Chapter 6

Fourier-Laplace integrals in more than one variable

In this chapter we generalize the work in the previous chapter on the asymptotic evaluation of the saddle point integral $\mathcal{I}(\lambda)$ to the case of more than one variable. In one variable, the comprehensive result Theorem 5.1 covers all degrees of degeneracy of the phase function (the parameter k) and all degrees of vanishing of the amplitude function (the parameter l). The range of possibilities for the phase function ϕ in higher dimensions is much greater. We will be concerned only with the quadratic case. In one dimension, this boils down to taking $k = 2$. In higher dimensions, we assume nonsingularity of the **Hessian matrix** $\mathcal{H} := \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x_j \, \partial x_k}\right)$. The Taylor series for any function ϕ is

$$\phi(\mathbf{x}) = \phi(\mathbf{0}) + \mathbf{x}^T \nabla \phi(\mathbf{0}) + \frac{1}{2}\mathbf{x}^T \mathcal{H} \mathbf{x} + O(|\mathbf{x}|^3) \, ,$$

hence the Hessian matrix represents (twice) the quadratic term in the phase and its nonsingularity is a generalization of nonvanishing of the quadratic term in the univariate case.

The initial part of the development is the same as in the univariate case. Let $S(\mathbf{x}) := x_1^2 + \cdots + x_d^2$ denote the standard quadratic, generalizing the special phase function x^2 in the univariate case. A result when A is monomial and ϕ is the standard quadratic (Corollary 6.5 below) is coupled with a big-O result (Proposition 6.6 below) allowing us to integrate term by term and obtain asymptotics for the standard phase function:

Theorem 6.1 (standard phase). *Let $A(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}}$ be any real analytic function defined on a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^d$. Let*

$$\mathcal{I}(\lambda) := \int_{\mathcal{N}} A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} \, d\mathbf{x} \, . \qquad (6.1)$$

Then

$$\mathcal{I}(\lambda) \sim \sum_n \sum_{|\mathbf{r}|=n} a_{\mathbf{r}} \beta_{\mathbf{r}} \lambda^{-(|\mathbf{r}|+d)/2}$$

as an asymptotic series expansion in increasing $|\mathbf{r}|$*, where* $\beta_{\mathbf{r}} = 0$ *if any* r_j *is odd, and*

$$\beta_{2\mathbf{m}} = (2\pi)^{-d/2} \prod_{j=1}^{d} \frac{(2m_j)!}{m_j! 4^{m_j}}$$

otherwise.

We then extend to a the following result concerning a general complex phase with the assumption that the real part has a strict minimum at the origin. Arguing again by contour deformation, we will prove:

Theorem 6.2 ($\mathrm{Re}\{\phi\}$ **has a strict minimum).** *Suppose that the real part of* ϕ *is strictly positive except at the origin and that its Hessian matrix* $\mathcal{H}$ *is nonsingular there. Let* A *be any analytic function not vanishing at the origin and define* $\mathcal{I}(\lambda)$ *by* (6.1)*. Then*

$$\mathcal{I}(\lambda) \sim \sum_{\ell \geq 0} c_\ell \lambda^{(-d-\ell)/2}$$

where

$$c_0 = A(\mathbf{0}) \frac{(2\pi\lambda)^{-d/2}}{\sqrt{\det \mathcal{H}}}$$

and the choice of sign is defined by taking the product of the principal square roots of the eigenvalues of $\mathcal{H}$*.*

The last set of results in the multivariate case departs from the framework of the univariate case. All the results in the univariate case assumed a strict minimum of the real part of the phase function, until the very end, when we proved results for strictly imaginary phase functions. These last results were proved quite differently, using C^∞ methods which allowed the introduction of bump functions. Having two completely different proofs for the two cases was not of great concern because the univariate case has a dichotomy: an analytic function on $\mathbb{R}$ whose real part has a minimum at zero either has a strict minimum there or has real part vanishing everywhere. In the multivariate case this is no longer true. Furthermore, we will need to integrate over regions such as rectangles and annuli, on which the real part of the phase is nonvanishing clear to the boundary. In one dimension this would lead to boundary terms large enough to appear in the asymptotics. In more than one variable, the boundary has positive dimension and boundary contributions are avoided when the phase is not stationary along the boundary.

There is some overhead even in stating these results. First, one must define appropriate chains of integration; these will be chains supported on Whitney stratified spaces. Next, to find the necessary deformations of these chains, one must use vector field constructions relying on semi-continuity notions for stratified spaces; these constructions are staples of stratified Morse theory. In the end, we prove the following result, which is more precisely stated as Theorem 6.16 below.

Theorem (critical point decomposition for stratified spaces). *Let A and ϕ be analytic functions on a neighborhood of a stratified space $\mathcal{M} \subseteq \mathbb{C}^d$. If ϕ has finitely many critical points on $\mathcal{M}$ then*

$$\mathcal{I}(\lambda) \sim (2\pi\lambda)^{-d/2} \sum_{\mathbf{x}} A(\mathbf{x}) e^{\lambda\phi(\mathbf{x})} \det(\mathcal{H}(\mathbf{x}))^{-1/2}$$

where $\mathcal{H}(\mathbf{x})$ is the Hessian matrix for ϕ at $\mathbf{x}$ and the sum is over critical points $\mathbf{x}$ at which the real part of ϕ is minimized.

6.1 Standard phase

As in the one-dimensional case, we begin with the simplest phase function and a monomial amplitude. We first state an explicit formula for the one-dimensional monomial integral in the case $k = 2$ and $l = 2n$.

Proposition 6.3.

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx = \beta_{2n} := \frac{1}{\sqrt{2\pi}} \frac{(2n)!}{n!\, 4^n}\,.$$

PROOF: For $n = 0$ this is just the standard Gaussian integral. By induction, assume now the result for $n - 1$. Integrate by parts to get

$$\begin{aligned} \int x^{2n} e^{-x^2}\, dx &= \int \frac{-x^{2n-1}}{2} \left(-2x\, e^{-x^2}\, dx\right) \\ &= \frac{2n-1}{2} \int x^{2n-2}\, e^{-x^2}\, dx \\ &= \frac{2n-1}{2} \frac{1}{\sqrt{2\pi}} \frac{(2n-2)!}{(n-1)!\, 4^{n-1}} \end{aligned}$$

by the induction hypothesis. This is equal to $(2\pi)^{-1/2}(2n)!\,/\,(n!\,4^n)$, completing the induction. □

Corollary 6.4.

$$\int_{-\infty}^{\infty} x^{2n}\, e^{-\lambda x^2}\, dx = \beta_{2n} \lambda^{-1/2-n}\,.$$

PROOF: Changing variables by $y = \lambda^{1/2} x$ yields

$$\int_{-\infty}^{\infty} \lambda^{-n} y^{2n} e^{-y^2} \frac{dy}{\lambda^{1/2}}\,.$$

□

Let $S(\mathbf{x}) := \sum_{j=1}^d x_j^2$ denote the standard quadratic.

Corollary 6.5 (monomial integral). *Let $\mathbf{r}$ be any d-vector of nonnegative integers. Then*

$$\int_{\mathbb{R}^d} \mathbf{x}^{\mathbf{r}} e^{-\lambda S(\mathbf{x})}\, d\mathbf{x} = \beta_{\mathbf{r}} \lambda^{-(d+|\mathbf{r}|)/2}$$

where $\beta_{\mathbf{r}} = \prod_{j=1}^d \beta_{r_j}$ if all the components r_j are even, and zero otherwise.

Proof: The integral factors into

$$\prod_{j=1}^{d} \left[\int_{-\infty}^{\infty} x_j^{r_j} \, e^{-\lambda r_j^2} \, dx_j \right] ,$$

reducing this to the result of Proposition 6.3. □

Proposition 6.6 (big-O estimate). *Let A be any smooth function satisfying $A(\mathbf{x}) = O(|\mathbf{x}|^r)$ at the origin. Then the integral of $A(\mathbf{x})e^{-\lambda S(\mathbf{x})}$ over any compact set K may be bounded from above by*

$$\int_K A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} \, d\mathbf{x} = O(\lambda^{-(d+r)/2})$$

Proof: Because K is compact and $A(\mathbf{x}) = O(|\mathbf{x}|^r)$ at the origin, it follows that there is some constant c for which $|A(\mathbf{x})| \leq c|\mathbf{x}|^r$ on all of K. Let K_0 denote the intersection of K with the ball $|\mathbf{x}| \leq \lambda^{-1/2}$ and for $n \geq 1$ let K_n denote the intersection of K with the shell $2^{n-1}\lambda^{-1/2} \leq |\mathbf{x}| \leq 2^n \lambda^{-1/2}$. On K_0 we have

$$|A(\mathbf{x})| \leq c\lambda^{-r/2}$$

while trivially

$$\int_{K_0} e^{-\lambda S(\mathbf{x})} \, d\mathbf{x} \leq \int_{K_0} d\mathbf{x} \leq c_d \lambda^{-d/2} .$$

Thus

$$\left| \int_{K_0} A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} \, d\mathbf{x} \right| \leq c' \lambda^{-(r+d)/2} .$$

For $n \geq 1$, on $K \cap K_n$, we have the upper bounds

$$\begin{aligned} |A(\mathbf{x})| &\leq 2^{rn} & c\lambda^{-r/2} \\ e^{-\lambda S(\mathbf{x})} &\leq & e^{-2^{n-1}} \\ \int_{K_n} d\mathbf{x} &\leq & 2^{rn} \, c_d \, \lambda^{-d/2} . \end{aligned}$$

Letting $c'' := c \cdot c_d \cdot \sum_{n=1}^{\infty} 2^{2rn} \, e^{-2^{n-1}} < \infty$, we may sum to find that

$$\sum_{n=0}^{\infty} \left| \int_{K_n} A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} \, d\mathbf{x} \right| \leq (c' + c'') \lambda^{-(r+d)/2} ,$$

proving the lemma. □

Proof of Theorem 6.1: Write $A(\mathbf{x})$ as a power series up to degree n plus a remainder term:

$$A(\mathbf{x}) = \left(\sum_{n=0}^{N} \sum_{|\mathbf{r}|=n} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} \right) + R(\mathbf{x})$$

where $R(\mathbf{x}) = O(|\mathbf{x}|^{N+1})$. Using Corollary 6.5 to integrate all the monomial terms and Proposition 6.6 to bound the integral of $R(\mathbf{x})e^{-\lambda S(\mathbf{x})}$ shows that

$$I(\lambda) = \sum_{n=0}^{N} \sum_{|\mathbf{r}|=n} a_{\mathbf{r}} \beta_{\mathbf{r}} \lambda^{-(n+d)/2} + O(\lambda^{-(n+1+d)/2})$$

which proves the asymptotic expansion. □

6.2 Real part of phase has a strict minimum

Let $\mathcal{N}$ be a neighborhood of the origin in $\mathbb{R}^d$. We say that the function $\phi : \mathcal{N} \to \mathbb{C}$ is analytic if ϕ is represented by a power series that converges on $\mathcal{N}$. Such a function may be extended to a holomorphic function on neighborhood $\mathcal{N}_{\mathbb{C}}$ of the origin in $\mathbb{C}^d$. Suppose $\phi(\mathbf{0}) = 0$ and the real part of ϕ is nonnegative on $\mathcal{N}$. The gradient of ϕ must vanish at the origin. We say that ϕ has a (quadratically) nondegenerate critical point at the origin if the quadratic part of ϕ is nondegenerate. Recall that the quadratic part of ϕ is a quadratic form represented by one half the Hessian matrix. By nondegeneracy of a quadratic form, we mean nonsingularity of the Hessian; by determinant of a quadratic form, we mean the determinant of half of the Hessian.

We review how the Hessian behaves under changes of variables. If $\psi : \mathbb{C}^d \to \mathbb{C}^d$ is a bi-holomorphic map, and if ϕ has vanishing gradient at $\psi(\mathbf{x})$ and Hessian matrix $\mathcal{H}$ there, then the Hessian matrix $\tilde{\mathcal{H}}$ of $\phi \circ \psi$ at $\mathbf{x}$ is given by

$$\tilde{\mathcal{H}} = J_\psi^T \mathcal{H} J_\psi \tag{6.2}$$

where J_ψ is the Jacobian matrix of the map ψ at $\mathbf{x}$.

The first key lemma is that, under the assumptoin of nondegeneracy of the Hessian, we can change variables so that ϕ becomes the standard quadratic form.

Lemma 6.7. *There is a bi-holomorphic change of variables* $\mathbf{x} = \psi(\mathbf{y})$ *such that* $\phi(\psi(\mathbf{y})) = S(\mathbf{y}) := \sum_{j=1}^d y_j^2$. *The differential* $J_\psi = d\psi(\mathbf{0})$ *will satisfy* $(\det J_\psi)^2 = (\det Q)^{-1}$.

Remark. This is known as the Morse lemma. The proof here is adapted from the proof of the real version given in [Ste93, VIII:2.3.2].

PROOF: Taking the last conclusion first, use (6.2) to see that the Hessian of the standard form S is equal to $J_\psi^T \mathcal{H} J_\psi$, where $\mathcal{H}$ is the Hessian matrix of ϕ. The Hessian of S is twice the identity matrix, so dividing by 2 and taking determinants gives $|J_\psi|^2 |Q| = 1$.

To prove the change of variables, the first step is to write

$$\phi(\mathbf{x}) = \sum_{j,k=1}^{d} x_j x_k \phi_{j,k}$$

where the functions $\phi_{j,k} = \phi_{k,j}$ are analytic and satisfy $\phi_{j,k}(\mathbf{0}) = (1/2)\mathcal{H}_{j,k}$. It is obvious from a formal power series viewpoint that this can be done because the summand $x_j x_k \phi_{j,k}$ can be any power series with coefficients indexed by the orthant $\{\mathbf{r} : \mathbf{r} \geq \delta_j + \delta_k\}$; these orthants cover $\{\mathbf{r} : |\mathbf{r}| \geq 2\}$, so we may obtain any function ϕ vanishing to order two; matching coeeficients on the terms of order precisely two shows that $\phi_{j,k}(\mathbf{0}) = (1/2)\mathcal{H}_{j,k}$.

More constructively, we may give a formula for $\phi_{j,k}$. There is plenty of freedom, but a convenient choice is to take

$$x_k x_k \phi_{j,k}(\mathbf{x}) := \sum_{|\mathbf{r}| \geq 2} \frac{r_j(r_k - \delta_{j,k})}{|\mathbf{r}|(|\mathbf{r}| - 1)} \, a_\mathbf{r} \mathbf{x}^\mathbf{r} \, .$$

For fixed $\mathbf{r}$, it is easy to check that

$$\sum_{1 \leq j,k \leq d} \frac{r_j(r_k - \delta_{j,k})}{|\mathbf{r}|(|\mathbf{r}| - 1)} = 1$$

whence $\phi = \sum x_j x_k \phi_{j,k}$. Alternatively, the following analytic computation from [Ste93] verifies that $\phi = \sum_{j,k} x_j x_k \phi_{j,k}$. Any function f vanishing at zero satisfies $f(t) = \int_0^1 (1-s) f'(s) \, ds$, as may be seen by integrating by parts (take $g(s) = -(1-s)$). Fix $\mathbf{x}$ and apply this with $f(t) = (d/dt)\phi(t\mathbf{x})$ to obtain

$$\phi(\mathbf{x}) = \int_0^1 \frac{d}{dt} \phi(t\mathbf{x}) \, dt = \int_0^1 (1-t) \frac{d^2}{dt^2} \phi(t\mathbf{x}) \, dt \, .$$

The multivariate chain rule gives

$$\frac{d^2}{dt^2} \phi(t\mathbf{x}) = \sum_{j,k} x_j x_k \frac{\partial^2 \phi}{\partial x_j \partial x_k}(t\mathbf{x}) \, ;$$

plug in $\phi = \sum_\mathbf{r} a_\mathbf{r} \mathbf{x}^\mathbf{r}$ and integrate term by term using $\int_0^1 (1-t) t^{n-2} \, dt = \frac{1}{n(n-1)}$ to see that $\phi = \sum_{j,k} x_j x_k \phi_{j,k}$.

The second step is an induction. Suppose first that $\phi_{j,j}(\mathbf{0}) \neq 0$ for all j. The function $\phi_{1,1}^{-1}$ and a branch of the function $\phi_{1,1}^{1/2}$ are analytic in a neighborhood of the origin. Set

$$y_1 := \phi_{1,1}^{1/2} \left[x_1 + \sum_{k > 1} \frac{y_k \phi_{1,k}}{\phi_{1,1}} \right] \, .$$

Expanding, we find that the terms of y_1^2 of total degree at most one in the terms $x_2, \ldots, x_d$ match those of ϕ and therefore,

$$\phi(\mathbf{x}) = y_1^2 + \sum_{j,k \geq 2} x_j x_k h_{j,k} \tag{6.3}$$

for some analytic functions $h_{j,k}$ satisfying $h_{j,k}(\mathbf{0}) = (1/2)\mathcal{H}_{j,k}$. Similarly, if

$$\phi(\mathbf{x}) = \sum_{j=1}^{r-1} y_j^2 + \sum_{j,k \geq r} x_j x_k h_{j,k}$$

then setting

$$y_r := \phi_{r,r}^{1/2} \left[x_r + \sum_{k>r} \frac{y_k \phi_{r,k}}{\phi_{r,r}} \right]$$

gives

$$\phi(\mathbf{x}) = \sum_{j=1}^{r} y_j^2 + \sum_{j,k \geq r+1} x_j x_k \tilde{h}_{j,k}$$

for some analytic functions $\tilde{h}_{j,k}$ still satisfying $h_{j,k}(\mathbf{0}) = (1/2)\mathcal{H}_{j,k}$. By induction, we arrive at $\phi(\mathbf{x}) = \sum_{j=1}^{d} y_j^2$, finishing the proof of the Morse Lemma in the case where each $\mathcal{H}_{j,j}$ is nonzero.

Finally, if some $\mathcal{H}_{j,k} = 0$, because $\mathcal{H}$ is nonsingular we may always find some unitary map U such that the Hessian $U^T \mathcal{H} U$ of $\phi \circ U$ has no vanishing diagonal entries. We know there is a ψ_0 such that $(\phi \circ U) \circ \psi_0 = S$, and taking $\psi = U \circ \psi_0$ finishes the proof in this case. □

Proof of Theorem 6.2: The power series allows us to extend ϕ to a neighborhood of the origin in $\mathbb{C}^d$. Under the change of variables ψ from the previous lemma, we see that

$$\begin{aligned} \mathcal{I}(\lambda) &= \int_{\psi^{-1}\mathcal{C}} A \circ \psi(\mathbf{y}) e^{-\lambda S(\mathbf{y})} (\det d\psi(\mathbf{y}))\, d\mathbf{y} \\ &:= \int_{\psi^{-1}\mathcal{C}} \tilde{A}(\mathbf{y}) e^{-\lambda S(\mathbf{y})}\, d\mathbf{y} \end{aligned}$$

where $\mathcal{C}$ is a neighborhood of the origin in $\mathbb{R}^n$ with the standard orientation. We need to check that we can move the chain $\psi^{-1}\mathcal{C}$ of integration back to the real plane.

Let $h(\mathbf{z}) := \mathrm{Re}\{S(\mathbf{z})\}$. The chain $\mathcal{C}' := \psi^{-1}(\mathcal{C})$ lies in the region $\{\mathbf{z} \in \mathbb{C}^d : h(\mathbf{z}) > 0\}$ except when $\mathbf{z} = 0$, and in particular, $h \geq \epsilon > 0$ on $\partial\mathcal{C}'$. Let

$$H(\mathbf{z}, t) := \mathrm{Re}\{\mathbf{z}\} + (1-t)\, i\, \mathrm{Im}\{\mathbf{z}\}.$$

In other words, H is a homotopy from the identity map to the map π projecting out the imaginary part of the vector $\mathbf{z}$. For any chain σ, the homotopy H induces a chain homotopy, $H(\sigma)$ supported on the image of the support of σ under the homotopy H and satisfying

$$\partial H(\sigma) = \sigma - \pi\sigma + H(\partial\sigma).$$

With $\sigma = \mathcal{C}'$, observing that $S(H(\mathbf{z}, t)) \geq S(\mathbf{z})$, we see there is a $(d+1)$-chain $\mathcal{D}$ with

$$\partial\mathcal{D} = \mathcal{C}' - \pi\mathcal{C}' + \mathcal{C}''$$

and $\mathcal{C}''$ supported on $\{h > \epsilon\}$. Stokes Theorem tells us that for any holomorphic d-form ω,

$$\int_{\partial\mathcal{D}} \omega = \int_{\mathcal{D}} d\omega = 0$$

and consequently, that

$$\int_{\mathcal{C}'} \omega = \int_{\pi\mathcal{C}'} \omega + \int_{\mathcal{C}''} \omega.$$

When $\omega = \tilde{A} e^{-\lambda S} \, d\mathbf{y}$, the integral over $\mathcal{C}''$ is $O(e^{-\lambda\epsilon})$, giving

$$\mathcal{I}(\lambda) = \int_{\pi\mathcal{C}'} \tilde{A}(\mathbf{y}) e^{-\lambda S(\mathbf{y})} \, d\mathbf{y} + O(e^{-\epsilon\lambda}) \, .$$

Up to sign, the chain $\pi\mathcal{C}''$ is a disk in $\mathbb{R}^d$ with the standard orientation plus something supported in $\{h > \epsilon\}$. To see this, note that π maps any real d-manifold in $\mathbb{C}^d$ diffeomorphically to $\mathbb{R}^d$ wherever the tangent space is transverse to the imaginary subspace. The tangent space to the support of $\mathcal{C}'$ at the origin is transverse to the imaginary subspace because $S \geq 0$ on $\mathcal{C}'$, whereas the imaginary subspace is precisely the negative d-space of the index-d form S. The tangent space varies continuously, so in a neighborhood of the origin, π is a diffeomorphism. Observing that $\tilde{A}(\mathbf{0}) = A(\mathbf{0}) \det(d\psi(\mathbf{0})) = A(\mathbf{0})(\det \mathcal{H})^{-1/2}$, finishes the proof up to the choice of sign of the square root.

The map $d\pi \circ d\psi^{-1}(\mathbf{0})$ maps the standard basis of $\mathbb{R}^d$ to another basis for $\mathbb{R}^d$. Verifying the sign choice is equivalent to showing that this second basis is positively oriented if and only if $\det(d\psi(\mathbf{0}))$ is the product of the principal square roots of the eigenvalues of $\mathcal{H}$ (it must be either this or its negative). Thus we will be finished by applying the following lemma (with $\alpha = \psi^{-1}$).

Lemma 6.8. *Let $W \subseteq \mathbb{C}^d$ be the set $\{\mathbf{z} : \mathrm{Re}\,\{S(\mathbf{z})\} > 0\}$. Pick any $\alpha \in GL_d(\mathbb{C})$ mapping $\mathbb{R}^d$ into $\overline{W}$. and let $M := \alpha^T \alpha$ be the matrix representing $S \circ \alpha$. Let $\pi : \mathbb{C}^d \to \mathbb{R}^d$ be projection onto the real part. Then $\pi \circ \alpha$ is orientation preserving on $\mathbb{R}^d$ if and only if $\det \alpha$ is the product of the principal square roots of the eigenvalues of M (rather than the negative of this).*

Proof: First suppose $\alpha \in GL_d(\mathbb{R})$. Then M has positive eigenvalues, so the product of their principal square roots is positive. The map π is the identity on $\mathbb{R}^d$ so the statement boils down to saying that α preserves orientation if and only if it has positive determinant, which is true by definition. In the general case, let $\alpha_t := \pi_t \circ \alpha$, where $\pi_t(\mathbf{z}) = \mathrm{Re}\,\{\mathbf{z}\} + (1-t)\mathrm{Im}\,\{\mathbf{z}\}$. As we saw in the previous proof, $\pi_t(\mathbb{R}^d) \subseteq \overline{W}$ for all $0 \leq t \leq 1$, whence $M_t := \alpha_t^T \alpha_t$ has eigenvalues with nonnegative real parts. The product of the principal square roots of the eigenvalues is a continuous function on the set of nonsingular matrices with no negative real eigenvalues. The determinant of α_t is a continuous function of t, and we have seen it agrees with the product of principal square roots of eigenvalues of M_t when $t = 1$ (the real case), so by continuity, this is the correct sign choice for all $0 \leq t \leq 1$; taking $t = 0$ proves the lemma. □

6.3 Localization

Our aim is to integrate $A(\mathbf{x})e^{-\lambda\phi(\mathbf{x})}$ over a compact chain $\mathcal{C}$. Consider, as an example, the chain $C \times I$ where C is the unit circle in $\mathbb{C}^1$ and I is the interval $[-1, 1]$. As an example, suppose the phase function is given by

$$\phi(e^{i\theta}, \alpha) := \theta^2 + i\alpha \, .$$

The real part of ϕ is nonnegatives but vanishes along the entire line segments $\{0\} \times I$. However, there is only one critical point, namely $(0, 0)$, because the gradient of ϕ has nonvanishing imaginary part elsewhere. How can we see that the main contribution to $\mathcal{I}(\lambda)$ occurs near $(0, 0)$? In this example, foliating by circles, we may use Lemma 5.9 to see that the integral is small away from the median circle, $C \times \{0\}$.

The point of this section is to give a general argument localizing the integral for $\mathcal{I}(\lambda)$ to neighborhoods of critical points. First, we must extend the definition of a critical point to spaces more general than manifolds. We then show that, away from critical points, we may deform the chain of integration to where the real part of the phase is strictly positive. In the example, $\mathcal{C}$ will be deformed to a new chain where the real part of ϕ vanishes only at $(0, 0)$. However, showing that the integrals over the old and new chains agree is somewhat tricky. Recall from (4.2) that if $H : \mathcal{C} \times [0, 1]$ is a homotopy between the original chain $\mathcal{C}$ and a new chain $\mathcal{C}'$ then

$$\partial H = \mathcal{C}' - \mathcal{C} + \partial\mathcal{C} \times \sigma$$

where σ is the standard 1-simplex. This last term may cause a headache if $\mathcal{C}$ has a nontrivial boundary and the real part of the phase has minima on $\partial\mathcal{C}$.

The main result of this section is Theorem 6.16, localizing the integral to critical points in the manner just described. To do this, we need to define a suitable class of chains, then develop some geometric properties of these. We begin with the classical notion of stratified spaces.

Whitney stratifications

Many interesting spaces, such as algebraic varieties, are not manifolds. The next best thing is if a space is built nicely out of parts that are manifolds. Let I be a finite partially ordered set and define an I-decomposition of a topological space Z to be a partition of Z into a disjoint union of sets $\{S_\alpha : \alpha \in I\}$ such that

$$S_\alpha \cap \overline{S_\beta} \neq \emptyset \iff S_\alpha \subseteq \overline{S_\beta} \iff \alpha \leq \beta\,.$$

Definition 6.9 (Whitney stratification). *Let Z be a closed subset of a smooth manifold $\mathcal{M}$. A **Whitney stratification** of Z is an I-decomposition such that*

(i) Each S_α is a manifold in $\mathbb{R}^n$.

(ii) If $\alpha < \beta$, if the sequences $\{x_i \in S_\beta\}$ and $\{y_i \in S_\alpha\}$ both converge to $y \in S_\alpha$, if the lines $l_i = \overline{x_i\, y_i}$ converge to a line l and the tangent planes $T_{x_i}(S_\beta)$ converge to a plane T of some dimension, then both l and $T_y(S_\alpha)$ are contained in T.

Associated with the definition of a stratification is the stratified notion of a critical point.

Definition 6.10 (smooth functions and their critical points). *Say that a function $\phi : \mathcal{M} \to \mathbb{C}$ on a stratified space $\mathcal{M}$ is smooth if it is smooth when restricted to each stratum. A point $p \in \mathcal{M}$ is said to be critical for the smooth function ϕ if and only if the restriction $d\phi_{|S}$ vanishes, where S is the stratum containing p.*

Whitney stratifications are ideal for the topological study of algebraic hypersurfaces because of the following classical result.

Proposition 6.11. *Every algebraic variety in $\mathbb{R}^d$ or $\mathbb{C}^d$ admits a Whitney stratification.*

The simplest example is a smooth manifold, $\mathcal{M}$. This is a Whitney stratified space with a single stratum, namely $\mathcal{M}$. The next simplest example is that of a space $\mathcal{V}$ for which one may find a finite subset E such that $\mathcal{V} \setminus E$ is a smooth manifold. The strata $(\mathcal{V} \setminus E, E)$ form a Whitney stratification. An algebraic variety $\mathcal{V}$ whose singular locus is a smooth manifold $\mathcal{V}'$, may be stratified as $(\mathcal{V} \setminus \mathcal{V}', \mathcal{V}')$. However, if the singular locus itself has a finite, nonempty singular locus, E, it is not always true that $(\mathcal{V} \setminus \mathcal{V}', \mathcal{V}' \setminus E, E)$ is a Whitney stratification of $\mathcal{V}$; one might need to decompose the middle stratum further. See the appendices for more detail.

The second Whitney condition is difficult to read and impossible to remember, but basically it says that the strata fit together nicely. A well known but difficult result is the local product structure of a stratified space: a point $\mathbf{p}$ in a k-dimensional stratum S of a stratified space $\mathcal{M}$ has a neighborhood in which $\mathcal{M}$ is homeomorphic to some product $S \times X$. According to [GM88], a proof may be found in mimoegraphed notes of Mather from 1970; it is based on Thom's Isotopy Lemma which takes up fifty pages of the same mimeographed notes.

Tangent vector fields

Our aim is to define a vector field along which to push a given embedding of a stratified space $\mathcal{M}$, so as to decrease the real part of ϕ everywhere, except at critical points, where we can do no better than to remain still. To do this, we begin with some basics about the tangent bundle.

The tangent space $T_{\mathbf{x}}(\mathcal{M})$ at a point $\mathbf{x}$ of the stratified space $\mathcal{M}$ is defined to be the tangent space $T_{\mathbf{x}}(S)$ where S is the stratum containing $\mathbf{x}$. To talk about continuity of vector fields, we need these spaces to fit together into a bundle. In the case where $\mathcal{M}$ is embedded in and inherits the analytic structure of $\mathbb{C}^d$, we may do precisely that. The local homeomorphism to a product, mentioned in the previous paragraph, is induced by the embedding. Each $T_{\mathbf{x}}(\mathcal{M})$ is natrually identified with a subspace of $T_{\mathbf{x}}(\mathbb{C}^d)$. A smooth section of the tangent bundle of $\mathcal{M}$ is simply a smooth vector field $f : \mathcal{M} \to \mathbb{C}^d$ such that $f(\mathbf{x}) \in T_{\mathbf{x}}(S)$ when $\mathbf{x}$ is in the stratum S. The product structure also gives us locally constant vector fields (though not in any natural way). the next two lemmas take advantage of this.

Lemma 6.12. *Let f be a smooth section of the tangent bundle to S, that is $f(s) \in T_s(S)$ for $s \in S$. Then each $s \in S$ has a neighborhood in $\mathcal{M}$ on which f may be extended to a smooth section of the tangent bundle.*

Proof: In a local paramterization of $\mathcal{M}$ by $S \times X$, given $s \in S$, one may transport any vector $\mathbf{v} \in T_s(S)$ to any tangent space $T_{(s,x)}(\mathcal{M})$. Extend f by $f(s,x) := f(s)$. □

Let $\mathcal{M}$ be a real stratified space embedded in $\mathbb{C}^d$. This means that each stratum S is a subset of $\mathbb{C}^d$ and each of the chart maps ψ from a neighborhood in $\mathbb{R}^k$ to some k-dimensional stratum $S \subseteq \mathbb{C}^d$ is analytic (the coordinate functions are convergent power series) with a nonsingular differential. It follows that ψ may be extended to a holomorphic map on a neighborhood of the origin in $\mathbb{C}^k$, whose range we denote by $S \otimes \mathbb{C}$; choosing a small enough neighborhood, we may arrange for $S \otimes \mathbb{C}$ to be a complex k-manifold embedded in $\mathbb{C}^d$.

Lemma 6.13 (vector field near a non-critical point). *Let $\mathbf{x}$ be a point of the stratum S of the stratified space $\mathcal{M}$ and suppose $\mathbf{x}$ is not critical for the function ϕ. Then there is a vector $\mathbf{v} \in T_{\mathbf{x}}(S \otimes \mathcal{M})$ such that $\mathrm{Re}\{d\phi(\mathbf{v})\} > 0$ at $\mathbf{x}$. Furthermore, there is a continuous section f of the tangent bundle in a neighborhood $\mathcal{N}$ of $\mathbf{x}$ such that $\mathrm{Re}\{d\phi(f(\mathbf{y}))\} > 0$ at every $\mathbf{y} \in \mathcal{N}$.*

Proof: By non-criticality of $\mathbf{x}$, there is a $\mathbf{w} \in T_{\mathbf{x}}(S)$ with $d\phi(\mathbf{w}) = u \neq 0$ at $\mathbf{x}$. Multiply $\mathbf{w}$ componentwise by $\overline{u}$ to obtain $\mathbf{v}$ with $\mathrm{Re}\{d\phi(\mathbf{v})\} > 0$ at $\mathbf{x}$. Use any chart map for $S \otimes \mathbb{C}$ near $\mathbf{x}$ to give a locally trivial coordinatization for the tangent bundle and define a section f to be the constant vector $\mathbf{v}$; then $\mathrm{Re}\{d\phi(f(\mathbf{y}))\} > 0$ on some sufficiently small neighborhood of $\mathbf{x}$ in S. Finally, extend to a neighborhood of $\mathbf{x}$ in $\mathcal{M}$ by Lemma 6.12. □

Although we are working in the analytic category, the chains of integration are topological objects, for which we may use C^∞ methods (in what follows, even C^1 methods will do). In particular, a partition of unity argument enhances the local result above to a global result.

Lemma 6.14 (global vector field, in the absence of critical points). *Let $\mathcal{M}$ be a compact stratified space and ϕ a smooth function on $\mathcal{M}$ with no critical points. Then there is a global section f of the tangent bundle of $\mathcal{M}$ such that the real part of $d\phi(f)$ is everrywhere positive.*

Proof: For each point $\mathbf{x} \in \mathcal{M}$, let $f_{\mathbf{x}}$ be a section as in the conclusion of Lemma 6.13, on a neighborhood $U_{\mathbf{x}}$. Cover the compact space $\mathcal{M}$ by finitely many sets $\{U_{\mathbf{x}} : \mathbf{x} \in F\}$ and let $\{\psi_{\mathbf{x}} : \mathbf{x} \in F\}$ be a smooth partition of unity subordinate to this finite cover. Define

$$f(\mathbf{y}) = \sum_{\mathbf{x} \in F} \psi_{\mathbf{x}}(\mathbf{y}) f_{\mathbf{x}}(\mathbf{y}) \, .$$

Then f is smooth; it is a section of the tangent bundle because each tangent space is linearly closed; the real part of $d\phi(f(\mathbf{y}))$ is positive because we took a convex combination in which each contribution was nonnegative and at least one was positive. □

Another partition argument gives the final version – the one we will actually use – of this result.

Lemma 6.15 (global vector field, vanishing only at critical points). *Let $\mathcal{M}$ be a compact stratified space and ϕ a smooth function on $\mathcal{M}$ with finitely many critical points. Then there is a global section f of the tangent bundle of $\mathcal{M}$ such that the real part of $d\phi(f)$ is nonnegative and vanishes only when $\mathbf{y}$ is a critical point.*

PROOF: Let $\mathcal{M}_\epsilon$ be the compact stratified space resulting in the removal of an ϵ-ball around each critical point of ϕ. Let f_ϵ be a vector field as in the conclusion of Lemma 6.15 with $\mathcal{M}$ replaced by $\mathcal{M}_\epsilon$. Let c_n be a positive real number, small enough so that the magnitudes of all partial derivitaves of $c_n f_{1/n}$ of order up to n are at most 2^{-n}. In the topology of uniform convergence of derivatives of bounded order, the series $\sum_n c_n f_n$ converges to a vector field f with the required properties. □

Saddle point theorem, final version

Let $\mathcal{M}$ be a compact stratified space of dimension d embedded in $\mathbb{C}^d$ and let $\phi : \mathcal{M} \to \mathbb{C}$ be analytic. Let $\mathbf{x}$ be an isolated critical point in a stratum S of dimension d. We have seen that ϕ extends holomorphically to a neighborhood of $\mathbf{x}$ in $\mathbb{C}^d$. Let $\mathcal{H}(\mathbf{x})$ denote the Hessian matrix for the function ϕ at $\mathbf{x}$. We expect the integral $\mathcal{I}(\lambda)$ of $e^{-\lambda\phi(\mathbf{x})}$ over $\mathcal{M}$ to have a contribution of $(2\pi\lambda)^{-d/2}/\sqrt{\det \mathcal{H}}$ near the point $\mathbf{x}$. Summing over $\mathbf{x}$ leads to the following result.

Theorem 6.16 (critical point decomposition for stratified spaces). *Let $\mathcal{M}$ be a compact stratified space of dimension d embedded in $\mathbb{C}^d$ and let A and ϕ be analytic functions on a neighborhood of $\mathcal{M}$. Suppose that ϕ has finitely many critical points on $\mathcal{M}$, all in strata of dimension d and all quadratically nondegenerate. Let G be the subset of these at which the real part of ϕ is minimized and assume without loss of generality that this minimal value is zero. Let $\mathcal{C}$ be a chain representing $\mathcal{M}$. Then then the integral*

$$\mathcal{I}(\lambda) := \int_{\mathcal{C}} A(\mathbf{z}) e^{-\lambda\phi(\mathbf{z})}\, d\mathbf{z}$$

has an asymptotic expansion

$$\mathcal{I}(\lambda) \sim \sum_{\ell=0}^{\infty} c_\ell \lambda^{-(d+\ell)/2} .$$

If A is nonzero at some point of G then the leading term is given by

$$c_0 = (2\pi)^{-d/2} \sum_{\mathbf{x}\in G} A(\mathbf{x}) e^{\lambda\phi(\mathbf{x})} (\det \mathcal{H}(\mathbf{x}))^{-1/2} . \tag{6.4}$$

PROOF: Let f be a tangent vector field as given by Lemma 6.15. Such a field gives rise to a differntial flow, which, informally, is the solution to $d\mathbf{p}/dt = f(\mathbf{p})$. To be more formal, let $\mathbf{x}$ be a point in a stratum S of $\mathcal{M}$. Via a chart map in a neighborhood of $\mathbf{x}$, we solve the ODE $d\Phi(t)/dt = f(\Phi(t))$ with initial condition $\Phi(0) = \mathbf{x}$, obtaining a trajectory Φ on some interval $[0, \epsilon_{\mathbf{x}}]$ that is supported on S. Doing this simultaneously for all $\mathbf{x} \in \mathcal{M}$ results in a map

$$\Phi : \mathcal{M} \times [0, \epsilon] \to \mathbb{C}^d$$

with $\Phi(\mathbf{x}, t)$ remaining in $S \otimes \mathbb{C}$ when $\mathbf{x}$ is in the stratum S. The map Φ satisfies $\Phi(\mathbf{x}, 0) = \mathbf{x}$ and $(d/dt)\Phi(\mathbf{x}, t) = f(\Phi(\mathbf{x}, t))$. The fact that this may be defined up to time ϵ for some $\epsilon > 0$ is a consequence of the fact that the vector field f is bounded and that a small neighborhood of $\mathcal{M}$ in $\mathcal{M} \otimes \mathbb{C}$ is embedded in $\mathbb{C}^d$.

The flow reduces the real part of ϕ everywhere except the critical points which are rest points. Consequently, it defines a homotopy $H(\mathbf{x}, t) := \Phi(\mathbf{x}, t/\epsilon)$ between $\mathcal{C}$ and a chain $\mathcal{C}'$ on which the minima of the real part of ϕ occur precisely on the set G. Recall that H induces a chain homotopy $\mathcal{C}_H$ with $\partial \mathcal{C}_H = \mathcal{C}' - \mathcal{C} + \partial \mathcal{C} \times \sigma$, where σ is a standard 1-simplex. Let ω denote the holomorphic d-form $A(\mathbf{z}) \exp(-\lambda \phi(\mathbf{z}))\, d\mathbf{z}$. Because ω is a holomorphic d-form in $\mathbb{C}^d$, we have $d\omega = 0$. Now, by Stokes' Theorem,

$$\begin{aligned} 0 &= \int_{\mathcal{C}_H} d\omega \\ &= \int_{\partial \mathcal{C}_H} \omega \\ &= \int_{\mathcal{C}'} \omega - \int_{\mathcal{C}} \omega - \int_{\partial \mathcal{C} \times \sigma} \omega. \end{aligned}$$

The chain $\partial \mathcal{C} \times \sigma$ is supported on a finite union of spaces $S \otimes \mathcal{C}$ where S is a stratum of dimension at most $d-1$. Recall (see Exercise 4.5) that the integral of ω vanishes over such a chain. Therefore, the last term on the right drops out and we have

$$\int_{\mathcal{C}} \omega = \int_{\mathcal{C}'} \omega \,.$$

Outside of a neighborhood of G the magnitude of the integrand is exponentially small, so we have shown that there are d-chains $\mathcal{C}_{\mathbf{x}}$ supported on arbitrarily small neighborhoods $\mathcal{N}(\mathbf{x})$ of each $\mathbf{x} \in G$ such that

$$\mathcal{I}(\lambda) - \sum_{\mathbf{x} \in G} \int_{\mathcal{C}_{\mathbf{x}}} \omega$$

is exponentially small. To finish that proof, we need only show that each $\int_{\mathcal{C}_{\mathbf{x}}} \omega$ has an asymptotic series in decreasing half-integral powers of λ whose leading term, when $A(\mathbf{x}) \neq 0$, is given by

$$c_0(\mathbf{x}) = (2\pi)^{-d/2} A(\mathbf{x}) e^{\lambda \phi(\mathbf{x})} (\det \mathcal{H}(\mathbf{x}))^{-1/2} \,. \tag{6.5}$$

The d-chain $\mathcal{C}_{\mathbf{x}}$ may by parametrized by a map $\psi_{\mathbf{x}} : B \to \mathcal{N}(\mathbf{x})$, mapping the origin to $\mathbf{x}$, where B is the open unit ball in $\mathbb{R}^d$. By the chain rule,

$$\int_{\mathcal{C}_{\mathbf{x}}} \omega = \int_B [A \circ \psi](\mathbf{x}) \exp(-\lambda [\phi \circ \psi(\mathbf{x})]) \det d\psi(\mathbf{x})\, d\mathbf{x}.$$

The real part of the analytic phase function $\phi \circ \psi$ has a strict minimum at the origin, so we may apply Theorem 6.2. We obtain an asymptotic expansion whose first term is

$$(2\pi\lambda)^{-d/2} [A \circ \psi](\mathbf{0}) (\det M_{\mathbf{x}})^{-1/2} \tag{6.6}$$

where $M_{\mathbf{x}}$ is the Hessian matrix of the function $\phi \circ \psi$. The term $[A \circ \psi](\mathbf{0})$ is equal to $A(\mathbf{x})$. The Hessian matrix of $\phi \circ \psi$ at the origin is given by $M_{\mathbf{x}} = d\psi(\mathbf{0})\, \mathcal{H}(\mathbf{x})\, d\psi(\mathbf{0})$. Thus

$$\det M_{\mathbf{x}} = (\det d\psi(\mathbf{0}))^2 \det \mathcal{H}(\mathbf{x})$$

and plugging into (6.6) yields (6.5), up to the choice of sign for each $\mathbf{x} \in G$. $\square$

Remark. In one dimension, let $\phi(z) = -z^2$ and let $\mathcal{M}$ be an interval about zero on the imaginary axis. Then $I(\lambda) = \int_{\mathcal{M}} e^{-\lambda\phi(z)}\, dz = \pm i/\sqrt{2\pi\lambda}$ according to whether $\mathcal{M}$ is oriented up or down the imaginary axis. There does not seem to be a canonical way to relate the sign choice on the square root to the eigenvalues of $\mathcal{H}$ and the orientation of $\mathcal{M}$. Nevertheless, it is easy to give a prescription for choosing the sign that involves choosing an arbitrary map. Let ψ parametrize $\mathcal{M}$ by a patch of $\mathbb{R}^d$ with the standard orientation; then we take

$$\det(\mathcal{H})^{-1/2} := (\det \mathcal{H}(\phi \circ \psi))^{-1/2} \det J_\psi$$

where the square root on the right is the product of principal square roots of the eigenvalues.

6.4 Examples and extensions

The following example occurs in Chapter 11 in connection the asymptotic evaluation of coefficients $a_{r,s}$ of the generating function $F(x,y) := \dfrac{1}{P(x,y)Q(x,y)}$ in the direction $s/r \sim \mu$.

Example 6.17. Let $\mathcal{M} = S \times I$ where I is the interval $[-1,1]$ and S is the circle $\mathbb{R}/(2\pi\mathbb{Z})$. Suppose the phase function ϕ has positive real part vanishing precisely on $\{0\} \times I$ with a unique critical point at $(0, p_0)$ and quadratic approximation

$$\phi(x,p) = Kx^2 + iLx(p - p_0) + O(|x|^3 + |p - p_0|^3) \tag{6.7}$$

near $(0, p_0)$, where $K > 0$ and L are real numbers. Note that the strip $\{0\} \times I$ on which the phase function vanishes extends out to the bounding circles of the cylinder $\mathcal{M}$, so we are not in a case where the magnitude of the integrand is small away from the critical point.

The Hessian matrix at $(0, p_0)$ is $\begin{bmatrix} 2K & L \\ L & 0 \end{bmatrix}$. The determinant of half the Hessian is equal to $K + L^2/4$, and from Theorem 6.16 we conclude that

$$\begin{aligned} \mathcal{I}(\lambda) &= \int_{\mathcal{N} \times I} e^{-\lambda\phi(\mathbf{x})}\, d\mathbf{x} \\ &\sim \frac{1}{2\pi\lambda\,\sqrt{K^2 + L^2/4}}\,. \end{aligned}$$

Boundary critical point

`will fill this in later, after writing conference paper`

Continuum of critical points

`will fill this in later, after writing conference paper`

Notes

In the case of purely real or imaginary phase, these results are fairly standard; see [BH86, Won89] for real phase or [Ste93] for imaginary phase. I have not seen the complex phase result Theorem 6.2 stated before. The remaining results in this section, though not entirely unexpected, are new. In particular, the existence of a deformation to localize to critical points even when the real part of the phase is not strictly minimized (Theorem 6.16) seems new. Such localization in the C^∞ category is certainly not new (the same method as in the proof of [Ste93, Theorem VIII:2.2] may be applied, for example) but the C^∞ results are weaker, giving rapid decay rather than exponential decay, and not allowing for further contour deformation after the localization. The results in this section will be published as a standalone paper, possibly in the Proceedings of the AMS Special Session on Algorithmic Probability and Combinatorics, edited by Lladser, Maier, Mishna and Rechnitzer.

Exercises

Exercise 6.1. Let $\mathcal{J}$ be an ideal of dimension k and suppose that in some neighborhood, the real zero set of $\mathcal{J}$ has co-dimension k. Let S denote the intersection of this zero set with a sufficiently small ball so that S is a manifold. Then $S \otimes \mathbb{C}$ is the intersection of a ball with the complex variety defined by $\mathcal{J}$.