Chapter 5

Fourier-Laplace integrals in one variable

This chapter is devoted to the proof of

$$\int_\gamma A(z)\exp(-\lambda\phi(z)) \sim A(z_0)\sqrt{\frac{2\pi}{\phi''(z_0)\lambda}}\exp(-\lambda\phi(z_0))\,, \tag{3.8}$$

The most general univariate result we will obtain is the following theorem, which will be proved in Section 5.2. It concerns the asymptotic evaluation of the integral of $A(z)\exp(-\lambda\phi(z))$ as $\lambda \to \infty$. The functions A and ϕ are called the **amplitude** and **phase** functions respectively (although in the case where $\phi = i\rho$ is purely imaginary, the phase usually denotes ρ rather than $i\rho$).

Theorem 5.1. *Let A and ϕ be analytic functions on a neighborhood $\mathcal{N} \subseteq \mathbb{C}$ of the origin. Let*

$$\begin{aligned} A(z) &= \sum_{j=0}^\infty b_j z^j \\ \phi(z) &= \sum_{j=0}^\infty c_j z^j \end{aligned}$$

be the power series for A and ϕ and let $l \geq 0$ and $k \geq 2$ be the indices of the least nonvanishing terms in the respective series, that is, $b_l \neq 0$, $c_k \neq 0$ and $b_j = 0$ for $j < l$ and $c_j = 0$ for $i < k$. Let $\gamma : [-\epsilon, \epsilon] \to \mathbb{C}$ be any smooth curve with $\gamma(0) = 0 \neq \gamma'(0)$ and assume that $\mathrm{Re}\{\phi(\gamma(t))\} \geq 0$ with equality only at $t = 0$. Denote

$$\begin{aligned} \mathcal{I}_+(\lambda) &:= \int_{\gamma|_{[0,\epsilon]}} A(z)\exp(-\lambda\phi(z))\,dz\,; \\ \mathcal{I}(\lambda) &:= \int_\gamma A(z)\exp(-\lambda\phi(z))\,dz\,; \\ C(k,l) &:= \frac{\Gamma((1+l)/k))}{k}\,. \end{aligned}$$

Then there are asymptotic expansions

$$\mathcal{I}_+(\lambda) = \sum_{j=l}^{\infty} a_j C(k,j)(c_k\lambda)^{-(1+j)/k} \tag{5.1}$$

$$\mathcal{I}(\lambda) = \sum_{j=l}^{\infty} \alpha_j C(k,j)(c_k\lambda)^{-(1+j)/k} \tag{5.2}$$

with the following explicit description.

(i) a_j *is a polynomial in the values* $b_l, \ldots, b_j, c_k^{-1}, c_{k+1}, \ldots, c_{k+j-l}$ *explicitly constructed in the proof, the first two values of which are* $a_l = b_l$ *and* $a_{l+1} = b_{l+1} - \frac{2+l}{k}\frac{c_{k+1}}{c_k}$.

(ii) the choice of k^{th} *root in the expression* $(c_k\lambda)^{-(1+j)/k}$ *is made by taking the principal root in* $x^{-1}(c_k\lambda x^k)^{1/k}$ *where* $x = \gamma'(0)$*;*

(iii) the numbers α_j *are related to the numbers* a_j *when* k *is even by*

$$\alpha_j = \begin{cases} 2a_j & j \text{ is even} \\ 0 & j \text{ is odd} \end{cases}$$

and when k *is odd by*

$$\alpha_j = \left(1 + \zeta^j\right) a_j$$

where

$$\zeta = -\exp\left(\frac{i\pi}{k}\operatorname{sgn}\operatorname{Im}\{\phi(\mathbf{v})\}\right).$$

Remarks. If $\phi(0) = \nu \neq 0$ but $\operatorname{Re}\{\phi(x)\}$ is still maximized at $x = 0$, then one may apply this result, replacing ϕ by $\phi - \nu$ and multiplying the outcome by $\exp(\lambda\nu)$. More significantly, if $\phi'(0) \neq 0$, then the hypothesis that $\operatorname{Re}\{\phi(x)\}$ is uniquely minimized at zero may be replaced by the weaker hypothesis that the minimum is achieved at zero. This requires methods from the next chapter.

To those unfamiliar with stationary phase methods, this result may seem difficult to decipher, but both the statement and proof are actually quite intuitive. When A and ϕ are real, the orders of magnitude of such integrals are evident from direct integration of the orders of magnitude. Changing variables to simplify the exponent produces a full asymptotic development of the integral. When the phase is complex, one can use integration by parts in order to cancel the oscillation, or one can reduce to the real case by a contour shift. The latter requires stronger hypotheses (analyticity rather than smoothness) but gives stronger results (exponentially small remainders rather than rapidly decreasing remainders). In order to give all of the intuition, I will take a route to the derivation that is longer than necessary. I will begin with a stripped down special case, in which direct integration suffices, then give the arguments that extend this to greater generality, giving parallel arguments in the smooth and analytic categories when appropriate.

5.1 Real integrands

The conclusion of Theorem 5.1 for real amplitude and phase functions would be an asymptotic expansion

$$\int_0^\epsilon A(x)\exp(-\lambda\phi(x))\,dx \sim \sum_{j=l}^\infty a_j\lambda^{-(1+j)/k}.$$

valid for real analytic functions A and ϕ with power series coefficients as in Theorem 5.1. The main result of this section, Theorem 5.5, yields this expansion, along with further information about a_j.

When working on the real line, complex analytic techniques are not needed, and consequently we need to assume only differentiability and not analyticity. We build the argument in three steps: first, take A and ϕ to be monomials; next keep the restriction on ϕ but remove the restriction on A; finally, remove the restriction on ϕ as well. The first step is accomplished via an exact computation, the second via a remainder estimate, and the third is deduced from the second by a change of variables.

A and ϕ are monomials

On the positive half-line, we can get away with a change of variables involving a fractional power. This allows us to handle the special case of monomial phase and amplitude by an exact integral, holding for any nonnegative real powers, α and β. Substitute $y=\lambda x^\alpha$ to get

$$\begin{aligned}\int_0^\infty x^\beta\exp(-\lambda x^\alpha)\,dx &= \int_0^\infty \left(\frac{y}{\lambda}\right)^{\beta/\alpha} e^{-y}\frac{1}{\alpha}\frac{y^{1/\alpha-1}}{\lambda^{1/\alpha}}\,dy\\ &= \frac{1}{\alpha}\lambda^{-(1+\beta)/\alpha}\int_0^\infty y^{\frac{1+\beta}{\alpha}-1}e^{-y}\,dy\end{aligned}$$

By the definition of the Γ-function, we therefore have the exact evaluation

$$\int_0^\infty x^\beta\exp(-\lambda x^\alpha)\,dx = C(\alpha,\beta)\,\lambda^{-(1+\beta)/\alpha} \tag{5.3}$$

$$C(\alpha,\beta) := \frac{\Gamma(\frac{1+\beta}{\alpha})}{\alpha}. \tag{5.4}$$

All the contribution to (5.3) comes from a neighborhood of zero: for any $\epsilon>0$ the contribution from $x\in[\epsilon,\infty)$ is exponentially small in λ so the integral over $[0,\epsilon]$ captures the value up to an exponentially small correction:

$$\left|\int_0^\epsilon x^\beta\exp(-\lambda x^\alpha)\,dx - C(\alpha,\beta)\,\lambda^{-(1+\beta)/\alpha}\right| \quad\text{decays exponentially.}$$

When β is an integer and α is an even integer, the corresponding two-sided integrals make sense as well:

$$\int_{-\infty}^\infty x^l\exp(-\lambda x^{2k})\,dx = \begin{cases} 2C(2k,l)\,\lambda^{-(1+l)/(2k)} & \text{if } l \text{ is even;}\\ 0 & \text{if } l \text{ is odd.}\end{cases} \tag{5.5}$$

ϕ is a monomial, A is anything

The results for monomials easily imply the following estimate.

Lemma 5.2 (big-O lemma). *Let $k, l > 0$ with k an integer. If A and ϕ are real-valued, piecewise smooth functions, with $A(x) = O(x^l)$ at $x = 0$, and $\phi(x) \sim x^k$ at $x = 0$ and vanishing in $[0, \epsilon]$ only at 0, then*

$$\int_0^\epsilon A(x) \exp(-\lambda\phi(x))\, dx = O(\lambda^{(-l-1)/k})$$

as $\lambda \to \infty$.

PROOF: Pick K such that $|A(x)| \le K|x|^l$ on $[-\epsilon, \epsilon]$ and δ such that $|\exp(-\lambda\phi(x))| \le \exp((\delta-\lambda)|x|^k)$ on $[0, \epsilon]$. Then

$$\begin{aligned} \left| \int_0^\epsilon A(x) \exp(-\lambda\phi(x))\, dx \right| &\le K \int_0^\epsilon x^l \exp((\delta - \lambda)x^k)\, dx \\ &= O(\lambda - \delta)^{-(1+l)/k} \end{aligned}$$

by (5.3). □

For monomial phase functions and general amplitude functions, we now have the following result.

Lemma 5.3. *Suppose that A is a real function with*

$$A(x) = \sum_{j=l}^{M-1} b_j x^j + O(x^M)$$

as $x \to 0$. Then

$$\int_0^\epsilon A(x) \exp(-\lambda x^k)\, dx = \sum_{j=l}^{M-1} b_j C(k, j) \lambda^{-(1+j)/k} + O(\lambda^{-(1+M)/k})$$

where $C(k, j) = \Gamma((1 + j)/k)/k$ are the constants computed in (5.3).

Remark. Note that the hypothesis on A is quite weak. In particular, A need not even be in the class C^1 (for example, take $A = x^M \sin(x^{-M})$). If A is represented by an infinite asymptotic series (convergent or not) then an infinite asymptotic expansion for the integral follows by applying the lemma for each M.

PROOF: Multiply the estimate

$$A(x) - \sum_{j=0}^{M-1} b_j x^j = O(x^M)$$

by $\exp(-\lambda\phi(x))$ and integrate. Using (5.3) to evaluate the integral of each monomial and Lemma 5.2 to bound the integral of the right-hand side gives

$$\left| \mathcal{I} - \sum_{j=0}^{M-1} \int_0^\epsilon b_j x^j \exp(-\lambda x^k)\, dx \right| = O(\lambda^{(-l-1)/k})$$

which is the conclusion of the lemma. □

General A and ϕ

A change of variables reduces the general case to Lemma 5.3. A bit of care is required to ensure we understand the asymptotic series for the functions involved in the change of variables.

Lemma 5.4. *Let $M \geq 2$ be an integer and let*

$$y(x) = c_1 x + \cdots + c_{M-1} x^{M-1} + O(x^M) \tag{5.6}$$

in a neighborhood of zero, where $c_1 \neq 0$. Then there is a neighborhood of zero on which y is invertible. The inverse function $x(y)$ has an expansion

$$x(y) = a_1 y + \cdots + a_{M-1} y^{M-1} + O(y^M)$$

with a_j polynomials in $c_1, \ldots, c_{j-1}$ and c_1^{-1}.

PROOF: Suppose $c_1 = 1$. From $y = x + O(x^2)$ we see that $y \sim x$ at zero, hence $x = y + O(x^2) = y + O(y^2)$. Now let $2 \leq n < M$ and suppose inductively that $x = y + a_2 y^2 + \cdots + a_{n-1} y^{n-1} + O(y^n)$, where $a_2, \ldots, a_{n-1}$ are polynomials in $c_2, \ldots, c_{n-1}$. Let a be an indeterminate, and plug in the value of y in (5.6) to the quantity

$$x - (y + a_2 y^2 + \cdots + a_{n-1} y^{n-1} + a y^n) .$$

The result is a polynomial in x, whose coefficients in degrees $1, \ldots, n-1$ vanish due to the induction hypothesis, plus a remainder of $O(x^M)$. The coefficient of the x^n term may be written as $a - P(a_2, \ldots, a_{n-1}, c_2, \ldots, c_n)$ where P is a polynomial. By induction, this is a polynomial in $c_2, \ldots, c_n$. Setting a_n equal to this polynomial, we see that

$$x_n - y - \sum_{j=2}^{n} a_n y^n = O(x^{n+1}) .$$

This completes the induction.

When $n = M - 1$, observing that $O(x^M) = O(y^M)$ completes the proof of the lemma for $c_1 = 1$. To remove the restriction on c_1, apply the case $c_1 = 1$ to represent x as a function of y/c_1, which shows that $x = \sum_{j=1}^{M-1} a_j y^j + O(y^M)$ with $c_1^j a_j$ a polynomial in $c_2, \ldots, c_j$. □

Theorem 5.5. *Let $M \geq k, l$ be integers. Suppose that A and ϕ are real functions with ϕ of class C^M and series*

$$\begin{aligned} A(x) &= \sum_{j=l}^{M-1} b_j x^j + O(x^M) \\ \phi(x) &= \sum_{j=k}^{M} c_j x^j + O(x^{M+1}) \end{aligned}$$

as $x \to 0$, where $b_l, c_k \neq 0$. Then as $\lambda \to \infty$, the quantity $\mathcal{I}(\lambda) := \int_0^\epsilon A(x) \exp(-\lambda \phi(x))\, dx$ has asymptotic expansion

$$\mathcal{I}(\lambda) \sim \sum_{j=l}^{M-1} a_j C(k,j)(c_k \lambda)^{-(1+j)/k} + O(\lambda^{-(1+M)/k}) \tag{5.7}$$

with $C(k,l) = \Gamma(\frac{1+l}{k})/k$ as in (5.4) and the terms a_j given by polynomials in $b_l, \ldots, b_j$ and $c_k^{-1}, c_{k+1}, \ldots, c^{k+j-l}$. The leading two terms are given by

$$\begin{aligned} a_l &= b_l\,; \\ a_{l+1} &= b_{l+1} - \frac{2+l}{k} \frac{c_{k+1}}{c_k}\,. \end{aligned} \tag{5.8}$$

PROOF: We employ the change of variables $y = (\phi(x))^{1/k}$. Writing

$$\phi(x) = c_k x^k \left(1 + \frac{c_{k+1}}{c_k} x + \cdots + \frac{c_M}{c_k} x^{M-k} + O(x^{M+1-k})\right)$$

we see that

$$y = c_k^{1/k} x \left(1 + \cdots + \frac{c_M}{c_k} x^{M-k} + O(x^{M+1-k})\right)^{1/k}. \tag{5.9}$$

Using the Taylor series for $(1+u)^{1/k}$ (that is, the binomial expansion), we see that

$$y = c_k^{1/k} \sum_{j=1}^{M} d_j x^j + O(x^{M+1})$$

where d_j are polynomials in $c_{k+1}, \ldots, c_M$ and c_k^{-1}.

By the previous lemma, the inverse function satisfies

$$x = \sum_{j=1}^{M} e_j \left(\frac{y}{c_k^{1/k}}\right)^j + O(y^{M+1}) \tag{5.10}$$

where e_j is a polynomial in $c_{k+1}, \ldots, c_j$. A function of class C^M with nowhere vanishing derivative has an inverse of class C^M, which justifies term by term differentiation, yielding

$$x'(y) = c_k^{-1/k} \sum_{j=1}^{M} j e_j \left(\frac{y}{c_k^{1/k}}\right)^{j-1} + O(y^M)\,.$$

The change of variables formula gives

$$\mathcal{I}(\lambda) = \int_0^{y(\epsilon)} \tilde{A}(y) \exp(-y^k)\, dy$$

where $\tilde{A}(y) = A(x(y))x'(y)$. Plugging in the series for x and x' into the definition of $\tilde{A}$ gives

$$\tilde{A}(y) = c_k^{-1/k} \sum_{j=l}^{M-1} \tilde{b}_j \left(\frac{y}{c_k^{1/k}}\right)^j + O(y^M)$$

where $\tilde{b}_j$ is a polynomial in $b_l, \ldots, b_j, c_k^{-1}, c_{k+1}, \ldots, c_j$, to be evaluated shortly. The existence of the expansion (5.7) now follows from the monomial exponent case (Lemma 5.3).

The leading terms (5.8) are computed as follows. The change of variables is

$$\begin{aligned} y &= c_k^{1/k} x \left(1 + \frac{c_{k+1}}{c_k} x + O(x^2)\right)^{1/k} \\ &= c_k^{1/k} x \left(1 + \frac{c_{k+1}}{k\, c_k} x + O(x^2)\right). \end{aligned}$$

Inverting and differentiating,

$$\begin{aligned} x &= \frac{y}{c_k^{1/k}} - \frac{1}{k} \frac{c_{k+1}}{c_k} \left(\frac{y}{c_k^{1/k}}\right)^2 + O(y^3); \\ x'(y) &= \frac{1}{c_k^{1/k}} - \frac{2}{c_k^{2/k}} \frac{c_{k+1}}{k\, c_k} y + O(y^2). \end{aligned}$$

Composing and multiplying shows the coefficients of $\tilde{A}(y) = A(x(y))x'(y)$ to be

$$\begin{aligned} \tilde{b}_l &= b_l \\ \tilde{b}_{l+1} &= b_{l+1} - \frac{l+2}{k} \frac{c_{k+1}}{c_k} \end{aligned}$$

and evaluating $\int \tilde{A}(y) \exp(-\lambda y^k)\, dy$ via Lemma 5.3 yields (5.8). □

5.2 Complex phase

Extending the results of the previous section to complex amplitudes is trivial - by linearity of the integral, the result holds separately for $\mathrm{Im}\{A\}$ and $\mathrm{Re}\{A\}$, and these may then be recombined to give the result for complex A. When it comes to complex phases, we are faced with a choice. If we assume A and ϕ are analytic in a neighborhood of zero, we are entitled to move the contour; this is the quickest justification for extending the conclusion to complex phases without much change in the formula and is the approach taken in this section.

PROOF OF THEOREM 5.1: Begin with the half-line integrals. Let $\gamma_+ : [0, \epsilon] \to \mathbb{C}$ denote the restriction $\gamma_{[0,\epsilon]}$ so that $\mathcal{I}_+ = \int_{\gamma_+} A(z) \exp(-\lambda\phi(z))\, dz$. Evaluate $\mathcal{I}_+$ as follows.

We employ the same change of variables $y = \phi(z)^{1/k}$ as in the proof of Theorem 5.5, only we will need to be careful in choosing a branch of the $1/k$ power. Formula (5.9) defines k different functions, one for each choice of the the k^{th} root in the expressions $c_k^{1/k}$. It follows from Lemma 5.4 that each of these k functions and their inverses are analytic in a neighborhood of the origin. We will need a notation for the **principal** k^{th} root. This is the analytic function from the plane minus the negative real half-line to the cone $K := \{z : -\pi/k < \arg(z) < \pi/k\}$ which we define $\mathfrak{p}(u^{1/k}) = z$ for the unique $z \in K$ such that $z^k = u$.

Let $\mathbf{v}$ denote any scalar multiple of $\gamma'(0)$. Near the origin, $\phi(z) \sim c_k z^k$ and the requirement that $\mathrm{Re}\{\phi\} \geq 0$ forces the quantity $\mathbf{v}$ to be in the windmill-shaped set of pre-images under $c_k z^k$ of the right half-plane. Define $f(x) := \mathfrak{p}(\phi(x)^{1/k})$. Since the path $\phi(\gamma_+(t))$ remains in the positive

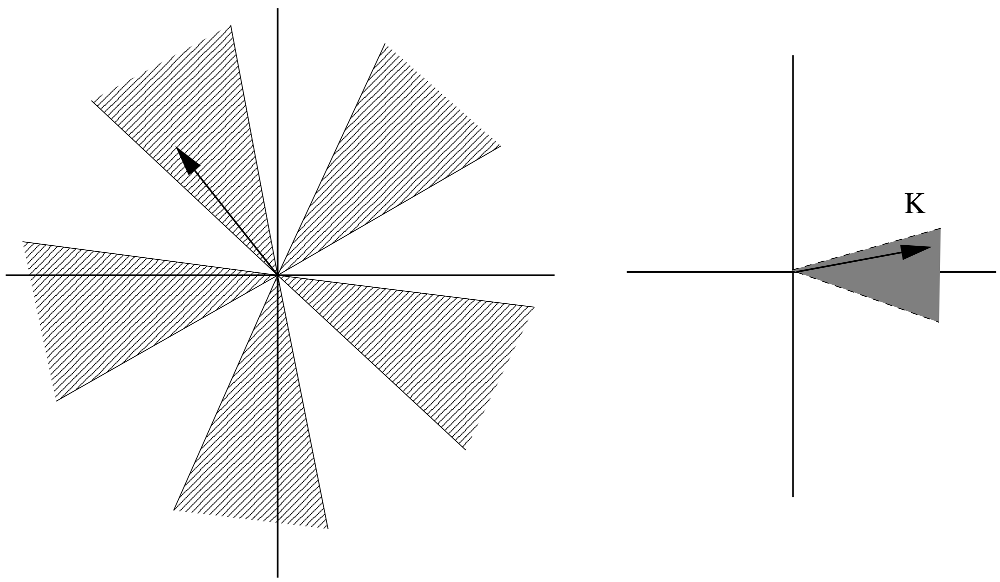

Figure 5.1: arrows represent $\mathbf{v}$ and $(d/dt)|_{t=0} f(\gamma^+(t))$

real half-plane for $0 < t \leq \epsilon$, it also remains in the slit plane, and hence maps the image of γ_+ bi-analytically to the cone K. With this choice of $1/k$ power, the change of variables (5.9) becomes

$$y = f(x) = \eta x \left(1 + \cdots + \frac{c_M}{c_k} x^{M-k} + O(x^{M-1-k})\right)^{1/k}$$

where $\eta = \mathbf{v}^{-1}\mathfrak{p}(c_k \mathbf{v}^k)^{1/k}$ and the branch of the $1/k$ power of the series in parentheses is the one which fixes 1. Thus $\eta = f'(0)$ and

$$(d/dt)_{t=0} f(\gamma_+(t)) = \eta \mathbf{v} \,. \tag{5.11}$$

The inverse function, $x = g(y)$, is given by taking $c_k^{1/k} = \eta$ in (5.10). This choice fulfills property (ii) in Theorem 5.1.

As in the proof on Theorem 5.5, we then have

$$\mathcal{I}_+ = \int_{\tilde{\gamma}} \tilde{A}(y) \exp(-\lambda y^k)\, dy$$

where $\tilde{\gamma} = f \circ \gamma_+$ is the image of γ_+ under the change of variables.

Let $p = f(\gamma(\epsilon))$ denote the endpoint of $\tilde{\gamma}$. Let $p' > 0$ denote the real part of p, let α be the line segment $[0, p']$ and let β denote the line segment $[p', p]$. The contour $\tilde{\gamma}$ is homotopic to $\alpha + \beta$

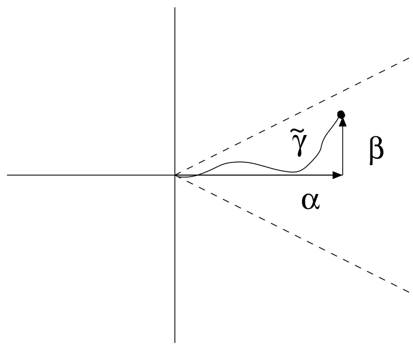

Figure 5.2: the path $\tilde{\gamma}$ in the cone K and the line segments α and β

(see figure 5.2), whence $\int_{\tilde{\gamma}} h(z)\,dz = \int_\alpha h(z)\,dz + \int_\beta h(z)\,dz$ for any analytic function h. On compact subsets of K, $\mathrm{Re}\,\{y^k\}$ is bounded from below by a positive constant. It follows that on β, there are positive C and ρ for which

$$\left|\tilde{A}(y)\exp(-\lambda y^k)\right| \leq Ce^{-\rho\lambda}\,.$$

(The reason we chose the principal value is so that β would lie inside K.) We conclude that

$$\mathcal{I}_+ = \int_\alpha \tilde{A}(y)\exp(-\lambda y^k)\,dy + R$$

for a remainder R that decays exponentially. Applying Theorem 5.5 (with complex amplitude) to $\int_\alpha$ gives the asymptotic series

$$\mathcal{I}_+ \sim \sum_{j=l}^{\infty} a_j C(k,j) c_k^{-(1+j)/k} \lambda^{-(1+j)/k}$$

satisfying (i) and (ii) of Theorem 5.1.

To evaluate the two-sided intergal, let γ_- denote the contour $\gamma|_{[-\epsilon,0]}$ oriented from 0 to $-\epsilon$. Then $\mathcal{I} = \mathcal{I}_+ - \mathcal{I}_-$ where $\mathcal{I}_- = \int_{\gamma_-} A(z)\exp(-\lambda\phi(z))\,dz$. Let $y = f_-(z)$ denote the relation between y and $z \in \gamma_-$ and let $\eta_- = f'_-(0)$. The evaluation of $\mathcal{I}_-$ breaks into two cases, depending on the parity of k.

Suppose first that k is even. Since $\phi(z) \sim c_k z^k$, the image of the smooth curve γ under ϕ does a U-turn at the origin, with the tangents to the images $\phi(\gamma_-(t))$ and $\phi(\gamma_+(t))$ coinciding at $t = 0$ (see figure 5.3). Mapping the right half-plane to K by the principal k^{th} root preserves this coincidence of tangents. Since $\gamma'_-(0) = -\mathbf{v}$, we see from (5.11) that

$$(\eta_-)(-\mathbf{v}) = \frac{d}{dt}\mathfrak{p}(\phi(\gamma_-(t))^{1/k})\Big|_{t=0} = \frac{d}{dt}\mathfrak{p}(\phi(\gamma_+(t))^{1/k})\Big|_{t=0} = \eta\mathbf{v}$$

and hence $\eta_- = -\eta$. It follows from the power series (5.9) that $f_-(z) = -f(z)$ as analytic maps in a neighborhood of the origin. Inverting, $g_-(y) = g(-y)$. A contour decomposition analogous to

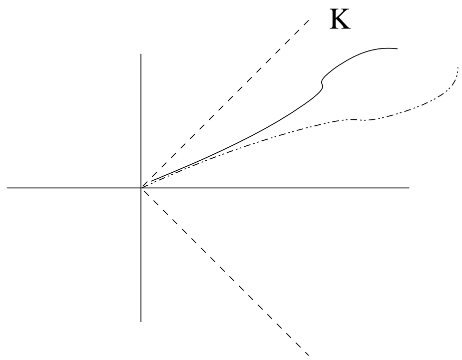

Figure 5.3: k is even: $\phi(\gamma_+)$ is shown solid, and $\phi(\gamma_-)$ is shown dotted

$\tilde{\gamma} = \gamma' + \gamma''$ shows that $\mathcal{I}_-$ is given, up to an exponentially small correction, by

$$\int A(g_-(y)) \exp(-\lambda y^k) \frac{d}{dy} g_-(y)\, dy$$

where the integral is over any short segment from the origin along the positive real axis. Since $g_-(y) = g(-y)$, we have

$$\begin{aligned} \mathcal{I}_- &= \int A(g(-y)) \exp(-\lambda y^k) \frac{d}{dy}(g(-y))\, dy \\ &= \int -\tilde{A}(-y) \exp(-\lambda y^k)\,. \end{aligned}$$

Hence $\mathcal{I} = \mathcal{I}_+ - \mathcal{I}_- = \int (\tilde{A}(y) + \tilde{A}(-y) \exp(-\lambda y^k)\, dy$. Therefore $\mathcal{I}$ has an expansion whose coefficients α_j vanish for odd j and are twice a_j for even j, completing the proof of the theorem in this case.

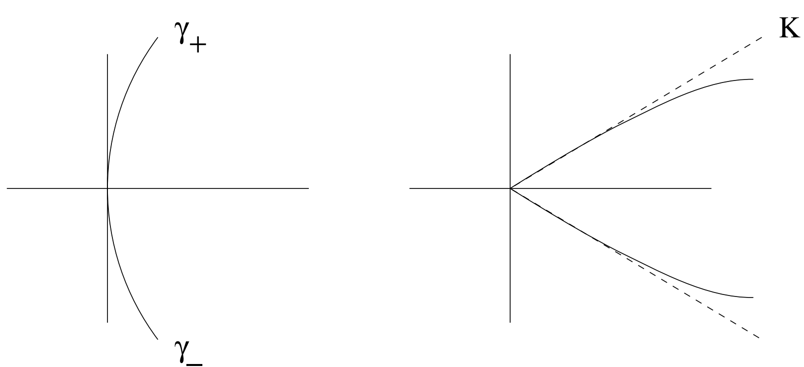

Figure 5.4: k is odd: $\phi(\gamma_+)$ and $\phi(\gamma_-)$ and their principal $1/k$ powers

When k is odd, the images of γ_+ and γ_- under ϕ point in opposite directions. Since both are in the closed right half-plane, this implies that one is in the positive imaginary direction and one is in the negative imaginary direction. The curve γ_+ will be in the positive imaginary direction if the sign of $\mathrm{Im}\{\phi(\mathbf{v})\}$ is positive and the latter occurs if the sign is negative. Mapping by the principal k^{th} root results in the argument of the tangent vector to $f_- \circ \gamma_-$ differing from the argument to the tangent to $f \circ \gamma_+$ by $-(\pi/k)\mathrm{sgn}\,\mathrm{Im}\{\phi(\mathbf{v})\}$. Again, $\gamma_-'(0) = -\mathbf{v}$ and therefore

$$\eta_- = \eta(-1)\exp(-i\pi\mathrm{sgn}\,\mathrm{Im}\{\phi(\mathbf{v})\}/k)\,.$$

Thus $A(g^{-1}(y))(d/dy)g^{-1}(y) = -\tilde{A}(y/\eta)$, hence

$$\mathcal{I} = \int \tilde{A}(y) + \tilde{A}((-\exp(i\pi[\mathrm{sgn}\,\mathrm{Im}\,\phi(x)]/k)y)\exp(-\lambda y^k)\,dy\,,$$

which completes the proof of the theorem in the case that k is odd. □

5.3 Classical methods: steepest descent (saddle point) and Watson' Lemma

This section contains some classical results that may be proved using the machinery of Sections 5.1 and 5.2. Lemma 5.3 with $k = 1$ is a special case of Watson's Lemma (smooth amplitude). The usual statement is

Proposition 5.6 (Watson's Lemma). *Let $A : \mathbb{R}^+ \to \mathbb{C}$ have asymptotic development*

$$A(t) \sim \sum_{m=0}^{\infty} b_m t^{\beta_m}$$

with $-1 < \mathrm{Re}\{\beta_0\} < \mathrm{Re}\{\beta_1\} < \cdots$ and $\mathrm{Re}\{\beta_m\} \uparrow \infty$. Then the Laplace transform has asymptotic series

$$L(\lambda) := \int_0^\infty A(t)e^{-\lambda t}\,dt \sim \sum_{m=0}^{\infty} b_m\Gamma(\beta_m+1)\lambda^{-(1+\beta_m)}$$

as $\lambda \to \infty$.

Remark. This result is similar in spirit to Darboux' Theorem (Theorem 3.4), and is a conceptual precursor to Darboux' Theorem.

Proof: We reproduce the argument from [BH86, Section 4.1]. It is by now obvious that we may replace the integral by an integral on $[0,\epsilon]$, introducing only an exponentially small error. Writing

$$A(t) = \sum_{m=0}^{N} b_m t^{\beta_m} + R_N(t)$$

for $R_N = O(t^{\mathrm{Re}\{\beta_{m+1}\}})$ at 0, we may integrate term by term to get the first N terms of the expansion (up to an exponenetially small correction for truncating the integral on $[0,\epsilon]$), then use Lemma 5.2 to see that the remainder satisfies

$$\left|\int_0^\epsilon R_n(t)e^{-\lambda t}\,dt\right| = O\left(\lambda^{-\mathrm{Re}\{\beta_m\}-1}\right)\,,$$

proving the proposition. □

Our Morse thoeretic approach to the evaluation of integrals subsumes the classical **saddle point** or **steepest descent** method. Nevertheless, having summarized how one computes asymptotics near a point where $\phi' = 0$, it would be wasteful to leave the scene without a brief discussion of the method of steepest descents as it is elementarily understood. I will present this as a method, stating no theorems but giving instructions and an example.

Consider again the integral

$$\mathcal{I}(\lambda) = \int_\gamma A(z)\exp(-\lambda\phi(z))\,dz$$

only now suppose that $\phi'(0)$ does not vanish on γ. As we have seen before, $\lambda^{-1}\log\mathcal{I}(\lambda)$ has a limsup of at most $\nu := \sup_{z\in\gamma}\mathrm{Re}\,\{\phi(z)\}$. This is sharp when ϕ' vanishes somewhere on the support of A in γ but otherwise is not expected to be sharp. The saddle point method says to deform the contour so as to pass through a point x where ϕ' vanishes. From our Morse theoretic analyses, we know that this can always be done and solves the problem of minimizing ν. The saddle point method, as commonly understood, merely says to attempt to deform the contour so as to pass through such a point. The phrase "steepest descent" comes from the fact that the real part of ϕ must have a local maximum on the contour at x, rather than a minimum or inflection point (if x is not a maximum then it cannot be the highest point on the contour, so we know from Morse theory that there is a higher critical point the contour must pass through). Having deformed the contour to pass through x in the right direction, one then applies Theorem 5.1 (with $\phi(0) \neq 0$ as in the remark following the theorem).

Example 5.7. Consider the univariate power series $f(z) = (1-z)^{-1/2}$. By the binomial theorem, this generates the numbers $a_n := (-1)^n\binom{-1/2}{n} \sim \sqrt{1/(\pi n)}$. Let us instead evaluate this via a contour integral. Since we understand meromorphic integrands the best, we change variables to $z = 1 - y^2$ with $dz = -2\,y\,dy$. Then, letting C be a small circle around the origin, oriented countercolockwise,

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_C z^{-n-1}(1-z)^{-1/2}\,dz \\ &= \frac{1}{2\pi i}\int_E (1-y^2)^{-n-1}y^{-1}\,(-2y)\,dy \\ &= \frac{i}{\pi}\int_E (1-y^2)^{-n-1}\,dy\,. \end{aligned}$$

As shown in figure 5.5, E is a small circle in the y-plane, oriented counterclockwise, around either the point $+1$ or the point -1, since either of these contours maps to a small contour around 0 in the z-plane. Let us take a circle around $+1$. In the y-plane, there is a critical point for $\phi(y) := -\log(1-y^2)$ at the origin. The contour E may be deformed to a contour $\tilde{E}$ passing through the origin in the downward direction. It is easy to see that any contour separating 1 and -1 must intersect the segment $(-1,1)$, so this deformation does indeed produce a minimax height contour. Changing variables to $y = -it$ gives $a_n = \frac{i}{\pi}\int (1+t^2)^{-n-1}\,(-i)\,dt$. Since $\phi(y) = \phi(-it) \sim -t^2$ near the origin, the integral then becomes asymptotically $\sqrt{2/(\pi n\phi''(0))} = \sqrt{1/(\pi n)}$.

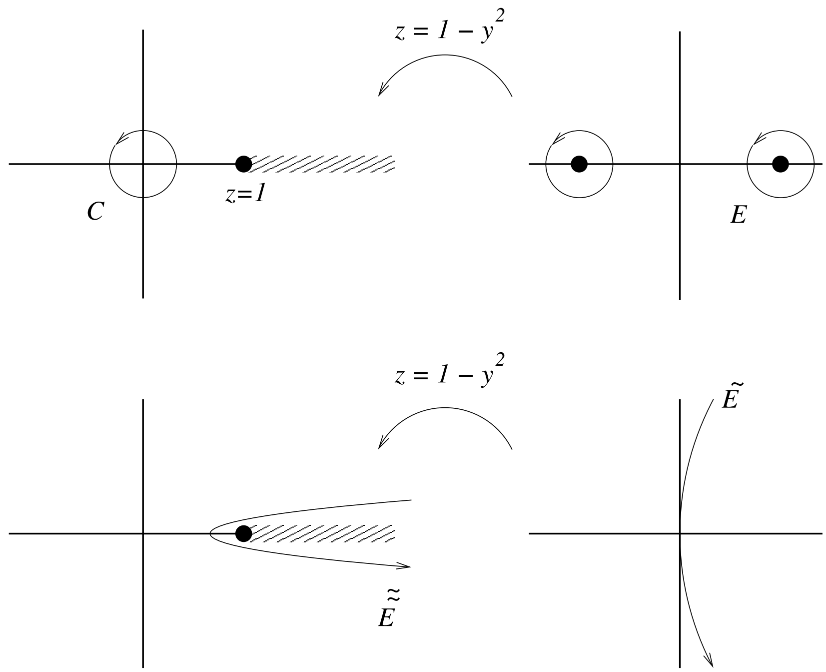

Figure 5.5: above: the contours C and E; below: $\tilde{E}$ and $\tilde{\tilde{E}}$

Remark 5.8. The function $(1-z)^{-1/2}$ is analytic on the slit plane $\mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 1\}$, which we may view as half of the Riemann surface $\mathfrak{R}$ obtained by gluing two copies of the slit plane, with the upper half of each attaching along the slit to the lower half of the other. The change of variables $z = 1 - y^2$ is the map from $\mathfrak{R}$ to $\mathbb{C}$. The saddle point contour $\tilde{E}$, when mapped to the z-plane, comes in to $+1$ along one copy of the slit, does a U-turn, and goes back along the other copy of the slit. Perturbing slightly gives a hairpin-shaped contour $\tilde{\tilde{E}}$ that may be drawn in the slit plane. This explains the origin of the Camembert contour in figure 3.1.

Problem. Use multivariate versions of this contour to extend results about meromorphic generating functions to functions with algebraic singularities.

5.4 Analytic versus C^∞ category

The result of the previous sections hold when the amplitude and phase are assumed only to be smooth, not necessarily analytic. This indicates there should be arguments that use smooth techniques, such as partitions of unity and integration by parts, rather than contour deformation. Such an approach to evaluating stationary phase integrals has been developed and used extensively by harmonic analysts, who are chiefly interested in the case where ϕ is purely imaginary. Note that this is not covered by the assumptions of Theorem 5.1: the contour decomposition requires that $\operatorname{Re}\{\phi\}$ be strictly positive away from zero. My chief reason for including this section is that any treatment of Fourier-Laplace integrals that bypasses smooth methods is pedagogically and historically incomplete. Results in this section will be used only once or twice in the analysis of generating functions.

When the exponent $i\phi(z)$ is imaginary, the modulus of the integrand is equal to $|A(z)|$, so it is no longer true that one may cut off the integral outside of an interval $[-\epsilon, \epsilon]$ and expect to introduce negligible remainders. Instead, one assumes that A has compact support, then uses smooth partitions of unity to reduce to integrals over small intervals. Note that neither partitions of unity nor compactly supported functions exist in the analytic category; however, when the contour of integration γ is a closed curve, any amplitude function has compact support on γ so both the analytic and the smooth methods apply and may be compared.

In this section we give asymptotics for

$$\mathcal{I}(\lambda) := \int_a^b A(z) \exp(i\lambda\phi(z))\, dz$$

where A and ϕ are smooth and A is supported on a compact sub-interval of (a, b); we use the term "phase" to denote ϕ rather than $i\phi$. The steps are similar to the steps of the proof of Theorem 5.5 except for the insertion of a localization step at the beginning and the introduction of a damping term in the step where both amplitude and phase are monomial.

- localization
- big-O estimate
- monomials (with damping)
- monomial phase
- full theorem

Localization lemma in C^∞

Following [Ste93] we begin with a localization principle.

Lemma 5.9 (localization lemma). *Suppose $\phi'(x) \neq 0$ for all $x \in (a, b)$. Then $\mathcal{I}(\lambda)$ is rapidly decreasing, that is,*

$$\mathcal{I}(\lambda) = O(\lambda^{-N}) \qquad \text{as } \lambda \to \infty$$

for any $N \geq 0$.

Proof: The smooth vanishing of A at the endpoints allows us to integrate by parts without introducing boundary terms. Integrate by parts with $dU = i\lambda\phi' e^{i\lambda\phi}\, dx$ and $V = A/(i\lambda\phi')$ to get

$$\mathcal{I}(\lambda) = -\int_a^b e^{i\lambda\phi(x)} \frac{d}{dx}\left(\frac{A}{i\lambda\phi'}\right)(x)\, dx.$$

For any $N \geq 1$ we may repeat this N times to obtain

$$\mathcal{I}(\lambda) = \int_a^b e^{i\lambda\phi(x)}(-\lambda^{-N})\mathcal{D}^N(A)(x)\,dx \tag{5.12}$$

where $\mathcal{D}$ is the differential operator $f \mapsto (d/dx)(f/i\phi')$. Letting

$$K_N = (b-a)\sup_{a\leq x\leq b}|\mathcal{D}^n A(x)|\,, \tag{5.13}$$

we see that

$$|\mathcal{I}(\lambda)| \leq \lambda^{-N}K_N$$

which proves that $\mathcal{I}$ is a rapidly decreasing function of λ. □

Remarks.

1. Compare this to the argument in the analytic case. There, if ϕ' is nowhere vanishing, the contour can be "pushed down" along the gradient flow so that the maximum of $\mathrm{Re}\,\{i\phi\}$ is strictly negative, resulting in an integral that decreases exponentially in λ.
2. Since ϕ' does not vanish, we may change variables to $y = \phi(x)$ and the conclusion is equivalent to the more familiar statement that the Fourier transform of the smooth function $\tilde{A}$ is rapidly decreasing.
3. While the lemma is stated only for purely imaginary phase functions, the same argument in fact shows that $\mathcal{I}(\lambda)$ is rapidly decreasing whenever the real part of ϕ is nonnegative and ϕ' is nonvanishing.

We call Lemma 5.9 the localization lemma for the following reason. Suppose we allow ϕ' to vanish on some finite set of points $x_1, \ldots, x_d \in [a,b]$. Then the contribution to $\mathcal{I}(\lambda)$ from any closed region not containing some x_i is rapidly decreasing, so the asymptotics for $\mathcal{I}(\lambda)$ may be read off as the sum of contributions local to each x_i. Indeed, for each i let $[a_i, b_i]$ be tiny intervals containing x_i, with all intervals disjoint and let $\xi_1, \ldots, \xi_d$ be a partition of unity subordinate to $\{[a_i, b_i] : 1 \leq i \leq d\}$. Once we see how to obtain asymptotics in a neighborhood of x_i containing no other critical points, we can write $A = A_0 + \sum_{i=1}^d A\xi_i$, so that the support of A_0 contains no x_i. By the localization lemma, $\int e^{i\lambda\phi(x)}A_0(x)\,dx$ is rapidly decreasing. It follows that as long as the integrals $\mathcal{I}_i(\lambda) := \int_a^b e^{i\lambda\phi(x)}A_i(x)\,dx$ sum to something not rapidly decreasing, the asymptotic development of $\mathcal{I}(\lambda)$ is gotten by summing the developments of $\mathcal{I}_i(\lambda)$.

Our main result for one-variable purely oscillating integrals will be the asymptotic development of integrals defined as follows.

Theorem 5.10. *Let ϕ and A be smooth real functions with A having compact support in (a,b) whose closure contains zero. Let $k \geq 2, l \geq 0$ be integers and suppose the power series for A and ϕ are given by $\{b_j\}$ and $\{c_j\}$ as in Theorem 5.5, with $c_k > 0$. Suppose that ϕ' vanishes in $[a,b]$ at 0 but nowhere else. Let $\tilde{A} := (A \circ g) \cdot g'$ where g is the inverse function to $x \mapsto (\phi/c_k)^{1/k}$. Then as $\lambda \to \infty$ there is an asymptotic development*

$$\mathcal{I}(\lambda) := \int_a^b A(x)\exp(-i\lambda\phi(x))\,dx \sim \sum_{j=l}^{\infty} \alpha_j C(k,j)(i\,c_k\,\lambda)^{-(1+j)/k}\,.$$

The coefficients α_j are obtained from the power series coefficients $a_0, \ldots, a_j$ for $\tilde{A}$ exactly as in part (iii) of Theorem 5.1. The constant in the $O(\lambda^{-(N+1)/k})$ remainder term is bounded by a continuous function of the suprema of the first $N+1$ derivatives of ϕ and A on the support of A. The $1/k$ power of $i\,c_k\lambda$ is the principal value.

The big-O lemma in C^∞

The following smooth counterpart to Lemma 5.2 is proved by showing that the main contribution comes from an interval of size $\lambda^{-1/k}$. The increase in length over the very short proof of Lemma 5.2 is due to the need to keep track of a partition of unity function and its derivatives.

Lemma 5.11. *If η is smooth and compactly supported and $l \geq 1$ and $k \geq 2$ are integers, then*

$$\left| \int_{-\infty}^{\infty} e^{i\lambda x^k} x^l \eta(x)\, dx \right| \leq C\lambda^{-(l+1)/k} \tag{5.14}$$

for a constant C depending on k, l and the first l derivatives of η.

An immediate corollary is:

Corollary 5.12. *Let $\phi(x) = x^k$. If a smooth function g vanishes in an neighborhood of 0, and decreases rapidly at infinity (or is compactly supported) then $\mathcal{I}(\lambda) = \int e^{i\lambda x^k} g(x)\, dx$ is rapidly decreasing.* □

Proof of lemma: Let α be a nonnegative smooth function equal to its maximum of 1 on $|x| \leq 1$ and vanishing on $|x| \geq 2$. Choose an $\epsilon > 0$ and rewrite (5.14) as

$$\int e^{i\lambda x^k} x^l \eta(x)\alpha(x/\epsilon)\, dx + \int e^{i\lambda x^k} x^l \eta(x)[1 - \alpha(x/\epsilon)]\, dx. \tag{5.15}$$

The absolute value of the first integrand is at most $|x|^l \cdot (\sup_{|x|\leq 2} |\eta(x)|) \cdot \mathbf{1}_{|x|\leq 2\epsilon}$, which yields an integral of at most $C_1\epsilon^{l+1}$ where $C_1 = \dfrac{2^{l+1}}{l+1} \sup_{|x|\leq 2} |\eta(x)|$.

The second integral will be done by parts, and to prepare for this we examine the iteration of the operator $D := (d/dx)(\cdot/x^{k-1})$ applied to the function $x^l\eta(x)(1 - \alpha(x/\epsilon))$. The result will be a sum of monomials, each monomial being a product of a power of x, a derivative of η, a derivative of α and a power of ϵ. In fact if (a, b, c, d) is shorthand for $x^a \eta^{(b)}(x)\alpha^{(c)}(x/\epsilon)\epsilon^d$, and $a \geq 0$, then

$$D(a, b, c, d) = (a - k + 1)(a - k, b, c, d) + (a - k + 1, b + 1, c, d) + (a - k + 1, b, c + 1, d - 1).$$

By induction, we see that $D^N(a, b, c, d)$ is the sum of terms $C \cdot (r, s, t, u)$ with $r + u \geq a + d - kN$, $s \leq b + N$, $t \leq c + N$, and C is bounded above by the factorial $\max\{kN, a\}!$. In particular, since $\epsilon \leq x$ we may replace positive powers of ϵ by the same power of x to arrive at the upper bound:

$$\left| D^N \left[x^l \eta(x)(1 - \alpha(x/\epsilon)) \right] \right| \leq \mathbf{1}_{|x|\geq\epsilon} C|x|^{l-kN} \tag{5.16}$$

where C is the product of $\sup_{j\le N,|x|\in(1,2)} \eta^{(j)}(x)$ and $\sup_{j\le N,|x|\in(1,2)} \alpha^{(j)}(x)$.

Now we fix an $N \ge 1$ and integrate the second integrand of (5.15) by parts N times, each time integrating $-ik\lambda x^{k-1}e^{i\lambda x^k}$ and differentiating the rest. The resulting integral is

$$\int e^{i\lambda x^k}(-ik\lambda)^{-N} D^N \left[x^l \eta(x)(1-\alpha(x/\epsilon))\right] dx.$$

By (5.16), the modulus of the integrand is at most $C\mathbf{1}_{|x|\ge\epsilon}|x|^{l-kN}(k\lambda)^{-N}$, which integrates to at most $C_2\lambda^{-N}\epsilon^{l-kN+1}$. Set $\epsilon = \lambda^{-1/k}$ and add the bounds on the two integrals to obtain an upper bound on $\int e^{i\lambda x^k} g(x)\,dx$ of $(C_1 + C_2)\lambda^{-(l+1)/k}$. We have also shown that C_1 and C_2 depend only on k, l, the first l derivatives of η and the first l derivatives of α. Thus, taking α to be a fixed, convenient function, the lemma is proved. □

A and ϕ are monomials and A is damped

In this section we prove the C^∞ version of Lemma 5.3. However, since a monomial amplitude function does not have compact support, we introduce a damping function which will later need to be removed. Define

$$\mathcal{I}(\lambda, k, l, \delta) := \int_{-\infty}^{\infty} e^{i\lambda x^k} e^{-\delta|x|^k} x^l \, dx \,. \tag{5.17}$$

Lemma 5.13. *As $\lambda \to \infty$, there is an asymptotic development*

$$\mathcal{I}(\lambda, k, l, \delta) \sim \lambda^{-(l+1)/k} \sum_{j=0}^{\infty} C(j, k, l, \delta)\lambda^{-j} \,.$$

The constants in the N^{th} remainder term remain bounded (in fact go to 0) as $\delta \to 0$.

Proof: Let $z = (\delta - i\lambda)^{1/k} x$, where we choose the principal branch of the $1/k$ power. The half-line integral, which we will denote $\mathcal{I}_+(\lambda, k, l, \delta)$, may be written as

$$\int_0^{\infty(\delta-i\lambda)^{1/k}} e^{-z^k}(\delta - i\lambda)^{-l/k} z^l \frac{dz}{(\delta - i\lambda)^{1/k}} \,.$$

Now we may rotate the contour back to the real line. Specifically, for fixed λ, as $M \to \infty$, the difference between the above integral taken from 0 to $M(\delta - i\lambda)^{1/k}$ and the integral along the positive real line segment $[0, M|\delta - i\lambda|^{1/k}]$ is the integral of an exponentially small function of M along an arc of length $O(M)$; the difference therefore goes to zero and we obtain

$$\mathcal{I}_+(\lambda, k, l, \delta) = (\delta - i\lambda)^{-(l+1)/k} \int_0^\infty e^{-x^k} x^l \, dx \,,$$

where the $(l+1)$ power of the principal $1/k$ power is used. The definite integral has value $C(k, l)$. Writing $(\delta - i\lambda)^{-(l+1)/k}$ as $(-i\lambda)^{-(l+1)/k}(1 + \delta i/\lambda)^{-(l+1)/k}$ and using the binomial theorem gives

$$\mathcal{I}_+(\lambda, k, l, \delta) = C(k, l)e^{i\pi(l+1)/(2k)}\lambda^{-(l+1)/k} \sum_{j=0}^{\infty} (i\delta)^j \binom{-(l+1)/k}{j} \lambda^{-j}.$$

Denote

$$C_+(j,k,l,\delta) = k^{-1}\Gamma\left(\frac{l+1}{k}\right) e^{i\pi(l+1+jk)/(2k)} \binom{-(l+1)/k}{j} \delta^j \tag{5.18}$$

and add the analagous computation for $\mathcal{I}_-(j,k,l,\delta)$ to prove the lemma with

$$C(j,k,l,\delta) := C_+(j,k,l,\delta) + C_-(j,k,l,\delta)\,,$$

the remainder terms evidently going to zero as $\delta \to 0$. □

ϕ is a monomial, A is anything

Theorem 5.14. *Let $\phi(x) = x^k$. Let A be smooth and compactly supported with 0 in the closed support and let $\{b_j\}$ be the power series coefficients at 0, with l denoting the index of the first nonvanishing term b_l. Then $\mathcal{I}_+ := \int_0^\infty A(x)\exp(i\lambda\phi(x))\,dx$ has asymptotic development*

$$\mathcal{I}_+ \sim \sum_{j=l}^{\infty} b_j C(k,j)(i\lambda)^{-(1+j)/k}\,.$$

The constant in the $O(\lambda^{-N/k})$ remainder term is bounded in terms of the suprema of the first N derivatives of A near 0. A similar result holds for the two-sided integral, $\mathcal{I}$, with coefficients α_j obtained by plugging b_j in for a_j in the conclusion to Theorem 5.1.

Proof: Let U be a smooth function that is 1 on the support of A and vanishes outside of a compact set. Fix $N \geq 1$ and $\delta > 0$ and define the polynomial $P(x) = P_{N,\delta}(x)$ to be the sum of the Taylor series for $e^{\delta x^k} A(x)$ through the x^N term. Let $b_{j,\delta}$ denote the Taylor coefficients of $P_{N,\delta}$. Define the normalized remainder term $R(x) = R_{N,\delta}(x)$ by $e^{\delta x^k} A(x) = P(x) + x^{N+1} R(x)$. Now represent I_+ as $B_1 + B_2 + B_3$ where

$$\begin{aligned} B_1 &:= \int_0^\infty e^{i\lambda x^k} e^{-\delta x^k} P(x)\,dx\,; \\ B_2 &:= \int_0^\infty e^{i\lambda x^k} x^{N+1} e^{-\delta x^k} R(x) U(x)\,dx\,; \\ B_3 &:= \int_0^\infty e^{i\lambda x^k} e^{-\delta x^k} P(x)(U(x)-1)\,dx\,. \end{aligned}$$

By Lemma 5.11 with $\eta(x) = e^{-\delta x^k} R(x) U(x)$ and $l = N+1$, we know that the magnitude of B_2 is bounded by $K\lambda^{-(l+2)/k}$. Similarly, by Corollary 5.12, we see that B_3 is rapidly decreasing as $\lambda \to \infty$. Furthermore, in both cases K may be bounded in terms of k, l and the first l derivatives of A, the bound being uniform over δ in a neighborhood of 0. It follows that the asymptotic series for $\mathcal{I}_+$, up to the $\lambda^{-(l+1)/k}$ term, may be obtained by taking $\delta \to 0$ in B_1.

Since P is a finite sum of monomials, we may use Lemma 5.13 to compute B_1, which recovers the hypothesis of the theorem. As before, we may sum results for $\mathcal{I}_+$ and $\mathcal{I}_-$ to prove the result for $\mathcal{I}$. □

General A and ϕ

Since $i\phi$ always lies along the imaginary axis, we may use a diffeomorphic change of variables to change ϕ into $ic_k x^k$, under which the contour remains along the imaginary axis (thus there is no need for arguments about moving the contour).

PROOF OF THEOREM 5.10: By assumption, $\phi(x) = c_k x^k(1 + \theta(x))$ where $\theta(x) = O(|x|)$. Let $y = x(1 + \theta(x))^{1/k}$. This is a diffeomorphism in a neighborhood of 0, and we write $x = g(y)$ to denote its inverse. Then $c_k y^k = \phi(x)$ and so we may change variables to write

$$\int e^{i\lambda\phi(x)} A(x)\, dx = \int e^{i\lambda c_k y^k} \tilde{A}(y)\, dy\, .$$

The result now follows from Lemma 5.3. □

Notes

My chief sources for Sections 1 and 2 were [BH86] and [Won89]; see also [Hen91], which was used heavily in Chapter 3. Although it certainly follows from the extensive analyses in, e.g., [BH86, Chapter 7], I have seldom seen Fourier and Laplace type integrals treated together, have never seen the statement of Theorem 5.1 in exactly this form and never seen a derivation by purely complex analytic methods.

Watson's Lemma may be found in many places. The version here agrees with the statements in [BH86, Section 4.1] and [Hen91, Section 11.5]. The method of steepest descent is described very nicely in [dB81]. My treatment is more akin to [Hen91, Section 11.8]; see also [BH86, Chapter 7], especially for a couple of the exercises.

Section 5.4 borrows heavily from [Ste93]. I have attempted to fill in some details. For instance, my proof of Lemma 5.11 is summarized as "A simple computation shows..." in [Ste93, bottom of page 335], which also omits details as to how the argument for $k = 2$ extends to greater values of k (see, e.g., [Ste93, VIII.1.3.3]). Despite its omission of detail in elementary arguments, Stein's book is a beautifully written, modern classic and is a recommended addition to anyone's bookshelf.

The use of complex analytic methods to prove Lemma 5.13 also follows [Ste93] (see 1.3.3 and step 1 of 1.3.1 in Chapter VIII).

Exercises

Exercise 5.1 (next term of the expansion). Let $k = 2$ and $l = 0$ in Theorem 5.1. The theorem then gives

$$\mathcal{I} = b_0\sqrt{\frac{\pi}{c_2}}\lambda^{-1/2} + a_2\lambda^{-3/2} + O(\lambda^{-5/2})\, .$$

Compute the coefficient a_2 in terms of b_0, b_1, b_2, c_2, c_3 and c_4.

Exercise 5.2 (Bessel function). The Bessel function is defined by

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ir \sin\theta - im\theta)\, d\theta$$

where m is a fixed parameter (you may assume it is a positive integer). Use Theorem 5.10 to find the two leading terms of an asymptotic series for $J_m(r)$ in decreasing powers of r.

Exercise 5.3 (Airy function). The Airy function is defined by

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xt+t^3/3)}\, dt\,.$$

Find an asymptotic expression for $\mathrm{Ai}\,(x)$ as $x \to \infty$ in $\mathbb{R}^+$. Step 1: change variables by $t = ix^{1/2}u$. Step 2: find the critical points and deform the contour to pass through one or more of them. Step 3: compute the expansion on a compactly supported interval, then argue that this converges as the limits of integration go to infinity.