Chapter 9

Decomposition into quasi-local cycles

In this chapter, Morse theory is our guide as we see how to make judicious choices of contours of integration in order to evaluate the Cauchy integral. At present, the status of the Morse theoretic arguments is just this: a guide. As a precondition for the use of stratified Morse theory, in the non-proper case (which is what we need), one must establish that the space under consideration admits a stratification compatible with the given height function. Unfortunately, we do not yet know how to do this in general (see Conjecture 9.1 below). One must also establish that the height function is Morse, which is not always true (though we can often get around this, using perturbations). Finally, the theory is quite technical and existing results in sources such as [GM88] are often difficult to lift because their hypotheses and conclusions have "big-picture" definitions, spread throughout many sections of the text. The upshot of this is that I will give a streamlined presentation geared toward showing what should be true. In subsequent chapters, the theorems whose truth we suspect will be proved via explicit deformations. An advantage is that these explicit deformations will be useful for computational purposes. This chapter begins with a summary of of the Morse theoretic concepts and facts we will need. A more detailed development is given in the Appendices.

9.1 Morse theory redux

Smooth Morse theory

Smooth (ordinary) Morse theory decomposes the topology of a manifold X equipped with a smooth height function $h : X \to \mathbb{R}$. The fundamental results are these. Say that $x \in X$ is a **critical point** if $dh(x) = 0$. The value $c := h(x)$ is a **critical value** and h is said to have distinct critical values if $h(x) \neq h(y)$ for critical points $x \neq y$. The function h is said to be a **Morse function** if at every critical point x, the matrix of second derivatives of h (in any chart map) is nonsingular. The signature of this matrix is called the **index** of x. Let X^c denote the subspace $\{x \in X : h(x) \le c\}$. The first fundamental result is (see Lemma 16.3

Lemma 9.1 (Morse lemma). *If a Morse function h has no critical values in $[a, b]$, then X^a is a strong deformation retract of X^b. In particular, the homotopy types of X^t are all naturally identified for $a \le t \le b$.* □

The second fundamental result tells us how the homotopy type of X^t changes as t increases past a critical value; see Theorem 16.4

Theorem 9.2 (Description of attachment). *Suppose that $h^{-1}[a, b]$ is compact and contains precisely one critical point $\mathbf{p}$, with critical value $h(\mathbf{p})$ strictly between a and b. Then the space X^b has the homotopy type of X^a with a λ-cell attached along its boundary, where λ is the index of the critical point $\mathbf{p}$; when $\lambda = 0$, the boundary is taken to be empty.* □

Stratified Morse theory

A definition of Morse function exists for compact stratified spaces. Essentially, it must be a smooth Morse function on each stratum and must obey a condition analogous to the second Whitney stratification condition; details are given in Definition 16.19. Let $h = h_{\hat{\mathbf{r}}} = \sum_{j=1}^d -\hat{r}_j \log |z_j|$. Let $\mathcal{M} := (\mathbb{C}^*)^d \setminus \mathcal{V}$ denote the domain of holomorphy of a rational function with pole variety $\mathcal{V}$. The space $\mathcal{M}$ is a manifold but it is not compact. There is, however, a notion of a stratified Morse function in the non-compact setting.

Definition 9.3 (stratified compactification). *Let X be any manifold and let h be any smooth function on X. A stratified Morse compactification of (X, h) is an embedding $X \hookrightarrow Z$ of X as a dense subset of a compact stratified space Z such that X is one stratum of Z and h extends continuously to an extended-real value Morse function $\overline{\overline{h}} : Z \to [-\infty, \infty]$. We will usually denote $\overline{\overline{h}}$ simply by h. If a stratified Morse compactification exists for (X, h), we will say that X is a* ***non-compact stratified space*** *with Morse function h.*

In order to apply the machinery of stratified Morse theory to our pair $(\mathcal{M}, h_{\hat{\mathbf{r}}})$ it is required to have a stratified compactification. One obvious step in consructing such a compactification is to add back the points of $\mathcal{V}$, or at least those not on the coordinate hyperplanes. The stratification of Z will induce a stratification of $\mathcal{V}$, so h must be a stratified Morse function on $\mathcal{V}$. We may also add back the coordinate hyperplanes giving h the value $+\infty$ there. So far, we will have constructed a complete space but not a compact one because we have not added any points at infinity. Most points at infinity will have $h = -\infty$ but some, near the coordinate hyperplanes will not. The most obvious way to finish the compactification is to embed $\mathcal{M}$ into the closure of the graph of h on $\mathcal{M}$. At this point, we do not know whether h satisfies Definition 16.19 on this compactification.

Conjecture 9.1. *There is a stratified Morse compactification (Z, h) of $(\mathcal{M}, h)$.*

Given a stratified space Z and a Morse function $h : Z \to \mathbb{R}$, we say that x is a **critical point** for h if it is a critical point for the restriction $h|_S$ of h to some stratum S. The analogue of the Morse lemma is as follows (see Lemma 16.20 below).

Lemma 9.4 (Stratified Morse lemma). *Let X be a non-compact stratified space with stratified compactification (Z, h). Let $a < b$ be real numbers and suppose the interval $[a, b]$ contains no critical values of h. If also $h^{-1}[a, b]$ is compact, then the inclusion $X^a \hookrightarrow X^b$ is a homotopy equivalence.* $\square$

The crowning achievement of stratified Morse theory is to explain how the topology of X^t changes as t increases past a critical value. It appears somewhat complicated, but only because the attachment type is in fact complicated! In what follows, X is the complement of an algebraic hypersurface, $\mathcal{V}$.

Given a point $\mathbf{p}$ in a stratum S, define the **normal slice** to S at $\mathbf{p}$, denoted $N_S(\mathbf{p})$, to be the intersection of X with a small disk D about $\mathbf{p}$ in the orthogonal complement to the tangent plane to S at $\mathbf{p}$: thus $N_S(\mathbf{p}) = D \setminus \mathcal{V}$. Here we think of the tangent plane and its orthogonal complement as an actual linear subspace of the ambient space. The definition of a stratification ensures that the slices $N_S(\mathbf{p})$ are all diffeomorphic as $\mathbf{p}$ varies over a stratum S. The boundary of the normal slice is called the **normal link**, denoted $L_S(\mathbf{p}) := \partial D \setminus \mathcal{V}$. Let $\mathbf{p}$ be the unique critical point with critical value $c \in [a, b]$. We define the normal Morse data and tangential Morse data at $\mathbf{p}$ respectively to be the topological pairs

$$\begin{aligned} \textsf{N-data} &:= (N_S(\mathbf{p}), N_S(\mathbf{p}) \cap X^a) \\ \textsf{T-data} &:= (B^\lambda, \partial B^\lambda) \end{aligned}$$

where B^λ denotes a ball of dimension λ and $d-\lambda$ is the index of the critical point $\mathbf{p}$. The retraction of $N_S(\mathbf{p})$ onto $L_S(\mathbf{p})$ shows that the normal Morse data is also homotopy equivalent to $(L(\mathbf{p}), \ell^-(\mathbf{p}))$, where $\ell^-(\mathbf{p})$ is $(L_S(\mathbf{p}), L_S(\mathbf{p}) \cap X^c)$ where $c = h(\mathbf{p})$ is the critical value. Excising $X^{c-} := \{y \in X^c : h(y) < c\}$ gives

$$\textsf{N-data} \simeq (\ell^+(\mathbf{p}), \ell^0(\mathbf{p})) \tag{9.1}$$

where $\ell^(\mathbf{p}) := L_S(\mathbf{p}) \cap h^{-1}[c, b]$ and $\ell^0(\mathbf{p}) := L_S(\mathbf{p}) \cap h^{-1}\{c\}$.

Theorem 9.5 (Description of attachment for the complement of a stratified space). *Let X be the complement in $\mathbb{R}^d$ or $\mathbb{C}^d$ of the stratified space $\mathcal{V}$. Let h be a Morse function on $\mathcal{V}$ and let $\mathbf{p}$ be a critical point of h. Let $[a, b]$ be an interval such that $\mathbf{p}$ is the only critical point for h with critical value in $[a, b]$. Then the space X^b has the homotopy type of X^a with a pair (Y, Z) attached by a map $\pi : Z \to X^a$, where*

$$(Y, Z) := \textsf{N-data} \times \textsf{T-data}\,.$$

Expanding this and using (9.1)*, the attachement type may be written as*

$$(\ell^(\mathbf{p}) \times B^\lambda\,,\, \ell^0(\mathbf{p}) \times B^\lambda \cup \ell^+(\mathbf{p}) \times \partial B^\lambda)\,.$$

Definition 9.6. *Let $H_{d,\mathbf{p}}(\mathcal{M})$ denote the homology of any pair with the homotopy type of (Y, Z) in the theorem.*

The theorem is actually a little more specific than is stated above. The homology of $(B^\lambda, \partial B^\lambda)$ is that of a λ-sphere: rank one in dimension λ and zero everywhere else. The Künneth product formula (Theorem 4.11 and Corollary 4.12) is simple when one of the factors is a homology λ-sphere. The homotopy equivalence of $(\mathcal{M}^b, \mathcal{M}^a)$ to $\mathsf{N\text{-}data} \times \mathsf{T\text{-}data}$ translates, via the Künneth formula, to a natural isomorphism between $H_d(\mathcal{M}^b, \mathcal{M}^a)$ and $H_{d-\lambda}(N_s(\mathbf{p}), N_S(\mathbf{p}) \cap \mathcal{M}^a)$. Locally, X is a cartesian product of $N_S(\mathbf{p})$ with S, and the isomorphism is via the cartesian product of a class $\mathcal{C}_\perp \in H_{d-\lambda}(N_S(\mathbf{p}), N_S(\mathbf{p}) \cap X^a)$ with the unique homology generator of $(S, S \cap \mathcal{V}^a)$. This unique generator may be taken to be $\mathcal{C}_{||} := (B^\lambda, \partial B^\lambda)$, where B^λ is a disk whose tangent space is the λ-dimensional subspace on which h is negative definite near $\mathbf{p}$. Note that $h < c - \epsilon$ on $\partial \mathcal{C}_{||}$. The theorem, therefore, tells us that $H_d(\mathcal{M}^b, \mathcal{M}^a)$ has a basis represented by chains $\mathcal{C}$, each of which is a direct sum, in a coordinate system local to $\mathbf{p}$,

$$\mathcal{C} = \mathcal{C}_{||} \times \mathcal{C}_\perp ,$$

where $\mathbb{C}_\perp$ is an absolute or relative cycle **local** to $\mathbf{p}$, in the sense that a cycle representative may be chosen in an arbitrarily small neighborhood of $\mathbf{p}$. We call the cycle $\mathcal{C}$ is what we call a **quasi-local** cycle, because $\mathbb{C}_\perp$ is a local cycle and $\mathbb{C}_{||}$ is a saddle-point contour reaching maximum height c at $\mathbf{p}$.

Pushing the contour down

Let us recall some notation from previous chapters. The function $F(\mathbf{z}) = G/H$ is a d-variable generating function with denominator vanishing at the singular variety $\mathcal{V}$. Given a vector index $\mathbf{r} \in (\mathbb{Z}^+)^d$, the d-form $\omega = \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) d\mathbf{z}/\mathbf{z}$ is the integrand of the Cauchy integral; its domain of holomorphy is $\mathcal{M} = (\mathbb{C}^*)^d \setminus \mathcal{V}$. The unitized vector $\mathbf{r}/|\mathbf{r}|$ is denoted $\hat{\mathbf{r}}$. The function $h_{\hat{\mathbf{r}}} := \sum_{j=1}^d -r_j \log |z_j|$ will often be abbreviated to h. The critical values of h on $\mathcal{M}$ are denoted $c_1 < \cdots < c_r$ and X denotes the pair $(\mathcal{M}^+, \mathcal{M}^-)$; here $\mathcal{M}^+ = \mathcal{M}^c = \{\mathbf{z} \in \mathcal{M} : h(\mathbf{z}) \leq c\}$ for an arbitrary $c > c_r$ (these all being homotopy equivalent) and $\mathcal{M}^- = \mathcal{M}^c$ for any $c < c_1$. We also denote by $(\mathcal{M}^j, \mathcal{M}^{j-1})$ any space $(\mathcal{M}^b, \mathcal{M}^a)$ with $c_{j+1} > b > c > a > c_{j-1}$; by the Fundamental lemma, all such pairs are naturally homotopy equivalent; thus $\mathcal{M}^r \simeq \mathcal{M}^+$ and $\mathcal{M}^0 \simeq \mathcal{M}^-$.

Let us now recall step (*iii*) of the six-step program given at the end of Section 1.3 for estimating coefficients of multivariate rational generating functions. The idea behind using Morse theory is that the integral will be easiest to evaluate once the chain of integration is repositioned so that the maximum modulus of the integrand is as small as possible. This is a well known heuristic: the magnitude of the integral cannot be much more than this minimax modulus, therefore, if the modulus of the integrand is much greater than this, there will be a lot of cancellation and the integral will be difficult to evaluate. We hope that the Morse theory will tell us how to "push down" the chain of integration to achieve the minimal maximum modulus. There is more reason to believe this than you may be able to tell from the brief summary of Morse theory that you have seen so far.

The method of proof of the fundamental lemmas is to construct homotopies following gradient flows, that literally push the chain down. Something more complicated happens when passing the critical values, where we have homological but not homotopic chains, and this is precisely where the Morse theoretical methods lose their computational effectiveness. In subsequent sections and chapters, we will work a number of examples that clarify the description the attachment given by Theorem 9.5. For now, let us see why such a description of the topology of $X = \mathcal{V}^c$ is useful in finding a contour homologous to T that minimizes the maximum of h over the contour. It should be stressed that the height function, hence the topological decomposition, depends on the direction of $\mathbf{r}$.

Lemma 9.7 (pushing down). *Given any homology class* $\alpha \in H_d(\mathcal{M}^j, \mathcal{M}^0)$ *there is a unique integer* $j \in [0, r]$ *and a unique homology class* $\alpha^\dagger \in H_d(\mathcal{M}^j, \mathcal{M}^{j-1})$ *such that*

(i) $\alpha^\dagger = 0$ *if and only if* $j = 0$*;*

(ii) the image of $\alpha^\dagger$ *induced by the inclusion* $(\mathcal{M}^j, \mathcal{M}^{j-1}) \hookrightarrow (\mathcal{M}^r, \mathcal{M}^{j-1})$ *is equal to the image* $\pi_*(\alpha)$ *of* α *under the projection* $(\mathcal{M}^r, \mathcal{M}^0) \to (\mathcal{M}^r, \mathcal{M}^j)$*.*

(iii) for all $r \geq j' > j$*, the image of* α *induced by the projection* $(\mathcal{M}^r, \mathcal{M}^0) \to (\mathcal{M}^r, \mathcal{M}^{j'})$ *is zero.*

Proof: Uniqueness of j is immediate because j is the greatest value among 1 and the set of j' such that the projection of α to $H_d(\mathcal{M}^r, \mathcal{M}^{j'-1})$ does not vanish. For the rest, proceed by induction on r. The base case is $r = 1$. If α is nonzero then $j = 1$ and $\alpha^\dagger = \alpha$, whereas if $\alpha = 0$ then $j = 0$. For the induction step, let α be any homology class in $H_d(X)$. If the projection of α onto $H_d(\mathcal{M}^r, \mathcal{M}^{r-1})$ is nonzero then we are forced to take j to be r and $\alpha^\dagger$ to be this projection. Otherwise, recall from the long exact homology sequence for the pair $(\mathcal{M}^r, \mathcal{M}^{r-1})$ that the kernel of the projection $(H_d(\mathcal{M}^r, \mathcal{M}^0) \to H_d(\mathcal{M}^r, \mathcal{M}^{r-1})$ is the image of the inclusion $(\mathcal{M}^{r-1}, \mathcal{M}^0) \hookrightarrow (\mathcal{M}^r, \mathcal{M}^0)$. We see also the long exact homology sequence that the inclusions ι^* in the following diagram are injections because H_d is the highest nonvanishing homology group. Thus α has a unique pullback β to $H_d(\mathcal{M}^{r-1}, \mathcal{M}^0)$. The value of r is one less for this pair, so by the induction hypothesis, there is a unique $j \leq r - 1$ and a unique $\beta^\dagger \in H_d(\mathcal{M}^j, \mathcal{M}^{j-1})$ mapping under ι^* to $\pi_*(\beta)$ in $H_d(\mathcal{M}^{r-1}, \mathcal{M}^{j-1})$. Taking $\alpha^\dagger = \beta^\dagger$ satisfies the conclusion of the lemma.

The top row is exact:

$$\begin{tikzcd}
 & H_d(\mathcal{M}^{r-1}, \mathcal{M}^0) \arrow[r, "\iota_*"] \arrow[d, "\pi_*"] & H_d(\mathcal{M}^r, \mathcal{M}^0) \arrow[r] \arrow[d, "\pi_*"] & H_d(\mathcal{M}^r, \mathcal{M}^{r-1}) \\
H_d(\mathcal{M}^j, \mathcal{M}^{j-1}) \arrow[r] & H_d(\mathcal{M}^{r-1}, \mathcal{M}^{j-1}) \arrow[r, "\iota_*"] & H_d(\mathcal{M}^r, \mathcal{M}^{j-1}) &
\end{tikzcd}$$

If $\alpha^{\dagger\dagger}$ is another such class, following along the bottom row of the diagram, it maps to $\pi_*(\alpha)$. By injectivity of ι^*, it maps to $\pi^*(\beta)$ in $H_d(\mathcal{M}^{r-1}, \mathcal{M}^{j-1})$. By uniqueness of $\beta^\dagger$, we see that $\alpha^{\dagger\dagger} = \alpha^\dagger$, establishing uniqueness and finishing the induction. □

We now apply this with $\alpha = [T]$, where T in the chain of integration in Cauchy's integral formula, taking the height function to be $h = h_{\hat{\mathbf{r}}}$ for some $\mathbf{r}$. We obtain a j and a homology class $\alpha^\dagger \in H_d(\mathcal{M}^j, \mathcal{M}^{j-1})$ such that $\alpha^\dagger$ maps to $\pi_*(\alpha) \in H_d(\mathcal{M}^r, \mathcal{M}^{j-1})$. Let $\mathcal{C}$ be a cycle representing $\alpha^\dagger$. The chain $\mathcal{C}$ is a relative cycle, meaning that $\partial\mathcal{C}$ is supported on $\mathcal{M}^{j-1}$. Equality of the images of α and $\alpha^\dagger$ in $H_d(\mathcal{M}^r, \mathcal{M}^{j-1})$ means that there is a chain $\mathcal{H}$ such that $\partial\mathcal{H} = T - \mathcal{C} - \mathcal{D}$ where $\mathcal{D}$ is supported in $\mathcal{M}^{j-1}$. For any holomorphic d-form ω, we know $d\omega = 0$ and therefore $\int_{\partial\mathcal{H}} \omega = \int_{\mathcal{H}} d\omega = 0$. The result of this is that

$$\int_T \omega = \int_{\mathcal{C}} \omega + \int_{\mathcal{D}} \omega \, .$$

Recall that the integrand ω has a magnitude at the point x of very roughly $\exp(h(x))$. We may retract $\mathcal{M}^j$ to $\mathcal{M}^{c_j+\epsilon}$, reducing the maximum value of h on the image of $\mathcal{C}$ under this retraction to at most $c_j + \epsilon$. The maximum value of h on $\mathcal{D}$ is less than c_j (and in fact may be reduced to $c_{j-1} + \epsilon$). Thus we have replaced T with a contour where $h < c_j + \epsilon$ together with a contour over which the integral of ω is negligible compared to the magnitude of ω on $\mathcal{C}$. This is as well as we can do because T is not homologous to a chain in $(\mathcal{M}^{j-1}, \mathcal{M}^0)$. Going back to the integral, we see that it is $O(\exp(c_j + \epsilon)|\mathbf{r}|)$ for any $\epsilon > 0$. Furthermore, the integral is represented by a quasi-local cycle at the point x with critical value c, that is, by an integral over a chain arbitrarily close to x with boundary in $\mathcal{M}^{c-\epsilon}$.

Let us call the value c_j provided by the pushing down lemma the **contributing exponent** for $\hat{\mathbf{r}}$, denote $c(\hat{\mathbf{r}})$, because we see that for any $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ of $\hat{\mathbf{r}}$ such that $\hat{\mathbf{s}} \in \mathcal{N}$ implies $|a_{\mathbf{r}}| = O(\exp[c(\hat{\mathbf{r}}) + \epsilon]|\mathbf{s}|)$. Up to this point, we have assumed that the Morse function h has distinct critical values at different critical points. If not, then in general $(\mathcal{M}^{b_j}, \mathcal{M}^{a_j})$ is a direct sum of groups $H_{d,\mathbf{p}}(\mathcal{M})$ defined by the relative homology of the pair $(\mathcal{M}^{b_j}, \mathcal{M}_{a_j} \cup \mathcal{N}(\mathbf{p})^c)$ where $\mathbf{p}$ ranges over critical points with value $c(\hat{\mathbf{r}})$ and $\mathcal{N}(\mathbf{p})$ denotes a small neighborhood of $\mathbf{p}$. The cycles $\alpha^\dagger_{\mathbf{p}}$ will be in the local homology group $H_{d,\mathbf{p}}(\mathcal{M})$ and $\alpha^\dagger$ will be an integer combination of these, thought of as an element of the direct sum.

Definition 9.8 (contributing critical points). *Let* $\mathsf{contrib} = \mathsf{contrib}(F, \hat{\mathbf{r}})$ *denote the set of critical points* $\mathbf{p}$ *for* $h_{\hat{\mathbf{r}}}$ *such that* $\alpha^\dagger_{\mathbf{p}}$ *is nonzero.*

9.2 Description of the quasi-local cycles

The Whitney stratification of a complex algebraic variety is a decomposition into algebraic subvarieties (see, e.g., [Mil68, Corollary 2.6]). Thus the strata have real dimension $2k$ for $0 \leq k \leq d$. The tangential Morse data for any stratum U is easy to compute. The function h is the real part of the complex analytic function $\tilde{h}(\mathbf{z}) := -\hat{\mathbf{r}} \cdot \log(\mathbf{z})$. It follows from the Cauchy-Riemann equations that the eigenvectors for the Hessian at a critical point $\mathbf{p}$ come in pairs with eigenvalues of equal magnitudes but opposite signs. In particular, half of the eigenvalues are positive and half are negative. Thus we have:

Proposition 9.9. *The tangential Morse data at any stratum of complex dimension k is $(B^k, \partial B^k)$.*

Putting this together with the discussion following Definition 9.6 yields:

Proposition 9.10. *If x is in a stratum of complex dimension k then $H_{d,x}(\mathcal{M})$ is isomorphic to the $(d-k)$-dimensional (top-dimensional) homology of the normal data, with the map backwards being given by the product with the pair $(B^k \times B^k, B^k \times \partial B^k)$ as described above.* □

Topology at a smooth point

Now suppose that $\mathbf{p}$ is a critical point of h with critical value c and is in the stratum S of smooth points. By the implicit function theorem, the normal slice to S is a punctured disk, B°. In fact the $2d-2$ local coordinates in the computation of ϕ may be completed to $2d$ local coordinates by the real and imaginary parts of H. Since $dh|_S$ is zero but dh never vanishes, we see that we may picture the punctured disk B° as sitting in a vertical plane with its puncture at height c. Referring to the definition of normal Morse data (Definition 16.26) we see that the normal data is $(B^\circ, B^{c-\epsilon})$.

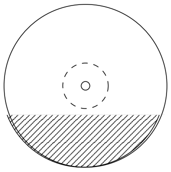

Figure 9.1: the pair $(B^\circ, B^{c-\epsilon})$

Figure 9.1 shows this pair, along with its homology generator. The homology of a punctured disk is generated by a small circle around the puncture in dimension 1 and by any point in dimension 0. Taking this relative to $B^{c-\epsilon}$ gives the reduced homology, which is purely 1-dimensional and is generated by the circle, shown by a dashed line in figure 9.1

We now have a complete description of the relative cycle $\mathcal{C}$ that generates $H_{d,\mathbf{p}}(\mathcal{M})$: it is the product of a small circle in the normal slice around the point where the normal slice intersects S with the $(d-1)$-dimensional ball in S of descending directions for h, modulo its boundary. Figure 9.2 shows an example when $d=2$. Here S has real dimension 2, whence $\mathbf{p}$ is an index-1 saddle for the height function. The tangential data is the arc of steepest descent from $\mathbf{p}$ and the relative cycle $\mathcal{C}$ is the product of the arc with a small circle in the normal slice, looking like a piece of macaroni draped over the saddle. The appearance of $\mathcal{C}$ intersecting S is an illusion due to lack of enough dimensions.

Subsequently, we will evaluate $\int_{\mathcal{C}} \omega$ as an iterated integral:

$$\int_{\mathcal{C}} \omega = \int_{\mathbf{x} \in \mathcal{C}_\parallel} \left(\int_{\mathcal{C}_\perp(\mathbf{x})} \omega \right) . \tag{9.2}$$

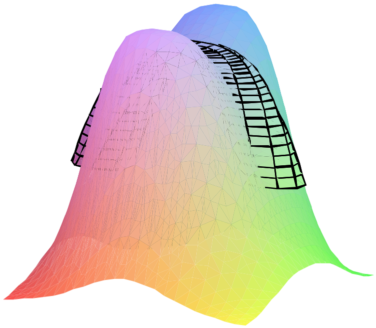

Figure 9.2: the relative cycle that generates $H_{2,\mathbf{p}}(\mathcal{M})$

The inner integral is, in the right coordinate system, a residue of the type one learns to evaluate in a first course on complex variables. The fact that $\mathbf{z}$ is a critical point for h on S will then imply that the outer integral is in the correct position to be evaluated as a saddle point integral. This will be carried out in Sections 10.1 and 10.2.

Topology at a multiple point

Say that $\mathbf{p}$ is a **multiple point** of $\mathcal{V}$ if there is a (local) product representation $\mathcal{V} = \prod_{j=1}^k H_j^{n_j}$ such that the varieties $\mathcal{V}_j := \{\mathbf{z} : H_j(\mathbf{z}) = 0\}$ intersect transversely. Necessarily, $1 \leq k \leq d$ and the stratum S containing $\mathbf{z}$ is a complex manifold of complex dimension $d - k$. We assume $k \geq 2$ because otherwise we are in the smooth case that we have already discussed.

The normal slice is homeomorphic to a ball B about the origin in $\mathbb{C}^k$ and its intersection with $\mathcal{M}$ is the complement of the coordinate hyperplanes in B. The normal Morse data is $(N(\mathbf{p}) \cap X, N(\mathbf{p})^{c-\epsilon} \cap X)$. The space $N(\mathbf{p}) \cap X$ is homeomorphic to the k-fold product of punctured disks $(B^\circ)^k$. This space has homology in every dimension up to k. There is a single generator for $H_k(N(\mathbf{p}) \cap X)$, namely a torus T that is the product of arbitrarily small circles around the origin in each coordinate.

Let $\tilde{X}$ be the k-fold product of the pairs $(B^\circ, (B^\circ)^{c-\epsilon})$. The homology of this k-fold product is that of a k-torus, so it has a single generator α in dimension k. Literally, the product is a pair $((B^\circ)^k, Y)$ where Y is the k-tuples with at least one coordinate having height at most $c - \epsilon$. The space $N(\mathbf{p})^{c-\epsilon}$ is a subset of Y, consequently there is a natural projection of the pair $(N(\mathbf{p}) \cap X, N(\mathbf{p})^{c-\epsilon} \cap X)$ onto $((B^\circ)^k, Y)$. The cycle T maps to α, whence T is nonzero and

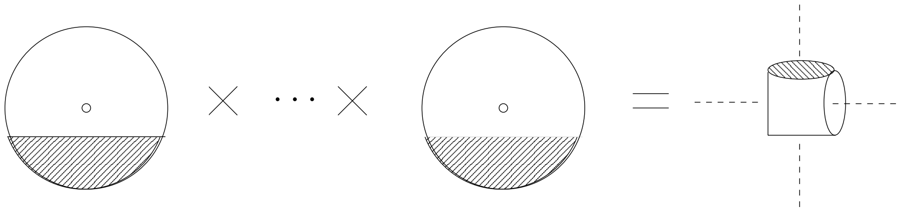

Figure 9.3: the normal morse data is a k-torus

generates $H_k(N(\mathbf{p}) \cap X, N(\mathbf{p})^{c-\epsilon} \cap X)$. Recall from the application of the Künneth formula in Proposition 9.10 that that the tangential Morse data has a single generator in dimension $d-k$. It follows that $H_{d,\mathbf{p}}(\mathcal{M})$ is generated by $T \times (B^{d-k}, \partial B^{d-k})$, where B^{d-k} is the ball in the downward directions in the stratum S. Thus the quasi-local cycle at $\mathbf{p}$ is a piece of generalized macaroni, whose cross-section is an arbitrarily small k-torus and which is draped over the generalized saddle, achieving a quadratically nondegenerate height maximum at $\mathbf{p}$.

Again, we will evaluate the integral as an iteratied integral

$$\int_{\mathcal{C}} \omega = \int_{\mathbf{x} \in \mathcal{C}_{\|}} \left(\int_{\mathcal{C}_{\perp}(\mathbf{x})} \omega \right) .$$

This time, the inner integral is a multivariate residue. The theory involved here is actually not much harder than the theory for univariate integrals. The computation looks very similar to the computation for smooth points and will be carried out in Chapter 11.

9.3 The critical point equations

Computing the critical points of h is surprisingly easy. Let $\mathsf{crit}(\mathbf{r})$, which depends only on $\hat{\mathbf{r}}$, denote the set of critical points of $h_{\hat{\mathbf{r}}}$. Let $\nabla_{\log}$ denote the logarithmic gradient

$$\nabla_{\log} H(\mathbf{z}) := \left(z_1 \frac{\partial H}{\partial z_1}(\mathbf{z}), \ldots, z_d \frac{\partial H}{\partial z_d}(\mathbf{z}) \right) ;$$

this is the gradient of H with respect to $\log \mathbf{z}$, or the gradient of $H \circ \exp$ at $\log \mathbf{z}$; it is also called the **moment map**.

Recall that each $\mathbf{z} \in \mathcal{V}$ is in some stratum S which is a complex k-manifold for some $0 \leq k \leq d-1$. The tangent space to S at $\mathbf{z}$ is naturally identified with $\mathbb{C}^k$, and we define the set of normals to $\mathcal{V}$ at $\mathbf{z}$ to be the complementary $(d-k)$-dimensional complex vector space. Define a linear space $\mathbf{L}(\mathbf{z})$ associated to each point by

Definition 9.11. *Let* $\mathbf{L}(\mathbf{z})$ *denote the logarithmic normal space to* $\mathcal{V}$ *at* $\mathbf{z}$, *namely the vectors* $(z_1 v_1, \ldots, z_d v_d)$ *as* $\mathbf{v}$ *ranges over normals to* $\mathcal{V}$ *at* $\mathbf{z}$.

If $\mathbf{z}$ is a smooth point of $\mathcal{V}$, then $\mathbf{L}(\mathbf{z})$ is the singleton subset of $\mathbb{CP}^{d-1}$ containing the direction $\nabla_{\log} H(\mathbf{z})$; it is possible for $\mathbf{z}$ to be a smooth point while $\nabla H(\mathbf{z})$ vanishes but then H is a power, in which case $\mathbf{L}(\mathbf{z})$ is the singleton containing $\nabla_{\log} \tilde{H}(\mathbf{z})$ where $\tilde{H}$ is the radical of H.

Suppose now that $\mathbf{z}$ is a **multiple point** of $\mathcal{V}$ with (local) product representation $\mathcal{V} = \prod_{j=1}^k H_j$ and the varieties $\mathcal{V}_j$ where H_j vanishes intersecting transversely. The subset of divisors vanishing at $\mathbf{z}$ depend only on the stratum containing $\mathbf{z}$. The logarithmic normal space to $\mathcal{V}$ at a multiple point $\mathbf{z}$ is the span of the vectors $\nabla_{\log} H_k(\mathbf{z})$. If $\mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}$ are d-vectors then membership in the linear span of these is given by $d-k$ linear equations

$$\det(v, w_\Phi^{(1)}, \ldots, w_\Phi^{(k)}) = 0 : \Phi = \Phi_1, \ldots, \ldots, \Phi_{d-k} \tag{9.3}$$

as Φ ranges over subsets of size $k+1$ of $1, \ldots, d$ and the subscript Φ denotes selecting coordinates whose indices are in Φ; one may always choose $\{\Phi_j\}$ so as to give independent linear equations; alternatively one may take

$$\det(Mv, Mw^{(1)}, \ldots, Mw^{(k)}) = 0$$

in the ring $\mathbb{C}[m_{ij} : 1 \le i \le k+1, 1 \le j \le d]$. This reduces to $d-k$ linear, albeit extremely messy equations.

When H is a polynomial, and when $\mathbf{z}$ is a smooth point or a multiple point then the following proposition follows directly from the definitions.

Proposition 9.12. *Suppose that H is a polynomial with no repeated factors. The point $\mathbf{z} \in \mathcal{V}$ is in $\mathsf{crit}_{\overline{\mathbf{r}}}$ if and only if $\overline{\mathbf{r}} \in \mathbf{L}(\mathbf{z})$. Smooth critical points are exactly the solutions to the d polynomial equations*

$$\begin{aligned} H &= 0 \\ r_d z_j \frac{\partial H}{\partial z_j} &= r_j z_d \frac{\partial H}{\partial z_d} \quad 1 \le j \le d-1 \end{aligned} \tag{9.4}$$

which do not have $\nabla H(\mathbf{z}) = \mathbf{0}$. If the stratum S of complex co-dimension k consists only of multiple points then there are $\Phi_1, \ldots, \Phi_{d-k}$ such that the critical points on S are exactly the solutions to the d equations

$$\begin{aligned} H_j &= 0\,, \quad 1 \le j \le k \\ \det(v_\Phi, w_\Phi^{(1)}, \ldots, w_\Phi^{(k)}) &= 0\,, \quad \Phi = \Phi_1, \ldots, \ldots, \Phi_{d-k} \end{aligned} \tag{9.5}$$

not lying on any substratum. □

The following examples illustrate the use of these equations.

Example 9.13 (binomial coefficients continued). The binomial coefficients $\binom{r+s}{r,s}$ have generating function $\dfrac{1}{1-x-y}$. With $H = 1-x-y$ we find the gradient is $(-1,-1)$ which never vanishes, so the variety $\mathcal{V}$ is smooth. The equations (9.4) are

$$\begin{aligned} 1 - x - y &= 0 \\ -sx &= -ry \end{aligned}$$

The solution is $x = \dfrac{r}{r+s}, y = \dfrac{s}{r+s}$. As a function of $\hat{\mathbf{r}}$, we have $\mathbf{z} = \hat{\mathbf{r}}$.

Example 9.14 (Delannoy numbers continued). Recall that the denominator for the Delannoy generating function is given by $H = 1 - x - y - xy$. The gradient is $(-1-y, -1-x)$. To check that $\mathcal{V}$ is smooth, we check that $-1-y, -1-x$ and H never simultaneously vanish. This is verified by the Gröbner basis computation `Groebner[Basis]([-1-x,-1-y,1-x-y-x*y],plex(x,y));`, which returns the basis `[1]`. See Sections 7.1 and 7.2 for more on the use of the Maple package `Groebner`. The critical point equations are the two following equations.

$$\begin{aligned} 1 - x - y - xy &= 0 \\ sx(1+y) &= ry(1+x) \end{aligned}$$

Solving this with

```
Groebner[Basis]([s*x*(1+y)-r*y*(1+x),1-x-y-x*y], plex(x,y));
```

yields $[sy^2 - s + 2ry, s - sy - r + rx]$. The first of the two polynomials is the elimination polynomial for y. Dividing through by $(r+s)$ we see these polynomials are homogeneous in (r,s). Solving the elimination polynomial for y gives

$$y = \frac{-r \pm \sqrt{r^2+s^2}}{s}.$$

Setting the second basis polynomial equal to zero gives x as a function of y: $x = (sy + r - s)/r$. One may avoid messing around with quadratics by computing an elimination polynomial for x directly:

$$x = \frac{-s \pm \sqrt{r^2+s^2}}{r}.$$

This gives four possible (x,y) pairs, of which two solve the second critical point equation: the two positive roots go together and the two negative roots go together. Thus $\mathsf{crit}_{\hat{\mathbf{r}}}$ contains the two points

$$\left(\frac{\sqrt{r^2+s^2} - s}{r}, \frac{\sqrt{r^2+s^2} - r}{s} \right) \text{ and } \left(\frac{-\sqrt{r^2+s^2} - s}{r}, \frac{-\sqrt{r^2+s^2} - r}{s} \right). \tag{9.6}$$

We will see later that the first of these determines the asymptotics of a_{rs}.

Example 9.15 (two intersecting planes). Let $H = H_1 H_2$ with $H_1 := 4 - 2x - y - z$ and $H_2 := 4 - x - 2y - z$. These two planes intersect in the line l containing the points $(0,0,4)$ and $(\frac{4}{3}, \frac{4}{3}, 0)$. Thus the strata are $S_1 := \mathcal{V}_1 \setminus l, S_2 := \mathcal{V}_2 \setminus l$ and $S_3 := l$. Critical points in the strata S_1 are obtained by solving the equations (9.4) for (x,y,z) in terms of (r,s,t). Solving

$$\begin{aligned} 4 - x - 2y - z &= 0 \\ tx &= rz \\ 2ty &= sz \end{aligned}$$

gives $x = 4\hat{r}, y = 2\hat{s}, z = 4\hat{t}$. Finding the critical point on S_2 is analogous and gives $x = 2\hat{r}, y = 4\hat{s}, z = 4\hat{t}$. Critical points on S_3 are obtained by solving (9.5). Solving

$$\begin{aligned} 4 - x - 2y - z &= 0 \\ 4 - 2x - y - z &= 0 \\ ryz + sxz - 3txy &= 0 \end{aligned}$$

yields $x = y = (4/3)(\hat{r} + \hat{s}), z = 4\hat{t}$, which is the unique point on l at which (r, s, t) lies in the plane spanned by the logarithmic tangents $(x, 2y, z)$ and $(2x, y, z)$ to the two planes.

9.4 Minimal points

Along with the unproved status of Conjecture 9.1, the chief obstacle to the stratified Morse method is the difficulty in computing the decomposition of $[T]$ into a basis of quasi-local cycles, or at least of computing the cycle $\alpha^\dagger$ where $\alpha = [T]$ is the torus in the multivariate Cauchy integral. Typically we can compute the critical points, and can compute the quasi-local cycles at each critical point. Usually we can asymptotically compute the integrals over these cycles, but the determination of which of these integrals dominates the asymptotics is exactly the determination of $\alpha^\dagger$, which we do not in general know how to do. There are some cases, however, in which we can indeed do this. One case, the smooth two variable case, is discussed in Section 10.6 below. The rest of this section concerns the only other class of cases which is well understood.

Definition 9.16 (minimal point). *Let $F = P/Q$ be a quotient of Laurent polynomials and let $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ be a Laurent expansion for F on a component B of $\mathsf{amoeba}(Q)^c$. Let $\mathcal{V} := \{\mathbf{z} : Q = 0\}$ denote the pole variety and let $h = h_{\hat{\mathbf{r}}}$ for some $\mathbf{r}$ in the positive orthant of $\mathbb{R}^d$. Say that a critical point $\mathbf{p}$ for h on the stratified space $\mathcal{V}$ is minimal if $\mathrm{ReLog}\,\mathbf{p} \in \partial B$.*

I should remark that the definition of minimal points found in [PW02, PW04] is slightly more restrictive. They assume that P and Q are ordinary polynomials, and that B is the component of $\mathsf{amoeba}(Q)^c$ containing the negative orthant, hence that $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ is an ordinary power series. They also assume that $\mathcal{V}$ intersects the torus containing $\mathbf{p}$ in finitely many points. The advantage of minimal points is that the torus T may be deformed by a simple explicit homotopy (a homothety) until it touches $\mathbf{p}$. It is then much easier to see whether or not the torus can be deformed past $\mathbf{p}$ and, assuming not, what is the topological contribution $\alpha^\dagger \in H_{d,\mathbf{p}}(\mathcal{M})$.

For any point $\mathbf{x} \in \partial B$, we know that the chain of integration can be deformed, within the domain of holomorphy of F, to a chain in $\mathcal{M}^{c+\epsilon}$ where $c = -\hat{\mathbf{r}} \cdot \mathbf{x}$. Therefore, if $c_1 < \cdots < c_r$ are as in Lemma 9.7 and j is as in the conclusion, then $c_j \leq m(\hat{\mathbf{r}}, B)$, the infimum of values of $-\hat{\mathbf{r}} \cdot \mathbf{y}$ on B. Say that $\hat{\mathbf{r}}$ is a **proper direction** if there is a unique point $\mathbf{x}_0(\hat{\mathbf{r}})$ on the boundary of B where $-\hat{\mathbf{r}} \cdot \mathbf{x}$ is minimized. Assume that $\hat{\mathbf{r}}$ is proper. There are two possibilities: either there exists a critical $\mathbf{p} \in \mathsf{contrib}$ with $\mathrm{ReLog}\,\mathbf{p} = \mathbf{x}_0$, or there is no such point. In the second case, we conclude immediately that $c(\hat{\mathbf{r}}) < m(\hat{\mathbf{r}}, B)$, whence the asympotitcs of $a_{\mathbf{r}}$ are not determined by minimal points. In this case, finding $\mathsf{contrib}(\hat{\mathbf{r}})$ requires some as of yet *ad hoc* topological computations. We now state conditions under which the first case holds.

Theorem 9.17. *Suppose* $\hat{\mathbf{r}}$ *is a proper direction with* $-\hat{\mathbf{r}} \cdot \mathbf{y}$ *minimized at* $\mathbf{y} = \mathbf{x}_0$*. Say that* $\mathbf{z} \in \mathcal{V}$ *is a* ***covering point*** *for* $\mathbf{x}_0$ *if* $\mathbf{z}$ *is in the torus* $T(\mathbf{x}_0) := \exp(\mathbf{x}_0 + i\mathbb{R}^d)$ *and if the image of an arbitrarily small neighbohood of* $\mathbf{z}$ *covers a neighborhood of* $\mathbf{x}_0$ *in* ∂B*.*

(i) If all coefficients $a_{\mathbf{r}}$ *are nonnegative then* $\exp(\mathbf{x}_0)$ *is a covering point for* $\mathbf{x}_0$*.*

(ii) Any covering point $\mathbf{z}$ *for* $\mathbf{x}_0$ *is a minimal critical point.*

Proof: First, we observe that the Laurent series for a meromorphic function cannot converge on the boundary of its domain of convergence. To see this, pick any $\mathbf{x} \in \partial B$. The meromorphic function F must have a singularity on the torus $T(\mathbf{x})$: if not, then F is holomorphic on a neighborhood of $T(\mathbf{x})$ and evaluating the coefficients by (8.3) over an appropriate torus $T(\mathbf{x}')$ would imply a bound on $|a_{\mathbf{r}}|$ that forced the domain of convergence of the Laurent series to strictly contain B, contradicting Theorem 8.4. Any singularity of F is a pole, meaning the values of F at nearby points are unbounded, implying non-convergence of the Laurent series. Now, non-negativity of the coefficients imply that $F(\exp(\mathbf{x})) \to +\infty$ as $\mathbf{x}' \to \mathbf{x}$ in B for any $\mathbf{x} \in \partial B$. Hence $\exp(\mathbf{x}) \in \mathcal{V}$ for every $\mathbf{x} \in \partial B$, and (i) follows.

To prove (ii), let $\mathbf{z}$ be a covering point for $\mathbf{x}$.

`finish this`

$\square$

Notes

Exercises