Chapter 8

Cones, Laurent series and amoebas

We introduce some notation that will be useful throughout these notes. Without ambiguity, we may extend the logarithm to vectors, by defining $\log \mathbf{z} := (\log z_1, \ldots, \log z_d)$ to be the coordinatewise logarithm. Similarly, the exponential function is extended, so that $\exp(\mathbf{x}) := (e^{x_1}, \ldots, e^{x_d})$. Another useful notation is the coordinatewise log-modulus, $\mathrm{ReLog}\, \mathbf{z} := \mathrm{Re}\, \{\log \mathbf{z}\} := (\log |z_1|, \ldots, \log |z_d|)$.

Our ultimate goal is to evaluate the multivariate Cauchy integral (1.4). First, though, a word of explanation about this chapter, which is comprised of a few more digressions, into cones and their duals, properties of Laurent series, and polynomial amoebas. The relevance of Laurent series as generating functions is self-evident. Section 2.1 gave a brief treatment of formal power series that barely touched on analytic properties. The reasons for the comparatively long treatment here of Laurent series are that (1) analytic properties are necessary to establish properties of the formal power series, (2) rigorous developments of these properties for Laurent series, though well known, appear in print rarely or never, and (3) these subsume results for ordinary power series, thus paying off several debts of rigor to preceding chapters. Amoebas are less central, but they provide a context in which the early results on **minimal points** of multivariate generating functions can be understood and generalized.

Regarding cones an dduality, these notions arise throughout Fourier theory. In our setting, denoting the logarithms of the radii of the torus of integration in Cauchy's integral formula by $b_1, \ldots, b_d$, the Cauchy integral becomes

$$T = T_{\mathbf{b}} := \{\mathbf{z} : |z_j| = e^{b_j} \text{ for all } 1 \le j \le d\} .$$

The change of variables $\mathbf{z} = \exp(\mathbf{b} + i\mathbf{x})$, $d\mathbf{z} = i^d \mathbf{z}\, d\mathbf{x}$, turns this into an integral over the flat torus $T_{\mathrm{flat}} := R^d / (2\pi\mathbb{Z})^d$:

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi}\right)^d e^{-\mathbf{r}\cdot\mathbf{b}} \int_{T_{\mathrm{flat}}} \exp(-i\mathbf{r}\cdot\mathbf{x}) \tilde{F}(\mathbf{x})\, d\mathbf{x} .$$

Here, $\tilde{F}(\mathbf{x}) := F(e^{\mathbf{b}+i\mathbf{x}})$ and we recognize the integral as a Fourier transform. The presence of the quantity $\mathbf{r}\cdot\mathbf{x}$ shows that the index vector $\mathbf{r} \in \mathbb{Z}^d$ plays a role dual to that of $\mathbf{x} \in \mathbb{R}^d$ (or $\mathbb{C}^d$). It will be helpful to keep this duality in mind and to build notation that reflects this. Accordingly, let $(\mathbb{R}^d)^*$ denote a copy of $\mathbb{R}^d$ with a basis dual to the standard basis of $\mathbb{R}^d$, and for $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{r} \in (\mathbb{R}^d)^*$, use the interchangeable notations $< \mathbf{r}, \mathbf{x} >$ or $\mathbf{r} \cdot \mathbf{x}$ to denote the pairing. We will denote vectors in $\mathbb{R}^d$ as column vectors and vectors in $(\mathbb{R}^d)^*$ as row vectors, so a third possibility for the inner product is the notation $\mathbf{rx}$.

8.1 Cones and dual cones

Let $\mathbf{L}$ be any convex open cone in $\mathbb{R}^d$. The (closed) convex dual cone $\mathbf{L}^* \subseteq (\mathbb{R}^d)^*$ is defined to be the set of vectors $\mathbf{v} \in (\mathbb{R}^d)^*$ such that $\mathbf{v} \cdot \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbf{L}$. Familiar properties of the dual cone are

$$L \subseteq M \quad \Rightarrow \quad L^* \supseteq M* \, ; \tag{8.1}$$

$$(L \cap M)^* \quad = \quad \mathsf{hull}(L^* \cup M^*) \, . \tag{8.2}$$

Suppose $\mathbf{x}$ is a point on the boundary of a convex set C. Then the intersection of all halfspaces that contain C and have $\mathbf{x}$ on their boundary is a closed affine cone with vertex $\mathbf{x}$ (a translation by $\mathbf{x}$ of a closed cone in $\mathbb{R}^d$) that contains C. Translating by $-\mathbf{x}$ and taking the interior gives the (open) **tangent cone** to C at $\mathbf{x}$, denoted by $\mathsf{tan}_{\mathbf{x}}(C)$. An alternative definition is

$$\mathsf{tan}_{\mathbf{x}}(C) = \{\mathbf{v} : \mathbf{x} + \epsilon \mathbf{v} \in C \text{ for all sufficiently small } \epsilon > 0\} \, .$$

The (closed) **normal cone** to C at $\mathbf{x}$, denoted $\mathbf{N^*}_{\mathbf{x}}(C)$, is the convex dual cone to the negative of the tangent cone:

$$\mathbf{N^*}_{\mathbf{x}}(C) = (-\mathsf{tan}_{\mathbf{x}}(C))^* \, .$$

Equivalently, it corresponds to the set of linear functionals on C that are maximized at $\mathbf{x}$, or to the set of outward normals to support hyperplanes to C at $\mathbf{x}$.

The following notation for the degree of vanishing of a function and the leading homogeneous part of a function at a point will be useful.

Definition 8.1 (degree of vanishing, homogeneous part). *For any locally analytic function* $f : \mathbb{C}^d \to \mathbb{C}$ *and any point* $\mathbf{z} \in \mathbb{C}^d$, *we let* $\deg(f, \mathbf{z})$ *denote the degree of vanishing of* f *at* $\mathbf{z}$*:*

$$\deg(f, \mathbf{z}) := \sup\{n : f(\mathbf{z} + \mathbf{w}) = O(|\mathbf{w}|^n)\} \, .$$

This is zero if $f(\mathbf{z}) \neq 0$ *and in general is the least degree of any term in the power series expansion of* $f(\mathbf{z} + \cdot)$. *When* $f(\mathbf{z}) \neq 0$, *then* $\deg(f, \mathbf{z})$ *is, by definition, zero. We let* $\mathsf{hom}(f, \mathbf{z})$ *denote the sum of all monomials of minimal degree in the power series for* $f(\mathbf{z} + \cdot)$ *and we call this the* ***homogeneous part of*** f ***at*** $\mathbf{z}$. *Thus*

$$f(\mathbf{z} + \mathbf{w}) = \mathsf{hom}(f, \mathbf{z})(\mathbf{w}) + O\left(|\mathbf{w}|^{\deg(f, \mathbf{z}) + 1}\right) \, .$$

When $\mathbf{z} = \mathbf{0}$, *we may omit* $\mathbf{z}$ *from the notation: thus,* $\mathsf{hom}(f) := \mathsf{hom}(f, \mathbf{0})$.

The term *tangent cone* has a different meaning in algebraic contexts, which we will require these as well. (The term *normal cone* has an algebraic meaning as well, which we will not need.) To avoid confusion, we define the **algebraic tangent cone** of f at $\mathbf{x}$ to be $\mathcal{V}_{\mathsf{hom}(f,\mathbf{z})}$ and denote this by $\mathsf{alg\text{-}tan}_{\mathbf{x}}(f)$. An equivalent but more geometric definition is that the algebraic tangent cone is the union of lines through $\mathbf{x}$ that are the limits of secant lines through $\mathbf{x}$; thus for a unit vector $\mathbf{u}$, the line $\mathbf{x}+t\mathbf{u}$ is in the algebraic tangent cone if there are $\mathbf{x}_n \in \mathcal{V}_f$ distinct from but converging to $\mathbf{x}$ for which $(\mathbf{x}_n - \mathbf{x})/||\mathbf{x}_n - \mathbf{x}|| \to \pm\mathbf{u}$. This equivalence and more is contained in the following results. We let S_1 denote the unit sphere $\{(z_1, \ldots, z_d) : |z_1|^2 + \cdots + |z_d|^2 = 1\}$ and let $S_r := rS_1$ denote the sphere of radius r.

Lemma 8.2 (algebraic tangent cone is the limiting secant cone). *Let Q be a polynomial vanishing to degree $m \geq 1$ at the origin and let $A = \mathsf{hom}(Q)$ be its homogeneous part; in particular,*

$$Q(\mathbf{z}) = A(\mathbf{z}) + R(\mathbf{z})$$

where A is a nonzero homogeneous polynomial of degree m and $R(\mathbf{z}) = O(|\mathbf{z}|^{m+1})$. Let Q_ϵ denote the polynomial

$$Q_\epsilon(\mathbf{z}) := \epsilon^{-m} Q(\epsilon\mathbf{z}) = A(\mathbf{z}) + R_\epsilon(\mathbf{z})$$

where $R_\epsilon(\mathbf{z}) = \epsilon^{-m} R(\epsilon\mathbf{z}) \to 0$ as $\epsilon \to 0$. Let $\mathcal{V}_\epsilon := \mathcal{V}_{Q_\epsilon} \cap S_1$ denote the intersection of $\{Q_\epsilon = 0\}$ with the unit sphere. Then $\mathcal{V}_\epsilon$ converges in the Hausdorff metric as $\epsilon \to 0$ to $\mathcal{V}_a \cap S_1$.

PROOF: On any compact set, in particular S_1, $R_\epsilon \to 0$ uniformly. If $\mathbf{z}^{(n)} \to \mathbf{z}$ and $\mathbf{z}^{(n)} \in \mathcal{V}_{1/n}$ then for each n,

$$|A(\mathbf{z}^{(n)})| = |Q_{1/n}(\mathbf{z}^{(n)}) + R_{1/n}(\mathbf{z}^{(n)})| = |R_{1/n}(\mathbf{z}^{(n)})| \to 0\,.$$

Hence $A(\mathbf{z}) = 0$ by continuity of A and we see that any limit point of $\mathcal{V}_\epsilon$ as $\epsilon \to 0$ is in $\mathcal{V}_A \cap S_1$. Conversely, fix a unit vector $\mathbf{z} \in \mathcal{V}_A$. The homogenous polynomial A is not identically zero, therefore there is a projective line through $\overline{\mathbf{z}}$ along which A has a zero of finite order at $\overline{\mathbf{z}}$. Back in affine space, there is a complex curve γ in the unit sphere along which A is holomorphic with a zero of some finite order k at $\mathbf{z}$. As $\epsilon \to 0$, the holomorphic function $\mathbb{R}_\epsilon$ goes to zero uniformly in a neighborhood of $\mathbf{z}$ in γ; hence there are k zeros of Q_ϵ converging to $\mathbf{z}$ as $\epsilon \to 0$, and therefore $\mathbf{z}$ is a limit point of $\mathcal{V}_\epsilon$ as $\epsilon \to 0$. $\square$

8.2 Laurent series

The ring of Laurent polynomials in d variables is the ring $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$. In what follows, it will be convenient to extend our scope to consider generating functions that are rational over the ring of Laurent polynomials. Not only does this allow us to deal more naturally with generating functions such as the Aztec Diamond generating function $\dfrac{z/2}{1 - (x + x^{-1} + y + y^{-1})z/2 + z^2}$, but it is the natural level of generality in which to discuss amoebas, which in turn are the best level of generality to discuss power series and their domains of convergence.

Let $\mathcal{L}(\mathbf{z})$ denote the complex vector space of formal linear combinations of monomials $\mathbf{z}^{\mathbf{r}}$ as $\mathbf{r}$ ranges over all of $\mathbb{Z}^d$. We call these formal Laurent series. The formal Laurent series are a module over the ring of Laurent polynomials: if f is a Laurent polynomial and G is a formal Laurent series, then the coefficient of $\mathbf{z}^{\mathbf{r}}$ in fG involves only finitely many terms of G and is therefore well defined. Note, however, that some elements of $\mathcal{L}(\mathbf{z})$ have nontrivial annihilators; for example, in one variable, if $G = \sum_{n \in \mathbb{Z}} z^n$ then $(1-z)G = 0$. Also, because the set of pairs (α, β) summing to $\gamma \in \mathbb{Z}^n$ is infinite, there is no natural product structure on $\mathcal{L}(\mathbf{z})$. We will see that convergent Laurent series are much better behaved, but first let's have a look at the canonical example of why rational functions may have more than one Laurent series representation.

Example 8.3 (Laurent series for $z/(1-z)$). Let G_1 be the Laurent series $\sum_{n \geq 1} z^n$; this is convergent on $\mathcal{D}_1 := \{z : |z| < 1\}$. Let $G_2 := \sum_{n \leq 0} -z^n$. Then G_2 is convergent on $\mathcal{D}_2 := \{z : |z| > 1\}$. We have $(1-z)G_1 = (1-z)G_2 = z$. Each series converges to $z/(1-z)$ uniformly on its domain. The intersection of the two domains is, of course, empty.

Turning now to the study of convergent Laurent series, let $\mathcal{D}$ be an open simply connected domain and let $\mathcal{L}(\mathbf{z})(\mathcal{D})$ denote the subspace of $\mathcal{L}(\mathbf{z})$ consisting of series that are absolutely convergent, uniformly on compact subsets of the domain $\mathcal{D}$. When discussing convergence of Laurent series, we shall always mean uniform convergence on compact sets. The following general facts about domains of convergence of Laurent series are stated (without proof) as Proposition 1.5 of [GKZ94, Chapter 6]. I have provided proofs of most of these because of the difficulty of finding such proofs in the literature; probably this is not essential to one's understanding and may be skipped on first reading.

Theorem 8.4. *(i) Let $G(z) = \sum_{\mathbf{r} \in \mathbb{Z}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ be a Laurent series. Then the open domain of convergence of G has the form $\mathcal{D} = \operatorname{ReLog}^{-1}(B)$ for some convex open set $B \subseteq \mathbb{R}^d$.*

(ii) The function g defined by the series G is holomorphic in $\mathcal{D}$.

(iii) Conversely, if g is a holomorphic function on $\mathcal{D} := \operatorname{ReLog}^{-1}(B)$, with B convex and open in $\mathbb{R}^d$, then there is a unique Laurent series $G \in \mathcal{L}(\mathbf{z})(\mathcal{D})$ converging to ϕ. The coefficients of G are given by Cauchy's integral formula:

$$a_{\mathbf{r}} := \left(\frac{1}{2\pi i}\right)^d \int_{T(\mathbf{x})} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} g(\mathbf{z})\, d\mathbf{z} \tag{8.3}$$

where $T(\mathbf{x})$ is the torus $\operatorname{ReLog}^{-1}(\mathbf{x})$ for any $\mathbf{x} \in B$.

With these facts established, we may define multiplication in $\mathcal{L}(\mathbf{z})(\mathcal{D})$ as follows. Let ι denote the identification map from $\mathcal{L}(\mathbf{z})(\mathcal{D})$ to the space of holomorphic functions on $\mathcal{D}$. Then ι is invertible and holomorphicity of the the product of holomorphic functions allows us to define $G \cdot H := \iota^{-1}(\iota(G) \cdot \iota(H))$. Similarly, if f is everywhere nonvanishing on $\mathcal{D}$ then there is a unique Laurent series identified with the holomorphic function $1/f$. We may therefore specify a formal Laurent series (such as the

Aztec Diamond generating function) as a quotient of Laurent polynomials, provided that we specify a domain on which the denominator is nonvanishing.

The proof of Theorem 8.4 requires the development of a few well known facts about series of holomorphic functions.

Proposition 8.5 (uniqueness). *Let* $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ *be a Laurent series converging uniformly to zero on the torus* $T_{\mathbf{x}} := \{\exp \mathbf{x} + i\mathbf{y} : \mathbf{y} \in \mathbb{R}^d\}$. *Then* $a_{\mathbf{r}} = 0$ *for all* $\mathbf{r}$.

Proof: Assume without loss of generality that $\mathbf{x} = \mathbf{0}$. Then $\sum_{\mathbf{r}} a_{\mathbf{r}} \exp(i\mathbf{r} \cdot \mathbf{y}) \to 0$ uniformly on $(\mathbb{R}/(2\pi\mathbb{Z}))^d$. Thus $\sum a_{\mathbf{r}} e^{i\mathbf{r}\cdot\mathbf{y}}$ is a Fourier series for the zero function. By uniqueness of Fourier series expansions, $a_{\mathbf{r}} = 0$ for all $\mathbf{r}$. $\square$

Proposition 8.6 (identity theorem). *If analytic functions* f *and* g *on a connected domain* $D \subseteq \mathbb{C}^n$ *agree on an open subset, then they agree on all of* D.

Proof: The set K where they agree is a closed subset of $\mathcal{D}$ because it is the inverse image of $\{0\}$ under the continuous function $f - g$. Let the closure of the interior of K be denoted $K' \subseteq K$. Choose any $\mathbf{z}_0 \in K'$. The functions f and g agree at $\mathbf{z}_0$, and by definition of analyticity, partial derivatives of all orders exist for each function at $\mathbf{z}_0$ and each function is equal to the limit of its Taylor expansion in a neighborhood of $\mathbf{z}_0$. The partial derivatives, hence the Taylor expansions, are determined by values in any open set with $\mathbf{z}_0$ on the boundary, hence by values in the interior of K, and hence are the same for the two functions. It follows that the two funtions agree in a neighborhood of $\mathbf{z}_0$. Thus, $\mathbf{z}_0$ is in the interior of K'. Since $\mathbf{z}_0 \in K'$ was arbitrary, K' is open. It is closed as well, so by connectedness of $\mathcal{D}$, we see that $K' = \mathcal{D}$. $\square$

Proposition 8.7. *The uniform limit of analytic functions on a domain* $D \subseteq \mathbb{C}^n$ *is analytic.*

Sketch of proof: Stokes' Theorem implies that the integral over $\partial\mathcal{C}$ of a holomorphic d-form must vanish. In fact the converse is true: if $\int_{\mathcal{C}} f \, d\mathbf{z}$ vanishes whenever $\mathcal{C}$ is the boundary of a $(d+1)$-simplex, then f is analytic. The integral of the uniform limit of functions is the limit of the integrals, which is zero, proving that the limit is holomorphic. $\square$

Proposition 8.8 (logarithmic convexity of domains of convergence). *Let* $F := \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ *be a formal Laurent series and let* $\mathcal{D}$ *be its open domain of convergence, that is the interior of the set of* $\mathbf{z}$ *for which* $\sum_{\mathbf{r}} |a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}| < \infty$. *Then* $\mathcal{D} = \mathrm{ReLog}^{-1}(B)$ *for some open convex set* $B \subseteq \mathbb{R}^d$.

Proof: Convergence depends on $\mathbf{z}$ only through the moduli of the components, hence the domain of convergence is invariant under $z_j \mapsto e^{i\theta} z_j$, hence is the union of tori $T(\mathbf{x}) = \{e^{\mathbf{x}+i\mathbb{R}^d}\}$ and hence equal to $\mathrm{ReLog}^{-1}(B)$ for some B. Clearly, if $\mathbf{x} \in B$ then

$$\sup_{\mathbf{r}} |a_{\mathbf{r}}| e^{\mathbf{r}\cdot\mathbf{x}} < \infty \,, \tag{8.4}$$

because a series with unbounded terms cannot converge uniformly. On the other hand, if

$$\sup_{\mathbf{r}} |a_{\mathbf{r}}| e^{\mathbf{r}\cdot\mathbf{x}'} < C \tag{8.5}$$

for all $\mathbf{x}'$ in some neighborhood of $\mathbf{x}$, then for some $\epsilon > 0$, this holds whenever $\mathbf{x}' = \mathbf{x} \pm \epsilon e_j$ and $1 \leq j \leq d$. Let $|\mathbf{r}|$ denote $\max_j |r_j|$. When the maximum value is $|r_j|$ and $r_j > 0$, we have $\exp(\mathbf{r} \cdot (\mathbf{x} + e_j)) = \exp(|\mathbf{r}|\epsilon)$ and when the maximum is $|r_j|$ and $r_j < 0$, we have $\exp(\mathbf{r} \cdot (\mathbf{x} - e_j)) = \exp(|\mathbf{r}|\epsilon)$. In either case,

$$\sup_{|\mathbf{r}| \geq k} |a_{\mathbf{r}}| e^{\mathbf{r} \cdot \mathbf{x}} \leq e^{-\epsilon k} C \, .$$

This is summable, therefore the interior of the set of $\mathbf{x}$ satisfying (8.4) is contained in the domain of convergence. We conclude that the open domain of convergence is the union over C of the set of $\mathbf{x}'$ satisfying (8.5). Fixing C, the set $B(C)$ of $\mathbf{x}'$ satisfying (8.5) is the intersection of open half spaces $H_{\mathbf{r}} := \{\mathbf{r} : \mathbf{r} \cdot \mathbf{x} < \log(C/|a_{\mathbf{r}}|)\}$ over the set of $\mathbf{r}$ for which $a_{\mathbf{r}} \neq 0$. We conclude that $B(C)$ is a convex set. For $C' > C$, the set $B(C')$ contains $B(C)$, hence $B = \cup_C B(C)$ is the increasing union of convex sets, and is therefore open. $\square$

PROOF OF THEOREM 8.4: We have just proved (i) in Proposition 8.8, while (ii) is Proposition 8.7 and uniqueness in (iii) is Proposition 8.5. It remains to show that (8.3) defines a series a Laurent series $G := \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^r r$ converging to g.

By holomorphicity, the integral (8.3) defining $a_{\mathbf{r}}$ is independent of the choice $T(\mathbf{x})$ of chain of integration. Fix $\mathbf{x} \in B$ and choose $\epsilon > 0$ small enough so that $\mathbf{x} \pm \epsilon e_j \in B$ for all $1 \leq j \leq d$. The modulus of $g(\mathbf{z})$ is bounded on the finite union of tori $T(\mathbf{x} \pm \epsilon e_j)$. By (8.3), and the same argument as in the previous proof, we see that $|a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}| \leq K \exp(-\epsilon |\mathbf{r}|)$ for all $\mathbf{z} \in T(\mathbf{x})$. For a slightly smaller value of ϵ, this holds for all $\mathbf{z} \in T(\mathbf{x}')$, for $\mathbf{x}'$ in some neighborhood $\mathcal{N}(\mathbf{x})$ of $\mathbf{x}$. If $K \subseteq B$ is any compact set, covering with finitely many neighborhoods $\mathcal{N}(\mathbf{x})$ shows that such a bound holds for all $\mathbf{z} \in \mathrm{ReLog}^{-1}(K)$. In particular, the series G converges uniformly on compact subsets of $\mathcal{D}$.

By Proposition 8.7, the function $\iota(G)$ is holomorphic on $\mathcal{D}$, being the uniform limit of the series of partial sums on compact subsets of $\mathcal{D}$. Once we show that $\iota(G) = g$ on some subset of $\mathcal{D}$ with nonempty interior, the theorem follows from the identity theorem.

Let B' be a closed rectangle $\prod_{j=1}^d [u_j, v_j]$ contained in the open domain of convergence. For $d = 1$, the proof that $\sum_{n=-\infty}^{\infty} a_n z^n$ converges uniformly to f when f is holomorphic on any annulus containing $\{z : e^a \leq |z| \leq e^b\}$ and $a_n := (2\pi i)^{-1} \int_\gamma z^{-n-1} f(z) \, dz$ may be found in most complex variable texts. For example, 1.11 of [Con78, Chapter V] uses the Cauchy kernel representation

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z} \, dw \\ &:= f_2(z) - f_1(z) \end{aligned}$$

where γ_1 and γ_2 are the inner and outer boundaries of the annulus respectively. The function f_2 is holomorphic on the disk of radius e^b and the function $f_1(1/z)$ is holomorphic on the disk of radius e^{-a}, so the usual power series expansions show that $f_2(z) = \sum_{n \geq 0} a_n z^n$ while $f_1(z) = \sum_{n \leq -1} a_n z^n$.

In d variables, when $B = \prod_{j=1}^d [u_j, v_j]$, we require a representation for $f(z)$ analogous to the Cauchy kernel representation, expressing f as the sum of 2^d integrals over the d-dimensional faces

$\mathrm{ReLog}^{-1}(B)$, which is a product of d annuli. The details are omitted; I will try to find somewhere this is written down so I can reference it. $\square$

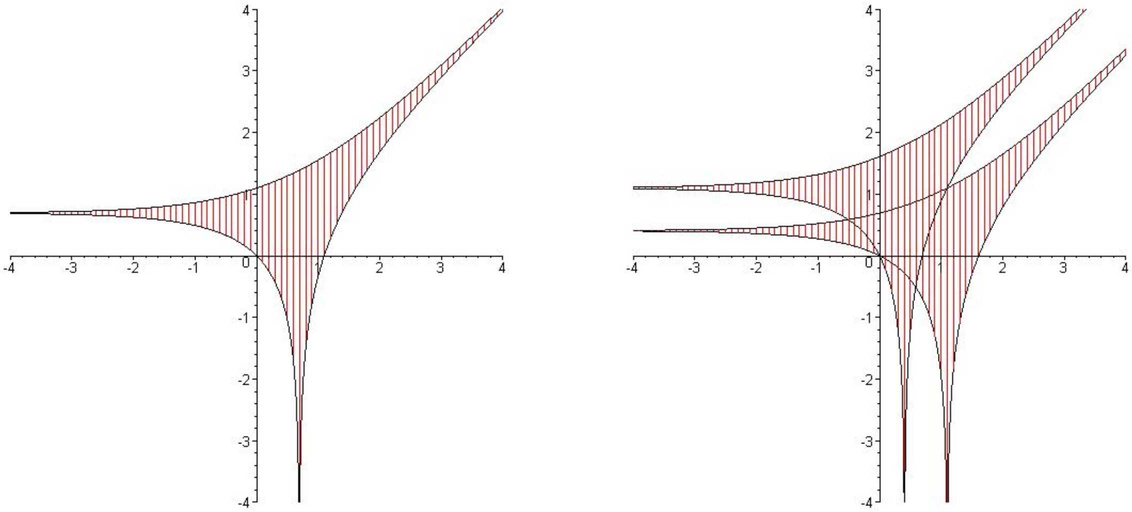
Remark 8.9. Let $K \subseteq \mathbb{R}^d$ be a cone containing the origin, such that $K + K := \{\mathbf{x} + \mathbf{y} : \mathbf{x}, \mathbf{y} \in K\}$ is a proper subset of K. Then each $\mathbf{x} \in K$ is in only finitely many of the sets $K, K+K, K+K+K, \ldots$. It follows that the space of formal Laurent series whose coefficients vanish off of $K \cap \mathbb{Z}^d$ is a ring (is closed under products). In fact a linear change of variables in $\mathbb{Z}^d$ induces a mapping from K into the formal power series ring. All facts about formal power series transfer to this case. For instance, if the coefficients $\{a_{\mathbf{r}}\}$ of the series G are supported on K and $a_{\mathbf{0}} = 0$ then $1/(1-G) := 1 + G + G^2 + \cdots$ is a well defined formal Laurent series inverting $(1-G)$, even if the coefficients of G grow sufficiently rapidly that G is not convergent on any domain.

8.3 Amoebas

If f is any Laurent polynomial, we define

$$\mathsf{amoeba}(f) := \{\mathrm{ReLog}\,\mathbf{z} : f(\mathbf{z}) = 0\}$$

to be the set of log-moduli of zeros of f. The simplest example is the amoeba of a linear function, such as $f = 2 - x - y$, shown in figure 8.1a. The amoeba of a product is the union of amoebas, as shown in figure 8.1b. The following result is stated in [GKZ94, Chapter 6], [The02, Theorem 2] and [FPT00, Theorem 1.1].



(a) $\mathsf{amoeba}(2 - x - y)$ (b) $\mathsf{amoeba}(3 - 2x - y)(3 - x - 2y)$

Figure 8.1: two amoebae

Proposition 8.10. *The connected components of* $\mathbb{R}^d \setminus \mathsf{amoeba}(f)$ *are convex. The components are in one-to-one correspondence with Laurent series expansions of* $1/f$*.*

Proof: Let $\{B_\alpha : \alpha \in I\}$ be the components of $\mathbb{R}^d \setminus \mathsf{amoeba}(f)$. Let G_α denote the Laurent series (8.3), which converges to the holomorphic function $1/f$ on B_α. Any $\mathbf{x} \in \partial B_\alpha$ is in $\mathsf{amoeba}(f)$, meaning that f vanishes somewhere on $T(\mathbf{x})$ so it is no Laurent expansion of $1/f$ can converge on $T(\mathbf{x})$. It follows that B_α is the domain of convergence of G_α, whence by part (i) of Theorem 8.4, B_α is convex. Applying parts (i), (ii) and (iii) of Theorem 8.4 in order to any Laurent expansion of $1/f$ shows that it is equal to some G_α. Finally, $G_\alpha \neq G_\beta$ for $\alpha \neq \beta$ because domains of convergence are convex and the convex hull of $B_\alpha \cup B_\beta$ intersects $\mathsf{amoeba}(f)$. $\square$

A **polytope** P is the convex hull $\mathsf{hull}(E)$ of a finite collection of points $E \subseteq \mathbb{R}^d$. The extreme points of $\mathsf{hull}(E)$, which form a subset of E, are called **vertices** of P. If $x \in \partial P$ then x is a vertex if and only if the normal cone $\mathbf{N}^*{}_x(P)$ has non-empty interior. It is easy to see that the interiors of $\mathbf{N}^*{}_x(P)$ are disjoint as x varies over the vertices of P and that the union of the closures is all of $\mathbb{R}^d$. [Note that I used $\mathbb{R}^d$ for the ambient space of the normal cone, rather than $(\mathbb{R}^d)^*$; that is because we will be considered polytopes in $(\mathbb{R}^d)^*$.]

The set of exponents of monomials of a Laurent poplynomial, f, is a finite subset $E \subseteq \mathbb{Z}^d \subseteq (\mathbb{R}^d)^*$. The **Newton polytope** is defined to be the convex hull in $(\mathbb{R}^d)^*$ of all exponents of monomials of f:

$$\mathsf{P}(f) := \mathsf{hull}\{\mathbf{r} : a_\mathbf{r} \mathbf{z}^\mathbf{r} \text{ is a nonzero monomial of } f\}\,.$$

Proposition 8.11 ([GKZ94, Ch. 6, Prop. 1.7 and Cor. 1.8]). *The vertices (extreme points) of* $\mathsf{P}(f)$ *are in bijective corresondence with connected components of* $\mathbb{R}^d \setminus \mathsf{amoeba}(f)$ *containing an affine convex cone with non-empty interior.*

Proof: Let $\mathbf{p} \in E$ be a vertex of $\mathsf{P}(f)$. We may write

$$f(\mathbf{z}) = a_\mathbf{p} \mathbf{z}^\mathbf{p} \cdot \left(1 + \sum_{\mathbf{p} \neq \mathbf{r} \in E} \frac{a_\mathbf{r}}{a_\mathbf{p}} \mathbf{z}^{\mathbf{r}-\mathbf{p}}\right) := a_\mathbf{p} \mathbf{z}^\mathbf{p}(1 + g(\mathbf{z}))\,.$$

the Laurent series

$$G = a_\mathbf{p}^{-1} \mathbf{z}^{-\mathbf{p}} \left(1 - g + g^2 - \cdots\right)$$

is formally well defined (see Remark 8.9).

> Claim: There is a translation $\mathbf{b} + \mathbf{N}^*{}_\mathbf{p}(\mathsf{P}(f))$ of the normal cone to the Newton polytope at $\mathbf{p}$, such that G converges to the holomorphic function $1/f$ for all $\mathbf{z}$ with $\mathrm{ReLog}\,\mathbf{z} \in \mathbf{b} + \mathbf{N}^*{}_\mathbf{p}(\mathsf{P}(f))$.
>
> Proof: By its explicit description, the series G converges to $1/f$ wherever $|g(x)| < 1$. Fix any $\mathbf{u}$ be in the interior of $\mathbf{N}^*{}_\mathbf{p}(\mathsf{P}(f))$, so by definition, $\mathbf{u} \cdot (\mathbf{r} - \mathbf{p}) < 0$ for any $\mathbf{r} \neq \mathbf{p}$

in the Newton polytope. Applying this to the finitely many $\mathbf{r} \neq \mathbf{p}$ in E, we see that we may choose $\lambda > 0$ large enough so that

$$\log \sum_{\mathbf{p} \neq \mathbf{r} \in E} |a_{\mathbf{r}}| + \lambda \sum_{\mathbf{p} \neq \mathbf{r} \in E} \mathbf{u} \cdot (\mathbf{r} - \mathbf{p}) < 0 \,.$$

This implies that $|g(\mathbf{z})| < 1$ whenever $\mathrm{ReLog}\,|\mathbf{z}| = \lambda \mathbf{u}$. In fact, if $\mathrm{ReLog}\,\mathbf{z} = \lambda \mathbf{u} + \mathbf{v}$ for $\mathbf{v} \in \mathbf{N}^*{}_{\mathbf{p}}(\mathsf{P}(f))$ then $\mathbf{v} \cdot (\mathbf{r} - \mathbf{p}) < 0$ implies that $|g(\mathbf{z})| < 1$ as well. We conclude that $|g(\mathbf{z})| < 1$ when $\mathrm{ReLog}\,\mathbf{z} \in \lambda \mathbf{u} + \mathbf{N}^*{}_{\mathbf{p}}(\mathsf{P}(f))$.

Now that we have a Laurent expansion of $1/f$ convergent on a translate of $\mathbf{N}^*{}_{\mathbf{p}}(\mathsf{P}(f))$, we let $B(\mathbf{p})$ denote the component of the complement of $\mathsf{amoeba}(f)$ containing this affine cone. If two cones have intersecting interior, then their affine cones intersect as well. The closures of the cones $\mathbf{N}^*{}_{\mathbf{p}}(\mathsf{P}(f))$ cover $\mathbb{R}^d$ as $\mathbf{p}$ varies over vertices of $\mathsf{P}(f)$, hence any projective cone with non-empty interior intersects some $\mathbf{N}^*{}_{\mathbf{p}}(\mathsf{P}(f))$. Translates of projective cones with intersecting interiors intersect, hence any component of the complement of $\mathsf{amoeba}(f)$ containing an affine cone with non-empty interior intersects some $B(\mathbf{p})$, hence is equal to some $B(\mathbf{p})$. To finish the argument, the reader is referred to the short argument in [GKZ94, page 196] showing that the components $B(\mathbf{p})$ are distinct for distinct vertices $\mathbf{p}$. $\square$

Let B be any convex set containing a translate of the projective cone K. If $\mathbf{x} \in \partial B$ then $\tan_{\mathbf{x}}(B) \supseteq K$. [To see this, note that there is a $\mathbf{b}$ such that $\lambda \mathbf{u} + \mathbf{b} \in B$ for any $\lambda > 0, \mathbf{u} \in K$; any half-space containing B is of the form $\{\phi < c\}$ for some $c \geq 0$ and ϕ linear, and we see that $\phi(u) \leq 0$, hence $\mathbf{u} \in B$.] From this, we obtain:

Proposition 8.12. *If the component B of the complement of $\mathsf{amoeba}(f)$ corresponds to a vertex of the Newton polytope, that is, if $B = B(\mathbf{p})$, then $\mathbf{x} \in \partial B(\mathbf{p})$ implies $\tan_{\mathbf{x}}(B) \subseteq \mathbf{N}^*{}_{\mathbf{p}}(\mathsf{P}(f))$.* $\square$

A case arising frequently with generating functions is when the Laurent polynomial f is an ordinary polynomial with nonzero constant term. In this case $\mathbf{0}$ is a vertex of $\mathsf{P}(f)$. All other points of $\mathsf{P}(f)$ are in the nonnegative orthant, so $N_{\mathbf{0}}(\mathsf{P}(f))$ contains the negative orthant. Another common case is when f has a nonzero constant term and there are weights $w_1, \ldots, w_d \in \mathbb{R}^+$ such that for any nonconstant monomial $a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ appearing in f, $\sum w_i m_i > 0$. This case is very similar to the previous case. Again $\mathbf{0}$ is a vertex of $\mathsf{P}(f)$. In both cases, assuming without loss of generality that the constant term of f is 1, one may write $1/f = 1/(1+g) = 1 - g + g^2 - \cdots$, in which each monomial appears only finitely often. In fact, an invertible affine change of coordinates $\mathbf{m} \mapsto L\mathbf{m}$ maps $\mathsf{P}(f)$ into the nonnegative orthant. For example, if $f = 1 - (x + x^{-1} + y + y^{-1})z/2 + z^2$ as in the Aztec diamond generating function, then $(i, j, k) \mapsto (i + k, j + k, k)$ maps $\mathsf{P}(f)$ into the nonnegative orthant. This corresponds to the change of variables $z = xyz'$, which maps f to the ordinary polynomial $1 - (x^2y + xy^2 + x + y)z'/2 + x^2y^2(z')^2$. In either of these cases, the domain of convergence of $1/f$ is $|g| \leq 1$. The compenent of the complement of the amoeba that corresponds to the vertex $\mathbf{0}$ of the Newton polytope is the one containing the (affine image of the) negative orthant.

Finally, I would like to close Part II, the "background material", part, by showing the connection to the first step in extracting asymptotics from multivariate rational generating functions. For any domain $B \subseteq \mathbb{R}^d$ and any $\mathbf{r} \in (\mathbb{R}^d)^*$, let

$$m(\mathbf{r}, B) := \inf\{-\mathbf{r} \cdot \mathbf{x} : \mathbf{x} \in B\}\,.$$

Proposition 8.13.

(i) If a function F is holomorphic in $\log^{-1}(B)$ then the Laurent expansion of $F = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ on B satisfies

$$a_{\mathbf{r}} = O(e^{\lambda|\mathbf{r}|})$$

for any $\lambda > m(\hat{\mathbf{r}}, B)$.

(ii) In particular, if F is a quotient of Laurent polynomials, then this holds where B is the component of $\mathbb{R}^d \setminus \mathsf{amoeba}(f)$ for a given Laurent expansion.

(iii) If F is a quotient of Laurent polynomials and $m(\hat{\mathbf{r}}, B) = -\infty$, then $a_{\mathbf{r}} = 0$. (In general, we can conclude only that $a_{\mathbf{r}}$ is super-exponentially decreasing.)

PROOF: For $\lambda > m(\hat{\mathbf{r}}, B)$ there exists $\mathbf{x} \in B$ with $-\hat{\mathbf{r}} \cdot \mathbf{x} < \lambda$. Cauchy's formula for $a_{\mathbf{r}}$, with chain of integration $T(\mathbf{x}) = \mathrm{ReLog}^{-1}(\mathbf{x})$, gives (i), with (ii) as a corollary. For the third part, we remark first that if $m(\hat{\mathbf{r}}, B) = -\infty$, then the function $\hat{\mathbf{r}} \cdot \mathbf{x}$ is unbounded from below on the set $B^{(\epsilon)}$ of points whose ϵ-neighborhood is contained in B. Hence we may choose $\mathbf{x}_n \in B^{(\epsilon)}$ with $\hat{\mathbf{r}} \cdot \mathbf{x}_n < -n$. There is a polynomial bound $\mathbf{z}^{-\mathbf{1}} F(\mathbf{z}) \le P(n)$ for $\mathbf{z} \in T(\mathbf{x}_n)$. The volume of the torus of integration is exactly $(2\pi)^n$; together with the estimates $\mathbf{z}^{-\mathbf{1}} F(\mathbf{z}) \le P(n)$ and $|\mathbf{z}^{-\mathbf{r}}| \le e^{-n|\mathbf{r}|}$ this implies

$$a_{\mathbf{r}} \le P(\mathbf{z}) e^{-|\mathbf{r}|n})$$

for any $n > 0$. The polynomial P does not depend on n, so taking $n \to \infty$ gives $a_{\mathbf{r}} = 0$. □

Notes

The study of amoebas and the origin of the term "amoeba" are generally credited to Gelfand, Kapranov and Zelevinsky [GKZ94]. This seminal text on discriminants devotes much of Chapter 6 to amoebas and Newton polytopes. Their development of the basic results in Section 6.1 begins by quoting without proof some basic facts about Laurent series akin to Theorem 8.4 (their Proposition 1.5 of Chapter 6). The reference they give, namely [Kra92], proves these only for ordinary power series, and the resulting wild-goose chase led me to write down more complete developments of these basic facts. I first learned many of these by sitting in on a graduate course given by L. Matusevich circa 2005. A more sophisticated, yet quite readable development may be found in [Rul01]. Other helpful sources include [The02, Mik01b, Mik01a].