M})$ and the union over k is denoted $E^*(\mathcal{M})$.

Functoriality

All of the preceding definitions take place in the category of smooth manifolds, where the arrows are smooth maps. This means that the bundles are independent of the local parametrization and choice of basis, and that a smooth map between manifolds induces a map, in the appropriate direction, between bundles. To make this more concrete, consider for example a smooth map $f : \mathbb{R}^d \to \mathbb{R}^n$. Suppose this is a chart map for a d-manifold $\mathcal{M} \subseteq \mathbb{R}^n$, so f is a diffeomorphism between a ball in $\mathbb{R}^d$ and neighborhood in $\mathcal{M}$. Let us see how f maps tangent vectors. A tangent vector may be thought of in several ways. The classical way to think of it is as a direction in which a derivative of a real function may be taken. More pictorially, we may think of a tangent vector in $T_p(\mathcal{M})$ as a limit of vectors $\vec{px}$. The tangent vector $\mathbf{v}$ in this depiction is the limit of $\vec{px}_t$ where $x_t = p + t\mathbf{v}$. Under f, this maps to $Df(p)(\mathbf{v})$, the image of $\mathbf{v}$ under the differential of the map f at p. For example, the tangent vector known as ∂x_1 maps to the tangent vector $\sum_{j=1}^n (\partial f_j / \partial x_1) \partial x_j$. More generally,
$$f_* \left(\sum_{i=1}^d a_j \partial x_i \right) = \sum_{i=1}^d \sum_{j=1}^n a_i \frac{\partial f_j}{\partial x_i} \partial x_j .$$

Let us next see what map f induces on the cotangent bundle. By definition, for a cotangent vector $u \in T_p^*(\mathcal{M})$ and for $\mathbf{v} \in T_p(\mathbb{R}^d)$, $f^*(u)(\mathbf{v}) = u(f_*(\mathbf{v}))$. Take $u = dx_j$, the linear function mapping ∂x_j to 1 and ∂x_m to 0 for $m \neq j$. Unraveling the definitions, we see that

$$f^*(dx_j)(\partial x_i) = \frac{\partial f_j}{\partial x_i}\,.$$

We may now use the naturality of the wedge product to see that

$$f^*(dx_{j_1} \wedge \cdots \wedge dx_{j_d}) = \left| \frac{\partial f_{j_k}}{\partial x_i} \right| dx_1 \wedge \cdots \wedge dx_d = J\, dV$$

where dV denotes $dx_1 \wedge \cdots \wedge dx_d$ and J is the Jacobian of the $(j_1, \ldots, j_d)$-coordinates of f with respect to the coordinates $x_1, \ldots, x_d$.

The differential operator

The ring structure on $\Lambda(\mathcal{M})$ is natural: a smooth map $\psi : \mathcal{M} \to \mathcal{N}$ induces a map $\psi^* : \Lambda(\mathcal{N}) \to \Lambda(\mathcal{M})$ and any commuting diagram of smooth maps induces a commuting diagram of the exterior algebras. Let $\{dx_1, \ldots, dx_n\}$ denote the standard basis for $T_{\mathbf{p}}^*(\mathbb{R}^n)$, where dx_i maps the j^{th} standard basis vector to δ_{ij}. The standard basis for p-forms is $\{dx_\Phi\}_\Phi$.

If $\mathcal{M} \subseteq \mathbb{R}^n$ is a d-manifold, then the inclusion $\iota : \mathcal{M} \to \mathbb{R}^n$ induces a pullback $\iota^* : \Lambda(\mathbb{R}^n) \to \Lambda(\mathcal{M})$. The pullback $\{\iota^*(dx_\Phi)\}_\Phi$ of the standard basis in each fiber $T_{\mathbf{p}}^*(\mathbb{R}_n)$ is also denoted dx_Φ.

Let $\mathcal{M}$ be a d-manifold in $\mathbb{R}^n$. We may define a unary operation d on $E^*(\mathcal{M})$ of degree 1 by defining d on $E^p(\mathcal{M})$ as follows. Define

$$d(f\, dx_\Phi) := \sum_{1 \le i \le d} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_\Phi\,.$$

Only summands with $i \notin \Phi$ will be nonzero. Extend this by linearity to all of $E(\mathcal{M})$. Then d has the following properties:[1]

(i) $d^2 = 0$;

(ii) if f is a zero form (a smooth function) then $df = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\, dx_i$;

(iii) If $\omega \in E^p(\mathcal{M})$ and $\eta \in E^q(\mathcal{M})$ then

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta)\,.$$

The operator d is natural in the sense that $d(\psi^*(\omega)) = \psi^*(d\omega)$.

[1] It may appear that d depends on the choice of local coordinatization, since, for example, $f_1 dx_1 + f_2 dx_2$ in one basis is a constant dy_1 in a basis chosen to have this as the first component in each $T_{\mathbf{p}}^*(\mathcal{M})$. In fact we require the basis for $T_{\mathbf{p}}^*(\mathcal{M})$ is required to be dual to the basis $\{\partial/\partial x_i\}$ of $T_{\mathbf{p}}(\mathcal{M})$, which carries with it the connection between bases in different fibers.

Integration of forms

For each $p \geq 1$, let Δ^p denote the standard p-simplex in $\mathbb{R}^p$ defined by

$$\{(x_1, \ldots, x_p) \in \mathbb{R}^p : x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{p} x_i \leq 1\}.$$

When $p = 0$, take Δ^p to be a single point. Associated with the standard simplices are $p+1$ ways of embedding Δ^p as a face of Δ^{p+1}: for $1 \leq i \leq p$, let κ_i^p embed by inserting a zero in the i^{th} position

$$\kappa_i^p(x_1, \ldots, x_p) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p)$$

and let κ_0^p embed into the diagonal face

$$\kappa_i^p(x_1, \ldots, x_p) = (1 - \sum_{i=1}^{p} x_i, x_1, \ldots, x_p).$$

Let $\mathcal{M}$ be a d-manifold in $\mathbb{R}^n$. For $0 \leq p \leq d$, a **(singular) p-simplex** in $\mathcal{M}$ is defined to be a smooth map $\sigma : \Delta^p \to \mathcal{M}$. Define the space $\mathbb{C}^p(\mathcal{M})$ of **p-chains** on $\mathcal{M}$ to be the space of finite formal linear combinations $\sum c_i \sigma_i$ of p-simplices in $\mathcal{M}$. Define the boundary $\partial\sigma$ of a p-simplex σ by

$$\partial\sigma := \sum_{i=0}^{p} (-1)^i \sigma \circ \kappa_i^{p-1}.$$

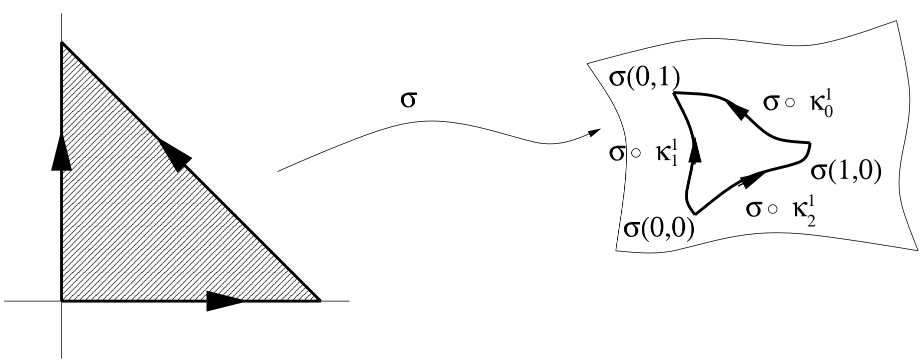

Figure 4.1: a singular 2-simplex and its boundary

Extend this linearly to the chains of $\mathcal{M}$. It is easy to verify (see Exercise 4.2) that

$$\partial^2 = 0. \tag{4.1}$$

For a domain $A \subseteq \mathbb{R}^n$, we define

$$\int_A f \, dx_1 \wedge \cdots \wedge dx_n := \int_A f \, dV$$

where dV is Lebesgue measure in $\mathbb{R}^n$. Now for any p-simplex $\sigma \in \mathcal{M}$, integration of a p-form ω over σ may be defined by

$$\int_\sigma \omega := \int_{\Delta^p} \sigma^*(\omega)$$

where σ^* is the pullback by σ of ω to the standard p-simplex. We may write $\sigma^*(dx_\Phi)$ more explicitly as $J\,dV$, where J is the Jacobian of the map $(x_1, \ldots, x_p) \mapsto (x_{i_1}, \ldots, x_{i_p})$. The integral $\int_{\mathcal{C}} \omega$ may be defined for any p-chain by extending linearly in $\mathcal{C}$. This allows us to integrate a p-form over any triangulable region of dimension p.

Stokes' Theorem now follows from definitions and some elementary computations (see [War83, Theorem 4.7]):

Theorem 4.1 (Stokes). *Let ω be a $p-1$-form ($p \geq 1$) on a manifold $\mathcal{M}$ of dimension at least p and let $\mathcal{C}$ be a p-chain on $\mathcal{M}$. Then*

$$\int_{\partial \mathcal{C}} \omega = \int_{\mathcal{C}} d\omega \,.$$

Some elementary properties of the integral and its relation to more elementary notions may be verified. The proofs of the following iterated integration properties are left as exercises.

Exercise 4.1. (i) Define a chain $\mathcal{C}$ corresponding to "the unit cube in $\mathbb{R}^n$" and prove that

$$\int_{\mathcal{C}} f\, dx_1 \wedge \cdots \wedge dx_n = \int_0^1 \cdots \left(\int_0^1 f(x_1, \ldots, x_n)\, dx_n \right) \cdots dx_1 \,.$$

(ii) Let $\mathcal{M}$ and $\mathcal{N}$ be respectively a p-manifold in $\mathbb{R}^m$ and a q-manifold in $\mathbb{R}_n$. Denote points in $\mathcal{M} \times \mathcal{N}$ by $(\mathbf{x}, \mathbf{y})$ and denote the projections $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$ and $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{y}$ by π and ρ respectively. Say that an element of the exterior algebra $\Lambda_k(\mathcal{M} \times \mathcal{N})$ is pure of degree (p, q) if it is the wedge $\pi^*\eta \wedge \rho^*\xi$ of forms of respective degrees p and q. What is the dimension of the space $\Lambda_k(p, q)$ of pure elements of degree (p, q) at a point $(\mathbf{x}, \mathbf{y})$?

(iii) Suppose $\omega = f(\mathbf{x}, \mathbf{y}) \pi^*\eta \wedge \rho^*\xi$ is a section of $\Lambda_k(p, q)$. Prove that if A is a p-chain in $\mathcal{M}$ and B is a q-chain in $\mathcal{N}$ then

$$\int_{A \times B} \omega = \int_A g(\mathbf{x}) \cdot \eta$$

where

$$g(\mathbf{x}) = \int_B f(\mathbf{x}, \mathbf{y}) \cdot \xi \,.$$

□

4.2 Differential forms in $\mathbb{C}^n$

The complex numbers may be identified with $\mathbb{R}^2$. Similarly, $\mathbb{C}^n$ may be identified with $\mathbb{R}^{2n}$; fix the identification that maps $\mathbf{z} = (z_1, \ldots, z_j) = (x_1 + iy_1, \ldots, x_j + iy_j)$ to $(x_1, y_1, \ldots, x_j, y_j)$. In this section, points of $\mathbb{R}^{2n}$ will be referred to by $(x_1, y_1, \ldots, x_n, y_n)$ rather than $(x_1, \ldots, x_{2n})$.

Let u be a map from $\mathbb{C}^n = \mathbb{R}^{2n}$ to $\mathbb{C}$ (we are not going to view the range as $\mathbb{R}^2$). The following are formal definitions only, though the symbols suggest the origins.

Definition 4.2. *Define*

$$\begin{aligned}
\frac{\partial u}{\partial z_j} &:= \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}\right) ; \\
\frac{\partial u}{\partial \overline{z}_j} &:= \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}\right) ; \\
dz &:= x + i\,dy \\
d\overline{z} &:= x - i\,dy\,, \\
\partial u &:= \sum_{j=1}^n \frac{\partial u}{\partial z_j}\,dz_j\,, \\
\overline{\partial} u &:= \sum_{j=1}^n \frac{\partial u}{\partial \overline{z}_j}\,d\overline{z}_j\,.
\end{aligned}$$

In terms of these notations, the d operator may be written

$$du = \partial u + \overline{\partial} u\,.$$

There is an intuitive notion of what it means for the map u to obey the complex structure: polynomials (and convergent power series) in the coordinates z_j should obey the complex structure, but functions such as $|z_1|$ that require decomposing into real and imaginary components $x_1^2 + y_1^2$ should not. This is formalized by the Cauchy-Riemann equations.

Definition 4.3. *Say that a p-form ω is holomorphic if $\overline{\partial}\omega = 0$. In particular, taking $p = 0$, this defines the notion of a holomorphic function from $\mathbb{C}^n$ to $\mathbb{C}$.*

One easily verifies that the coordinate functions are holomorphic and that holomorphic functions are closed under sums, products, limits in C^1 and applications of the implicit function theorem. This implies holomorphicity of rational functions, the exponential, the logarithm and so forth.

In particular, the holomorphic functions form a subring of $C^\infty(\mathbb{R}^d)$. A basis for $\Lambda_1(\mathbb{R}^{2n})$is given by $\{dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n\}$. The ring generated by the subcollection $\{dz_1, \ldots, dz_n\}$ over the ring of holomorphic functions on $\mathcal{M}$ is called the ring of holomorphic forms on $\mathcal{M}$. It is easy to check that the operator d preserves holomorphicity — indeed if $\overline{\partial}\omega = 0$ then $d\omega = \partial\omega$ which is evidently holomorphic. The notation $d\mathbf{z} := dz_1 \wedge \cdots \wedge dz_n$ denotes the **holomorphic volume form** in $\mathbb{C}^n$. It is an n-form in $\mathbb{R}^{2n}$, thus *middle-dimensional*, but is the highest dimensional holomorphic form in $\mathbb{R}^{2n}$.

One can go backwards: if u is holomorphic in a centered polydisk then it may be represented by an absolutely convergent power series and the terms of the power series may be extracted, leading to (see [Hör90, (2.2.3) and Theorem 2.2.1]):

Theorem 4.4 (multivariate cauchy formula). *Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ be a d-variate power series holomorphic in an open polydisk $\mathbf{D}$ containing the origin. Let $T = \prod_{i=1}^{d} \gamma_i$ be a product of circles γ_i which bound disks D_i of radii b_i, such that the polydisk $\prod_{i=1}^{d} D_i$ is a subset of $\mathbf{D}$. Then the multivariate Cauchy formula 1.4 holds:*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z}$$

where $d\mathbf{z} := dz_1 \wedge \cdots \wedge dz_d$ is the holomorphic volume d-form.

The foregoing definitions for complex forms are based on the definitions for real forms in twice the dimension. However, when dealing with holomorphic forms, one often works in the category of complex analytic spaces and holomorphic maps. In this case, the same results that hold for differential forms on real n-manifolds often hold for holomorphic forms on complex manifolds of the same dimension, n. See, for example, Exercises 4.3 and 4.4.

4.3 Algebraic topology

Chain complexes and homology theory

Our motivation for studying homology theory is as follows. A differential form ω is said to be **closed** if $d\omega = 0$. Many of the forms we care about are closed. For example, if ω is any holomorphic d-form in $\mathbb{C}^d$ then $\overline{\partial} u$ vanishes by holomorphicity and ∂u vanishes because there are no holomorphic $(d+1)$-forms, hence ω is closed.

A chain $\mathcal{C}$ is a **cycle** if $\partial \mathcal{C} = 0$. A chain $\mathcal{C}$ is a **boundary** if $\mathcal{C} = \partial \mathcal{D}$. The boundaries are a subset (in fact a sub-vector space) of the cycles because $\partial^2 = 0$. Suppose we wish to integrate a closed p-form ω over a boundary $\mathcal{C} = \partial \mathcal{D}$. By Stokes' Theorem,

$$\int_{\mathcal{C}} \omega = \int_{\mathcal{D}} d\omega = \int_{\mathcal{D}} 0 = 0\,.$$

By linearity, therefore, $\int_{\mathcal{C}} \omega$ depends only on the equivalence class of $\mathcal{C}$ in the quotient space of cycles modulo boundaries. Homology theory is the study of this quotient space. It turns out there is a benefit to maintaining generality. Although we care ultimately about the space of singular chains on a manifold, we define chain complexes and the homology functor in a purely algebraic way.

Definitions 4.5.

*(i) A **chain complex** $\mathcal{C}$ is a collection $\{C_n : n = 0, 1, 2, \ldots\}$ of complex vector spaces, not necessarily finite dimensional, together with a boundary operator ∂ which is a linear map ∂_n from the n-**chains** C_n to C_{n-1} satisfying $\partial^2 = 0$ (meaning that $\partial_n \circ \partial_{n+1} = 0$ for every n). By definition, $\partial = 0$ on C_0.*

(ii) The group of ***cycles*** *$Z_n \subseteq C_n$ is the kernel of ∂_n and the group B_n of* ***boundaries*** *is the image of ∂_{n+1}.*

(iii) The n^{th} homology group of $\mathcal{C}$ is defined by

$$H_n(\mathcal{C}) = \frac{Z_n}{B_n} .$$

The notation $H_*(X)$ is used to refer collectively to $H_n(X)$ for all n.

Remark 4.6. This is sometimes called **homology with coefficients in** $\mathbb{C}$, to distinguish it from the analogous construction with coefficients in $\mathbb{Z}$. While the theory with coefficients in $\mathbb{Z}$ is richer, taking coefficients in a field better suits the purposes of computing integrals (also, see Remark 4.8 below).

When $\mathcal{C}$ is the complex of singular chains of a d-manifold (or indeed any topological space) $\mathcal{M}$, then $H_k(\mathcal{C})$ is denoted $H_k(\mathcal{M})$ and is called the k^{th} (singular) homology group of $\mathcal{M}$. One thinks of the rank of the homology group $H_n(X)$ as indicating how many cycles there are that don't bound anything; for example, the rank of $H_1(X)$ should be the number of inequivalent circles that can be drawn on X and do not bound a disk in X; the rank of $H_1(X)$ for a connected space X should be zero if and only if X is simply connected.

A map between topological spaces (maps in this category are continuous) induces a natural map on the singular chain complexes (maps in this category commute with ∂). A map between chain complexes induces a natural map on the homology groups. Composing with the inverse, we see that a homeomorphism induces an isomorphism between the homology groups. Thus the homology groups of a topological space are topological invariants.

A homotopy is a map $H : X \times [0,1] \to Y$. When $H_0 := H(\cdot, 0)$ is a homeomorphism, we say that H is a homotopy between the image Y_0 of H_0 and the image Y_1 of $H_1 := H(\cdot, 1)$. Within the space Y, the spaces Y_0 and Y_1 are topologically equivalent. For example, if Y_1 is a single point, then Y_0 can be shrunk to a point inside Y and is a topologically trivial subspace. One way to see why this is true is to examine the homotopy at the chain level. Let $\mathcal{C}$ be a chain supported on X and for $j = 1, 2$ let $\mathcal{C}_j$ be the image of $\mathcal{C}$ under H_j. Let $\mathcal{C}_H$ denote the $(d+1)$-chain on the space $X \times [0,1]$ which is the produt of $\mathcal{C}$ with the standard 1-simplex, σ. Then

$$\partial \mathcal{C}_H = \mathcal{C}_1 - \mathcal{C}_0 + \partial \mathcal{C} \times \sigma . \tag{4.2}$$

When $\mathcal{C}$ is a cycle, this shows $\mathcal{C}_0$ and $\mathcal{C}_1$ to differ by a boundary, meaning that they are in the same homology class.

More general than a homeomorphism is a homotopy equivalence. Say that a map $f : X \to Y$ is a **homotopy equivalence** if there is a map $g : Y \to X$ such that $f \circ g$ is homotopic to the identity in Y and $g \circ f$ is homotopic to the identity in X. Homotopic maps induce equal maps on homology, hence homotopy equivalent spaces have naturally identical homology. To see this, one proves, on the chain level, that a homotopy equivalence between topological spaces induces a **chain homotopy equivalence** between the singular chain complexes, which induces again an isomorphism between the homology groups (see, e.g., [Mun84, Theorems 12.4, 30.7]).

While the singular chain complex is infinite dimensional, one may also isolate certain subcomplexes whose inclusion into the singular chain complex induces an isomorphism on homology. For example, a topological space is a **cell complex** if it may be built from cells homeomorphic to closed simplices of various dimensions, by identifying the boundaries in certain prescribed ways. The corresponding **cellular chain complex** $\mathcal{D}$ has a vector space of k-chains of dimension n_k equal to the number of k-cells, and the boundary map is given by the identifications. The singular homology is equal to the homology of $\mathcal{D}$ [Mun84, Theorem 39.4], hence it is easy to compute homology groups of a space expressed as a cell complex. Another consequence is that a space built from cells of dimension at most d has vanishing homology above dimension d. We will be interested in cell complexes as spaces over which we integrate differential forms. Let X be a cell complex of dimension d and chain complex $\mathcal{D}$. Each d-simplex corresponds naturally (up to sign) with a generator for the d-chains of $\mathcal{D}$. A sum of all the generators, with any signs, is called a **representing chain** for X.

An **exact sequence** of abelian groups is a sequence of maps

$$\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$$

where the image of each map is equal to the kernel of the next. A **short exact sequence** is a sequence

$$0 \to X \to Y \to Z \to 0$$

that is, the first map is injective, the last is surjective, and $Z \cong Y/\text{Image}\,(X)$. A short exact sequence of chain complexes is a map of chain complexes which is a short exact sequence on the n-chains for each n. The useful fact about short exact sequences of chain complexes is that they give rise to long exact homology sequences.

Theorem 4.7 (the long exact homology sequence). *Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence of chain complexes. Then there is a long exact sequence*

$$\cdots H_{n+1}(B) \to H_{n+1}(C) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \cdots$$

where the maps $H_n(A) \to H_n(B) \to H_n(C)$ are induced by the maps on chains. The maps ∂_* : $H_*(C) \to H_{*-1}(A)$ (in speech, the "boundary-star operators") have an explicit natural definition as well.

The proof is a "diagram chase" and is left as an exercise (Exercise 4.7 or see [Mun84, Theorem 23.3]).

Remark 4.8. When X, Y, Z are finite dimensional complex vector spaces of dimensions k, l, m respectively, then the short exact sequence $0 \to X \to Y \to Z \to 0$ means that $l = k + m$. One may therefore write Y as a direct sum $X \oplus Z$. However, this **splitting** is not natural: X embeds naturally in Y, but there is no canonical choice of coset representatives for Y/Z.

Relative homology and attachments

Let $Y \subseteq X$ be topological spaces. The inclusion of $Y \hookrightarrow X$ induces an inclusion of chain complexes $\mathcal{C}(Y) \hookrightarrow \mathcal{C}(X)$. Let $\mathcal{C}(X/Y)$ denote the quotient complex whose n-chains are the quotient group $C_n(X)/C_n(Y)$.

Definition 4.9 (relative homology). *The relative homology of a pair (X,Y) is defined to be $H_*(\mathcal{C}(X/Y))$.*

The long exact homology sequence for the short exact sequence $0 \to X \to Y \to X/Y \to 0$ is called the long exact sequence for the pair (X,Y). One may think of relative homology roughly as the homology of X if the space Y were to be shrunk to a point - we are looking for cycles that do not bound, but are willing to count a chain as a cycle if its boundary is in Y. In fact, if Y is nicely embedded in X (Y is a deformation retract of an open neighborhood of Y in X) then

$$\pi : (X,Y) \twoheadrightarrow (X/Y, Y/Y) \text{ is a topological quotient } \Rightarrow \pi_* \text{ is an isomorphism} \tag{4.3}$$

where π_* is the map induced by π on homology (see [Mun84, Exercise 39.3]).

Figure 4.2 shows a relative cycle in (X,Y). An important feature of relative is the **excision property**:

$$H_n(X,Y) = H_n(X \setminus U, Y \setminus U)$$

if U is in the interior of Y. Informally, the relative homology of (X,Y) "cannot see" the interior of Y.

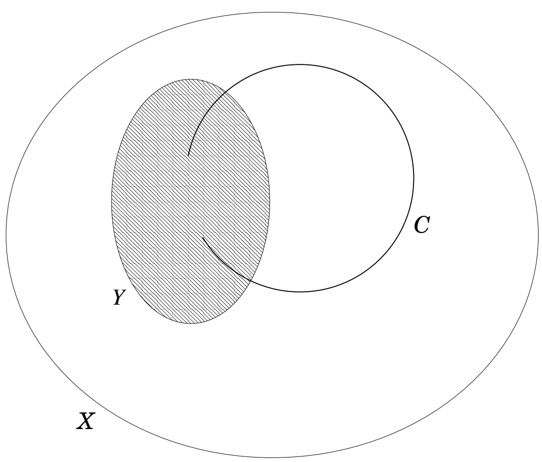

Figure 4.2: C is a relative cycle in $\mathcal{C}(X,Y)$

Definition 4.10 (attachment). *The attachment of a space Y to a space X along a closed subset $Y_0 \subseteq Y$ by the map $\phi : Y_0 \to X$ is the topological quotient $(X \biguplus Y)/\phi$ obtained from the disjoint union of X and Y by identifying each $\mathbf{y} \in Y_0$ with $\phi(\mathbf{y}) \in X$. The triple (Y, Y_0, ϕ) is known as the **attachment data**.*

Relative homology may be used to compute the homology of an attachment when the homology of the components is known. Let B be the attachment of Y to X by ϕ. In B, the set $X \setminus \phi(Y_0)$ is in the interior of X, so there are isomorphisms

$$H_*(Y, Y_0) \cong H_*(Y/Y_0, Y_0/Y_0) \cong (Y/\phi, Y_0/\phi) \cong H_*(B, X)\ , \tag{4.4}$$

the first two isomorphisms following from (4.3) and the last by excision. There is a lemma, known as the "five lemma", which states that in the following commutative diagram, if the horizontal rows are exact and all but the middle vertical arrows are known to be isomorphisms, then the middle vertical arrow is also an isomorphism.

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

The ∂_* operator in the sequence for the pair (B, X) may be written in terms of the ∂_* operator for the pair (Y, Y_0) as $\phi_* \circ \partial_* \circ \iota_*^{-1}$ where ι_* is the isomorphism obtained in (4.4).

$$\begin{array}{ccc}
H_{n+1}(B, X) & \longrightarrow & H_n(X) \\
\iota_* \uparrow \cong & & \uparrow \phi_* \\
H_{n+1}(Y, Y_0) & \xrightarrow{\partial_*} & H_n(Y_0)
\end{array}$$

The five lemma then shows that the homology group $H_n(B)$ is determined by diagram

$$H_{n+1}(B, X) \stackrel{\phi_*\partial_*\iota_*}{\to} H_n(X) \to H_n(B) \to H_n(B, X) \stackrel{\phi_*\partial_*\iota_*}{\to} H_{n-1}(X)\,.$$

In other words, if we understand the homology groups $H_*(Y, Y_0)$ and $H_*(X)$ and the maps $\phi_* : H_*(Y_0) \to H_*(X)$, then we may compute the homology of the whole attachment. For this reason, the pair (Y, Y_0) in the attachment data should be thought of as a topological pair.

Products

The product $\mathcal{C} = \mathcal{C}' \otimes \mathcal{C}''$ of chain complexes is defined by letting $C_n = \bigoplus_{k=0}^n C'_k \otimes C''_{n-k}$, where a basis for the tensor product $C'_k \otimes C''_{n-k}$ is given by $\bigcup_{k=0}^n \sigma \times \tau$, as σ ranges over a basis for C'_k and τ ranges over a basis for C''_{n-k}. The boundary operator is defined by $\partial(\sigma \times \tau) = (\partial\sigma) \times \tau + \sigma \times (\partial\tau)$. In this way, the singular chain complex of a product space is just the product of the chain complexes. The product of the category of pairs of topological spaces is defined by

$$(X', Y') \times (X'', Y'') = (X' \times X'', X' \times Y'' \cup Y' \times X'') \,.$$

The singular chain complex $\mathcal{C}_X/\mathcal{C}_Y$ for a pair (X, Y) which is the product of (X', Y') and (X'', Y'') is then equal to the product of the singular chain complexes for the pairs (X', Y') and (X'', Y'').

The homology of a product is given by the Künneth formula. With coefficients in $\mathbb{C}$, the formula relatively simple.

Theorem 4.11 (Künneth product formula). *There is a natural isomorphism*

$$\bigoplus_{p+q=n} H_p(\mathcal{C}') \times H_q(\mathcal{C}'') \to H_n(\mathcal{C}' \otimes \mathcal{C}'') \,.$$

Applying this to the singular chain complexes $\mathcal{C}'$ and $\mathcal{C}''$ for two spaces X' and X'' gives an identical looking formula for $H_n(X' \times X'')$. If $\mathcal{C}'$ and $\mathcal{C}''$ are the relative chain complexes for pairs (X', Y') and (X'', Y'') then one obtains

Corollary 4.12 (Künneth formula for pairs).

$$H_n(X' \times X'', X' \times Y'' \cup Y' \times X'') = H_n((X', Y') \times (X'', Y'')) \cong \bigoplus_{p+q=n} H_p(X', Y') \times H_q(X'', Y'') \,.$$

Cohomology

Given a chain complex $\mathcal{C}$, one may replace each vector space $\mathcal{C}_n$ by its dual $\mathcal{C}^n$, the elements of which are called **co-chains**. The maps ∂_n induce maps δ^n in the other direction. Thus we have the **co-chain complex**

$$\cdots \to \mathcal{C}^{n-1} \stackrel{\delta^{n-1}}{\to} \mathcal{C}^n \stackrel{\delta_n}{\to} \mathcal{C}^{n+1} \to \cdots$$

in which $\delta^n \circ \delta^{n-1} = 0$ for all n. The quotient of the kernel of δ^n (the cocyles) by the image of δ^{n-1} (the coboundaries) is called the n^{th} **cohomology group** of $\mathcal{C}$ and is denoted $H^N(\mathcal{C})$.

It is easy to verify that value of a cocycle ν evaluated at a cycle σ depends only on the cohomology class $[\nu]$ of ν and the homology class $[\sigma]$ of σ. This defines a product $\langle \omega, \eta \rangle$ for $\omega \in H^n(\mathcal{C})$ and $\eta \in H_n(\mathcal{C})$. In fact if X is a cell complex then this is a pairing and $H^n(X)$ is naturally the dual space of $H_n(X)$.

Any p-form ω may be naturally identified with the co-chain of degree p defined by $\mathcal{C} \mapsto \int_{\mathcal{C}} \omega$. Using the definition of δ and Stokes' Theorem we have

$$\delta\omega(\mathcal{C}) := \omega(\partial\mathcal{C}) := \int_{\partial\mathcal{C}} \omega = \int_{\mathcal{C}} d\omega = d\omega(\mathcal{C}) \,,$$

in other words, $\delta\omega = d\omega$. The cocycles are thus the closed forms, and the observation at the beginning of this section may be stated as the following theorem.

Theorem 4.13 (integral depends only on homology class). *Let ω be a closed p-form holomorphic on a domain $\mathcal{M} \subseteq \mathbb{C}^n$. If $p = n$ then ω is always closed. Let $\mathcal{C}$ be a singular p-cycle on $\mathcal{M}$. Then $\int_{\mathcal{C}} \omega$ depends on $\mathcal{C}$ only via the homology class $[\mathcal{C}]$ of $\mathcal{C}$ in $H_p(\mathcal{M})$ and on ω only by the cohomology class $[\omega]$ of ω in $H^p(\mathcal{M})$.*

Often we will be integrating a form of a specific type,

$$\omega = \exp(\lambda\, \phi(\mathbf{z}))\, \eta\,,$$

where η is a holomorphic k-form and the real part of ϕ is bounded above by c on Y. Let $\mathcal{C}$ and $\mathcal{C}'$ be chains representing the same relative cycle in $H_k(X, Y)$.

Proposition 4.14. *In this case, as $\lambda \to \infty$,*

$$\int_{\mathcal{C}} \omega = \int_{\mathcal{C}'} \omega + O\left(e^{\lambda c'}\right)$$

for any $c' > c$.

Proof: By definition, the difference between $\mathcal{C}$ and $\mathcal{C}'$ is a relative boundary:

$$\mathcal{C} - \mathcal{C}' = \partial \mathcal{D} + \mathcal{C}''$$

with $\mathcal{C}''$ supported on Y. Thus

$$\begin{aligned} \int_{\mathcal{C}} \omega - \int_{\mathcal{C}'} \omega &= \int_{\mathcal{D}} d\omega + \int_{\mathcal{C}''} \omega \\ &\leq e^{\lambda c} \int_{\mathcal{C}''} |\eta| \end{aligned}$$

because $d\omega = 0$. □

Notes

As mentioned at the start of Section 4.1, the more modern and general treatment of differentiable manifolds is to define the underlying space to be an arbitrary abstract set, together with a set of parametrizations of subsets by open balls in $\mathbb{R}^d$, such that compositions $\phi^{-1} \circ \psi$ of parametrizations ϕ and ψ are smooth maps on their domain in $\mathbb{R}^d$. This is undoubtedly more natural, since the embedding of the manifold in $\mathbb{R}^n$ plays no role in its properties. The use of this review in briefly informing readers who do not already know the material dictates, however, that a shorter path be taken. The embedding in $\mathbb{R}^n$ allows tangent vectors to be defined geometrically rather than as derivations. This seems to me the only way that they can be digested on the first pass. For a comparison of the geometric and abstract definitions, see [Lee03, Chapter 3].

The material in this chapter is all standard graduate level calculus, but the first two sections of this chapter owe an organizational debt to Warner's text [War83].

Exercises

Exercise 4.2 ($\partial^2 = 0$). Verify that $\partial^2 = 0$ in equation (4.1) by proving that $\kappa_j^{p+1} \circ \kappa_j^p = \kappa_{j+1}^{p+1} \circ \kappa_i^p$.

Exercise 4.3. (a) Let $f : X \to \mathbb{R}$ be a smooth map on a d-manifold X for which df is everywhere nonvanishing. Let $\mathcal{M}$ be the zero set of f and let $\iota : \mathcal{M} \to X$ denote the inclusion map. Prove that for any $(d-1)$-form η, $\iota^*(\eta) = 0$ if and only if $\eta \wedge df$ vanishes on $\mathcal{M}$. Hint: use the implicit function theorem to coordinatize X with first coordinate f and use functoriality of $\wedge$ to reduce to the case $f = x_1$. (b) Repeat this for $k \leq d$ functions $f_1, \ldots, f_k$, whose transverse intersection defines a smooth surface $\mathcal{M}$ of co-dimension k.

Exercise 4.4. Do Exercise 4.3 replacing $\mathbb{R}$ by $\mathbb{C}$, that is, X is a complex d-manifold and $f : X \to \mathbb{C}$ is analytic. Hint: you can copy the proof, only you need the complex form of the implicit function theorem in order to be sure your coordinates are holomorphic; see [Hör90, Theorem 2.1.2].

Exercise 4.5. Let $\mathcal{C}$ be a d-chain supported on a submanifold of $\mathbb{C}^d$ of dimension less than d. Show that $\int_{\mathcal{C}} \omega$ vanishes for any holomorphic d-chain ω.

Exercise 4.6. Define a 2-form ω in $\mathbb{R}^3$ by $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$. Define a 2-chain $\mathcal{C}$ that is "the unit sphere" and compute $\int_{\mathcal{C}} \omega$ directly from the definitions. Now figure out a shortcut to the same computation using Stokes' Theorem.

Exercise 4.7 (long exact sequence). Define the ∂_* operator in Theorem 4.7 and give a proof of the theorem.