Chapter 10

Smooth point asymptotics

Let $F = G/H$ be a rational generating function with pole variety $\mathcal{V}$. Let $\hat{\mathbf{r}}_*$ be a unit vector in the positive orthant. Suppose that $\hat{\mathbf{r}}_*$ is a proper direction and that $\mathbf{z}_* \in \mathcal{V}$ is a smooth, minimal, critical point for the direction $\hat{\mathbf{r}}_*$. Let B be the component of $\mathsf{amoeba}(Q)^c$ corresponding to the expansion $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. This chapter concerns aymptotic formulae for $a_{\mathbf{r}}$ when $\hat{\mathbf{r}}$ as $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}}$ varying over a neighborhood of $\hat{\mathbf{r}}_*$. Throughout the chapter, we fix the functions F, G, H and the vector $\hat{\mathbf{r}}_*$. For $\mathbf{x} \in \mathbb{R}^n$, denote by $T(\mathbf{x})$ the torus $\exp(\mathbf{x} + i\mathbb{R}^d)$. Thus $T(\mathrm{ReLog}\, \mathbf{z})$ is the torus through $\mathbf{z}$. A preliminary step is to check that $|\mathbf{z}_*^{-\mathbf{r}_*}|$, which appears in the Cauchy integrand, is of the right exponential order to contribute non-negligibly to $a_{\mathbf{r}}$.

Proposition 10.1. *If $\hat{\mathbf{r}}_*$ is a proper direction then $-\hat{\mathbf{r}}_* \cdot \mathbf{x}$ is minimized on B at $\mathbf{x} = \mathrm{ReLog}\, \mathbf{z}$. Consequently (see Proposition 8.13), $a_{\mathbf{r}} = O(|\mathbf{z}_* + \epsilon|)^{-\mathbf{r}_*}$ for any $\epsilon > 0$.*

PROOF: `supply this`. □

In equation (9.2), we begin to carry out step (*vi*) of the program in Section 1.3 by writing the Cauchy integral as an iterated integral. Now we carry out this computation for real. We will do this in two ways. This is not only to reflect the historical development, but also because the methods are useful in different ways. The first method is elementary and direct. It leads to formulae that are explicit, which, though not easy to read or remember, are of great benefit when computing. The second method is stated in more canonical, coordinate-free terms. This yields greater understanding and has been useful for problem-solving, or when we have wanted to extend the theory. Computations in the second framework usually devolve to the first framework once coordinates are chosen, though occasionally the coordinate-free framework allows us to simplify before going to coordinates.

10.1 Residue integral: an explicit construction

The explicit approach was first carried out in [PW02], where the assumption was made that $\mathcal{V}_1 := \mathcal{V} \cap \mathbf{T}(\text{ReLog}\, \mathbf{z}_*)$ was finite (this assumption is referred t in [PW02] as **finite minimality**). In order to clarify the exposition (primarily the notation), we assume in fact that $\mathbf{z}_*$ is the only point of $\mathcal{V}$ in $T(\text{ReLog}\, \mathbf{z}_*)$ (referred to in [PW02] as **strict minimality**). The following construction, quoted from [PW02, Lemma 4.1], represents $a_\mathbf{r}$ as a saddle point integral local to $\mathbf{z}_*$, as $|\mathbf{r}| \to \infty$ as $\hat{\mathbf{r}}$ varies in some neighborhood of $\hat{\mathbf{r}}_*$.

Consider $\mathbf{z}_*$ to be fixed for the remainder of this construction and let $\mathbf{x}_* := \text{ReLog}\, \mathbf{z}_*$. Because $\mathbf{z}_*$ is a smooth point, we know $\nabla H(\mathbf{z}_*) \neq \mathbf{0}$ and we may pick a coordinate k such that $\partial H / \partial z_k(\mathbf{z}_*) \neq 0$. Let $\beta := |(z_*)_k|$ denote the modulus of the k^{th} coordinate of $\mathbf{z}_*$. This approach parametrizes $\mathcal{V}$ locally by the $(d-1)$ coordinates other than the k^{th} one. It will be convenient to let a superscript circle denote the projection of a d-vector onto these $d-1$ coordinates, and to write $\mathbf{z} = (\mathbf{z}^\circ, z_k)$ even though the second coordinate is not appended but insterted into position k. In keeping with this notation, we write T° for the $(d-1)$-dimensional torus $T(\mathbf{x}_*^\circ)$ through $\mathbf{z}_*^\circ$.

Let g parametrize $\mathcal{V}$ by $\mathbf{z}^\circ$ in a neighborhood of $\mathbf{z}_*$. Let me be more explicit about this. By the implicit function theorem, there are a $\delta \in (0, \beta)$, a neighborhood $\mathcal{N}$ of $\hat{\mathbf{z}}_*^\circ$ in T° and an analytic function $g : \mathcal{N}$ such that for $\mathbf{z}^\circ \in \mathcal{N}$, we have

(i) $H(\mathbf{z}^\circ, g(\mathbf{z}^\circ)) = 0$;

(ii) $\beta \leq |g(\mathbf{z}^\circ)| < \beta + \delta$ with equality only if $\mathbf{z}^\circ = \mathbf{z}_*^\circ$;

(iii) $H(\mathbf{z}^\circ, w) \neq 0$ if $w \neq g(\mathbf{z}^\circ)$ and $|w| < \beta + \delta$.

Let C_1 denote the circle of radius $\beta - \delta$ centered at the origin of the complex plane and let C_2 denote the circle of radius $\beta + \delta$. Write the Cauchy integral as an iterated integral:

$$a_\mathbf{r} = \left(\frac{1}{2\pi i}\right)^d \int_{T^\circ} (\mathbf{z}^\circ)^{-\mathbf{r}^\circ} \left[\int_{C_1} w^{-r_k} F(\mathbf{z}^\circ, w) \, \frac{dw}{w}\right] \frac{d\mathbf{z}^\circ}{\mathbf{z}^\circ} \, . \tag{10.1}$$

The key observation is that the inner integral is small away from $\mathbf{z}_*^\circ$. Indeed, for each fixed $\mathbf{z}^\circ \neq \mathbf{z}_*^\circ$, the function $F(\mathbf{z}^\circ, \cdot)$ has radius of convergence greater than β. hence the inner integral is $O(|w_*| + \epsilon)^{-r_k}$ for some $\epsilon > 0$. By continuity of the radius of convergence, a single $\epsilon > 0$ may be chosen for any compact set K not containing $\mathbf{z}_*^\circ$. Going back to the Cauchy integral, we see that

$$|\mathbf{z}_*^\mathbf{r} \, (a_\mathbf{r} - I)| \to 0 \tag{10.2}$$

exponentially rapidly, where I is the integral in (10.1) with T° replaced by any neighborhood of $\mathbf{z}_*^\circ$ in T°.

We have not made use of g yet, but we do so now, choosing the neighborhood that defines I to be the neighbrohood $\mathcal{N}$ on which properties (i)–(iii) hold for g. We will compare I to another integral I' in which the contour C_1 for the inner integral is replaced by C_2:

$$\begin{aligned} I &:= \left(\frac{1}{2\pi i}\right)^d \int_{\mathcal{N}} (\mathbf{z}^\circ)^{-\mathbf{r}^\circ} \left[\int_{C_1} w^{-r_k} F(\mathbf{z}^\circ, w) \, \frac{dw}{w}\right] \frac{d\mathbf{z}^\circ}{\mathbf{z}^\circ} \\ I' &:= \left(\frac{1}{2\pi i}\right)^d \int_{\mathcal{N}} (\mathbf{z}^\circ)^{-\mathbf{r}^\circ} \left[\int_{C_2} w^{-r_k} F(\mathbf{z}^\circ, w) \, \frac{dw}{w}\right] \frac{d\mathbf{z}^\circ}{\mathbf{z}^\circ} \, . \end{aligned}$$

The inner integrand has a unique pole in the annulus $\beta - \delta \leq |w| \leq \beta + \delta$, occuring at $w = g(\mathbf{z}^\circ)$. Therefore, the difference between the inner integrals is $(2\pi i) R(\mathbf{z}^\circ)$, where

$$R(\mathbf{z}^\circ) = g(\mathbf{z}^\circ)^{-r_k} \Psi(\mathbf{z}^\circ)$$

and $\Psi(\mathbf{z}^\circ)$ is the residue in w of the function $F(\mathbf{z}^\circ, w)/w$ at the pole $w = g(\mathbf{z}^\circ)$. From the magnitude of the integrand defining I' we see that

$$|\mathbf{z}_*^{\mathbf{r}} I'| \to 0 \tag{10.3}$$

exponentially in $|\mathbf{r}|$ for $\hat{\mathbf{r}}$ in some neighborhood of $\hat{\mathbf{r}}_*$. Putting together (10.2) and (10.3) proves the following estimate.

Theorem 10.2. *Define*

$$\Xi := I - I' = \left(\frac{1}{2\pi i}\right)^{d-1} \int_{\mathcal{N}} (\mathbf{z}^\circ)^{-\mathbf{r}^\circ} g(\mathbf{z}^\circ)^{-r_k} \Psi(\mathbf{z}^\circ) \frac{d\mathbf{z}^\circ}{\mathbf{z}^\circ} \, , \tag{10.4}$$

Under the condition that $\mathbf{z}_*$ is a smooth, strictly minimal, critical point for $\hat{\mathbf{r}}_*$, the quantity $a_{\mathbf{r}}$ is well estimated by Ξ in the sense that there is a neighborhood of $\mathbf{r}_*$ such that

$$|\mathbf{z}_*^{\mathbf{r}} \, (a_{\mathbf{r}} - \Xi)| \to 0$$

exponentially rapidly in $|\mathbf{r}|$ as $\hat{\mathbf{r}}$ varies over this neighborhood. □

Next, we put this into a more recognizable format by changing variables to $z_j = (z_*)_j e^{i\theta_j}$. Apply this for $j = 1, \ldots, k-1, k+1, \ldots, d$, using $dz_j = i z_j d\theta_j$. Let $\mathcal{N}'$ denote the diffeomorphic image of $\mathcal{N}$ under this change of variables, in other words, an arbitrarily small neighborhood of the origin in $\mathbb{R}^{d-1}$, and let f and ψ denote the functions $\log(g/g_*)$ and ψ respectively after the change of variables, centered by $(i/r_k)\mathbf{r}^\circ \cdot \theta$:

$$\begin{aligned} f(\theta) &:= \log \frac{g(\mathbf{z}_*^\circ \exp(i\theta))}{g(\mathbf{z}_*^\circ)} + \frac{i}{r_k} \mathbf{r}^\circ \cdot \theta \\ \psi(\theta) &:= \Psi(\mathbf{z}_*^\circ \exp i\theta) \, . \end{aligned} \tag{10.5}$$

Corollary 10.3. *Another expression for Ξ in the previous theorem is*

$$\Xi = \left(\frac{1}{2\pi}\right)^{d-1} \mathbf{z}_*^{-\mathbf{r}} \int_{\mathcal{N}'} e^{-r_k f(\theta)} \, \psi(\theta) \, d\theta \, .$$

When $\hat{\mathbf{r}} = \hat{\mathbf{r}}_*$, the function f vanishes to order at least two at the origin, as required for saddle point integration.

Proof: Computing $(2\pi)^{d-1}\Xi$ after the change of variables, and recalling $g(\mathbf{z}_*^\circ) = (z_*)_k$, we find that

$$\begin{aligned}(2\pi)^{d-1}\,\Xi &= \int_{\mathcal{N}'} \left(\mathbf{z}_*^\circ\, e^{i\theta}\right)^{\mathbf{r}^\circ} \left[g(\mathbf{z}_*^\circ)\frac{g(\mathbf{z}^\circ)}{g(\mathbf{z}_*^\circ)}\right]^{r_k} \Psi\left(\mathbf{z}_*^\circ e^{i\theta}\right)\,d\theta \\ &= \mathbf{z}_*^{-\mathbf{r}} \int_{\mathcal{N}'} \exp\left[-r_k\left(i\mathbf{r}^\circ\cdot\theta + \log\frac{g(\mathbf{z}^\circ)}{g(\mathbf{z}_*^\circ)}\right)\right]\,\psi(\theta)\,d\theta \\ &= \mathbf{z}_*^{-\mathbf{r}} \int_{\mathcal{N}'} e^{-r_k f(\theta)}\,\psi(\theta)\,d\theta\,,\end{aligned}$$

proving the first statement. It is obvious that $f(\mathbf{0}) = 0$. To see that $\nabla f(\mathbf{0})$ vanishes as well, we use the definition to compute

$$\frac{\partial f}{\partial\theta_j}(\mathbf{0}) = i\frac{r_j}{r_k} + \frac{i(z_*)_j\,\frac{\partial g}{\partial z_j}(\mathbf{z}_*^\circ)}{g(\mathbf{z}_*^\circ)}\,. \tag{10.6}$$

Implicitly diffentiating $H(\mathbf{z}^\circ, g(\mathbf{z}^\circ)) = 0$ we see that (in subscript partial derivative notation) $g_j = -H_j/H_k$; substituting this into (10.6) and using $g(\mathbf{z}_*^\circ) = (z_*)_k$ yields

$$\frac{\partial f}{\partial\theta_j}(\mathbf{0}) = i\frac{r_j}{r_k} - i\,\frac{H_j}{H_k}\,\frac{(z_*)_j}{(z_*)_k}\,.$$

The verification is finished by recalling from the critical point equations that $\nabla_{\log}H(\mathbf{z}_*) = \lambda\mathbf{r}_*$ and plugging in $\lambda(r_j)_*$ for $H_j(z_*)^j$. □

Example 10.4 (binomial coefficients continued further). Recall from Example 9.13 that $F = 1/(1-x-y)$ and that

$$\mathbf{z}_* := (x_*, y_*) = \hat{\mathbf{r}}_* = \left(\frac{r}{r+s}, \frac{s}{r+s}\right)\,.$$

Neither partial derivative of $H = 1-x-y$ vanishes. Picking $k = 2$, we parametrize $\mathcal{V}$ by $(x, g(x))$ where $g(x) = 1-x$. We compute

$$\Psi(x) = \operatorname{Res}\,(\frac{1}{y(1-x-y)}; y = 1-x) = \frac{1}{1-x}\,\operatorname{Res}\,(\frac{1}{1-x-y}; y = 1-x) = \frac{-1}{1-x}\,.$$

Hence

$$\psi(x) = \frac{-1}{1-x_*e^{i\theta}}$$

and

$$f(x) = i\theta\,\frac{r}{s} + \log\frac{1-x_*e^{i\theta}}{1-x_*}\,.$$

From this expression, the conclusion of Corollary 10.3 is not obvious:

$$\frac{d^2}{d\theta^2}\left(i\theta\frac{r}{s} + \log\frac{1-(r/s)e^{i\theta}}{1-(r/s)}\right) = 0\,.$$

10.2 The residue form: A more natural, less explicit construction

Our second derivation of the saddle-residue integral requires the notion of a residue form. For smooth points, we need only simple residue forms, coresonding to the fact that the inner integral in (10.1) is univariate. A **meromorphic** d-form is one that can be written locally as $(P/Q)\, d\mathbf{z}$ with P and Q holomorphic.

Proposition 10.5. *Let ξ be a meromorphic form, written as $(P/Q)\, d\mathbf{z}$ on a domain $\mathcal{D}$. Let $\mathcal{V}_Q$ be the zero set of Q and suppose that Q has a simple zero everywhere on $D := \mathcal{D} \cap \mathcal{V}_Q$. Then*

$$dQ \wedge \theta = P\, d\mathbf{z} \tag{10.7}$$

always has a holomorphic solution, and the following uniqueness holds: for any representation of ξ as $(P/Q)\, d\mathbf{z}$ and any holomorphic solution θ to (10.7), the restriction $\eta := \iota^*\theta$ induced by the inclusion $D \xrightarrow{\iota} V_Q$ is always the same. We define the residue of ξ on D to be the form η, denoted

$$\mathrm{Res}\,(\xi; D) := \eta\,.$$

Proof: Uniqueness follows from Exercise 4.4: if θ_1 and θ_2 are two solutions then $dQ \wedge (\theta_1 - \theta_2) = 0$ hence $\iota^*\theta_1 = \iota^*\theta_2$.

To prove existence, suppose first that $Q(\mathbf{z}) = z_1$. Then a solution to (10.7) is simply $\theta = P(\mathbf{z})\, dz_2 \wedge \cdots \wedge dz_d$. In the general case, proceeding as in Exercise 4.4, use the complex implicit function theorem [Hör90, Theorem 2.1.2] to find a bi-holomorphic map ψ from a neighborhood of $\mathbf{p}$ to $\mathbb{C}^d$ with first coordinate Q. Use the special case to solve $dz_1 \wedge \theta_0 = P \circ \psi^{-1}\, d\mathbf{z}$; then $\theta := \psi^*(\theta_0)$ solves (10.7). $\square$

For computation, one must introduce coordinates. Although the residue form is natural, and its integral (corresponding to the outer integral in (10.1)) is a scaler and also natural, splitting a form into components on orthogonal spaces is not. Before continuing with the theory, let us have a look at how the residue form might best be understood. The space of holomorphic $(d-1)$-forms on the $(d-1)$-manifold $D \subseteq \mathcal{V}$ is one-dimensional over the space of holomorphic functions. This is the space where $\iota^*\theta$ lives, so the form $\mathrm{Res}\,(\omega; \mathbf{p})$ is a varying multiple $g(\mathbf{z})\xi$ for any generator ξ. The space of holomorphic $(d-1)$-forms on $\mathbb{C}^d$ is, on the other hand, d-dimensional, whence the kernel of ι^* is $(d-1)$-dimensional. This makes it difficult, without further structure, to pick θ in a natural way. Geometrically, one might take advantage of orthogonality of $\mathcal{V}$ to the vector ∇H: pick θ to annihilate any $(d-1)$-tuple of tangent vectors containing one perpendicular to D. This defines θ up to a scalar multiple, after which (10.7) completes the specification of θ.

On the other hand, G and H are defined in terms of the specific variables $z_1, \ldots, z_d$ and it may make sense to sacrifice some naturality in order to remain in these coordinates. Accordingly, we have the following coordinate representation of both θ and $\iota^*\theta$ in which, conveniently, dz_j denotes both a 1-form on $\mathbb{C}^d$ and its pullback to $\mathcal{V}$.

Proposition 10.6. *On a domain in $\mathcal{V}$ where $\partial H/\partial z_1$ does not vanish,*

$$\text{Res}\,(\omega) = \frac{G}{\partial H/\partial z_1}\, dz_2 \wedge \cdots \wedge dz_d\,. \tag{10.8}$$

An analogous result holds with any z_j in place of z_1:

$$\text{Res}\,(\omega) = (-1)^j \frac{G}{\partial H/\partial z_j}\, dz_2 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_d\,.$$

Proof: Taking θ to be the RHS of (10.8), we compute

$$dH \wedge \theta = \left(\sum_{j=1}^d \frac{\partial H}{\partial z_j}\, dz_j\right) \wedge \left(\frac{G}{\partial H/\partial z_1}\, dz_2 \wedge \cdots \wedge dz_d\right)$$

in which sum all terms but one drop out, leaving $G\, d\mathbf{z}$ as desired. The proof for z_j is the same. ☐

Let us apply this to the form $\omega = x^{-r-1}y^{-s-1}(G(x,y)/H(x,y))\, dxdy$ which is the integrand in the two variable Cauchy integral. Wherever $H_x \neq 0$, we may choose the description

$$\text{Res}\,(\omega) = x^{-1-1}y^{-s-1}\frac{G}{H_y}\, dy\,.$$

Example 10.7 (binomial coefficients continued even further). For the binomial coefficients, $G = 1$ and $H = 1 - x - y$, which gives us $\theta = -x^{-r-1}y^{-s-1}\, dy$. Mapping by ι^* does not change the formula. Thus,

$$\text{Res}\,(\omega) = -x^{-r-1}y^{-s-1}\, dy\,.$$

Example 10.8 (Delannoy numbers continued). Here, $F = \dfrac{1}{1 - x - y - xy}$. We have $G = 1, H = 1 - x - y - xy$ and hence

$$\text{Res}\,(\omega) = -x^{-r-1}y^{-s-1}\frac{dy}{1+y}\,.$$

Since the residue lives on $\mathcal{V}$, we may rewrite this any way we like using the relation $1-x-y-xy = 0$. For example,

$$\text{Res}\,(\omega) = x^{-r-1}y^{-s-1}\frac{x\, dy}{y-1}\,.$$

The purpose of a residue form is to decompose integrals into iterated integrals. Recall that the complex orthogonal complement to the tangent space of a smooth complex hypersurface has an orientation induced by the complex structure.

Theorem 10.9 (residue integral theorem). *Let η be meromorphic in a domain $\mathcal{D}$ with a simple pole on the set D. Let $\mathcal{C}$ be a d-chain in $\mathcal{M}$ that is locally the product of a $(d-1)$-chain A supported on D with a circle γ in the normal slice to D, oriented by the complex structure. Then*

$$\int_{\mathcal{C}} \eta = 2\pi i \int_A \text{Res}\,(\eta, D)\,. \tag{10.9}$$

Proof: Decomposing A into a union of chains decomposes each side into a sum, so we may assume without loss of generality that $\mathcal{C}$ is a global product and $\eta = (G(\mathbf{z})/H(\mathbf{z}))\,d\mathbf{z}$. We first prove the result in the case where $H(\mathbf{z})$ is the coordinate function z_1. Compute the integral $\int_{\gamma\times A}(G/z_1)d\mathbf{z}$ as an iterated integral:

$$\begin{aligned}\int_{\gamma\times A}\frac{G}{z_1}\,d\mathbf{z} &= \int_A\left(\int_{\gamma\times\{(z_2,\ldots,z_d)\}}\frac{G}{z_1}dz_1\right)dz_2\wedge\cdots\wedge dz_d\\ &\qquad\text{(formally justified by Exercise 4.1)}\\ &= \int_A(2\pi i)G(0,z_2,\ldots,z_d)\,dz_2\wedge\cdots\wedge dz_d\\ &\qquad\text{by the residue theorem,}\\ &= 2\pi i\int_A \operatorname{Res}(G\,d\mathbf{z}/z_1)\end{aligned}$$

by Proposition 10.6.

The rest is functoriality. Let U be a neighborhood of $\mathbf{p}$ in $(\mathbb{C}^*)^d$ and let $\psi : \mathbb{C}^d \to U$ be a holomorphic map whose inverse gives a coordinatization with the first coordinate H, that is $(\psi^{-1}(\mathbf{z}))_1 = H(\mathbf{z})$. A representative for the class of $\mathcal{C}$ is $\psi_*(\gamma\times A)$. We have

$$\int_{\mathcal{C}}\eta \sim_c \int_{\psi_*(\gamma\times A)}\eta = \int_{\gamma\times A}\psi^*\eta\,.$$

The form $\psi^*\eta$ is a holomorphic multiple of $d\mathbf{z}/z_1$, so applying the previous computation gives

$$\int_{\mathcal{C}}\eta \sim_c 2\pi i\int_A \operatorname{Res}(\psi^*\eta)\,.$$

The residue map is functorial, that is, $\operatorname{Res}(\psi^*\eta) = \psi^*(\operatorname{Res}(\eta))$ (see Exercise 10.1). Hence,

$$\int_{\mathcal{C}}\eta \sim_c 2\pi i\int_A \psi^*(\operatorname{Res}(\eta)$$

which is, by definition, $\int_{\psi_*(A)}\operatorname{Res}(\eta)$. ☐

We will apply this to integrate with $\eta = \omega$, the Cauchy integrand, and $\mathcal{C} = \mathcal{C}_\perp \times \mathcal{C}_\parallel$, the quasi-local cycle. In general, our high-level, Morse theoretic description of the topology of $\mathcal{M}$ requires assumptions we cannot verify, however, when the highest critical points are smooth, minimal points, then we can verify the Morse theoretic description by the following explicit construction.

Suppose $\mathbf{z}^*$ is a (not necessarily strictly) minimal, smooth critical point for h in direction $\hat{\mathbf{r}}_*$. Let $\mathbf{x}'$ be a point inside B and choose $\mathbf{x}''$ outside of B with $\mathbf{x}_*$ on the line segment joining $\mathbf{x}'$ and $\mathbf{x}''$. Let $\mathbf{H} : T_{\text{flat}} \times [0,1]$ denote the homotopy

$$\mathbf{H}(\mathbf{y},t) = \exp(\mathbf{x}'' + t(\mathbf{x}'' - \mathbf{x}') + i\mathbf{y})$$

taking $T(\mathbf{x}')$ to $T(\mathbf{x}'')$.

Lemma 10.10. *If $\mathbf{x}', \mathbf{x}''$ are sufficiently close to $\mathbf{x}_*$, then the image of $\mathbf{H}$ intersects $\mathcal{V}$ transversely.*

PROOF: Let $\mathbf{z} \in \mathcal{V}_1 := \mathcal{V} \cap T(\mathbf{x}_*)$. Then $T_{\mathbf{z}}(\mathcal{V})$ is the complex orthogonal complement to the vector $\vec{\alpha} := \nabla H(\mathbf{z}_*)$. Let W denote the purely imaginary vector space (viewed as a d-dimensional real vector space); the tangent space $T_{\mathbf{z}}(\mathbf{H})$ is the direct sum of W with the one-dimensional subspace spanned by the pointwise product $\vec{\beta}$ of $\exp(\mathbf{x}_*)$ with $(\mathbf{x}'' - \mathbf{x}')$. If $\vec{\alpha}$ is not a complex scalar multiple of a real vector, then $T_{\mathbf{z}}(\mathcal{V}) \cup W$ already spans the whole space. If $\vec{\alpha}$ is a scalar multiple of a real vector $\mathbf{v}$, then $T_{\mathbf{z}}(\mathcal{V}) \cup T_{\mathbf{z}}(\mathbf{H})$ spans the whole space unless $\mathbf{v}$ is parallel to β. This would violate minimality because $\mathbf{x}'$ lies in B. We conclude that $\mathcal{V}$ and $\mathbf{H}$ intersect transversely at all points of $T(\mathbf{x})$. The set at which $\mathcal{V}$ and $\mathbf{H}$ fail to intersect transversely is closed, therefore, choosing $\mathbf{x}''$ close enough to $\mathbf{x}_*$, the entire intersection is transverse (the choice of $\mathbf{x}'$ does not matter because $\mathcal{V}$ does not intersect $\operatorname{ReLog}^{-1}(B)$ at all). ☐

Let σ be a chain representing the intersection of $\mathcal{V}$ and $\mathbf{H}$; then $\partial\sigma$ is supported by $\partial\mathbf{H} \cap \mathcal{V}$ which is a subset of $T(\mathbf{x}'')$. This shows that σ is a relative cycle in the pair $(\mathcal{M}, \mathcal{M}_a)$. Next, we construct a tubular neighborhood of σ: using one or more local coordinate patches as needed, let Σ be the product of σ with a small disk D in the 2-plane orthogonal to $\mathcal{V}$ (smoothly piecing together coordinate patches). The boundary of $\Sigma \cap \mathbf{H}$ is the sum of $\partial D \times \sigma$ with a chain supported in $\mathcal{M}^a$. Now let Z be a singular chain representing $\mathbf{H} \setminus \Sigma$. The chain Z is supported on the domain of holomorpy of the integrand ω. Its boundary ∂Z is equal to $T(\mathbf{x}) - \partial D \times \sigma$ plus a chain supported on $\mathcal{M}^a$. This shows that the projection $\pi_*([T])$ of $[T]$ to $H_d(\mathcal{M}, \mathcal{M}^a)$ is homologous to $\partial D \times \sigma$. It follows that this class does not depend on the parameters of the construction such as $\mathbf{x}', \mathbf{x}'', a$. Consequently, the class of σ in $H_{d-1}(\mathcal{V}, \mathcal{V} \cap \mathcal{M}^a)$ is well defined.

Definition 10.11 (intersection class). *The class of $[\sigma]$ in $H_{d-1}(\mathcal{V}, \mathcal{V} \cap \mathcal{M}^a)$ is called the intersection class of $[T]$ in $\mathcal{V}$. It is an instance of the* ***Thom isomorphism****.*

Summing up, we have the following theorem.

Theorem 10.12. *Let $F = G/H$ and let $\mathbf{x}_*$ be the unique minimizer of $\hat{\mathbf{r}}_* \cdot \mathbf{x}$ on the closure of the component B of $\mathsf{amoeba}(H)^c$. Suppose that $\mathcal{V}_H$ is smooth on a neighborhood of $T(\mathbf{x}_*)$. Then the class $[T]$ projects in $H_d(\mathcal{M}, \mathcal{M}^a)$ to the class $\gamma \times \sigma$ where γ is the class of the circle ∂D, oriented counterclockwise about the origin in the natural orientation of the normal slice to $\mathcal{V}$.* ☐

Putting this together with Theorem 10.9, we arrive at the following result.

Theorem 10.13. *Let $F = G/H$ and let $\mathbf{x}_*$ be the unique minimizer of $\hat{\mathbf{r}}_* \cdot \mathbf{x}$ on the component B of the complement of $\mathsf{amoeba}(H)$. Suppose that $\mathcal{V}_H$ is smooth on a neighborhood of $T(\mathbf{x}_*)$. Let σ be the intersection class for T on $\mathcal{V}$. Then for some $\epsilon > 0$,*

$$\left| a_{\mathbf{r}} - \frac{1}{(2\pi i)^d} \int_\sigma \operatorname{Res}(\omega) \right| = O\left(e^{(c-\epsilon)|\mathbf{r}|}\right) .$$

Proof: Applying first Cauchy's integral theorem, then Theorem 10.12 and Proposition 4.14 we obtain

$$\begin{aligned} a_{\mathbf{r}} &= \int_T \omega \\ &= \int_{\mathcal{C}_{\|} \times \mathcal{C}_{\perp}} \omega + O\left(e^{(c-\epsilon)|\mathbf{r}|}\right) . \end{aligned}$$

The residue integral theorem shows that

$$\int_{\gamma \times \sigma} \omega = 2\pi i \int_{\sigma} \operatorname{Res}(\omega) , \tag{10.10}$$

finishing the proof. □

To finish off the discussion of the residue integral in the smooth case, we would like to see that it is indeed a saddle point integral. Morse theory tells us that the chain σ may be deformed in the direction of lower values of $h_{\hat{\mathbf{r}}_*}$ everywhere except at the critical points. Because there is only one stratum, again an explicit deformation is readily available. Let X be the gradient vector field for $-h_{\hat{\mathbf{r}}_*}$ on $\mathcal{V}$. Let $\Phi(\mathbf{z}, t)$ satisfy $\Phi(\mathbf{z}, 0) = \mathbf{z})$ and $(d/dt)\Phi(\mathbf{z}, t) = X(\Phi(\mathbf{z}, t))$. This is known as the (downward) **gradient flow** on $\mathcal{V}$ for the objective function h. The rest points for this flow are precisely the critical points of h on $\mathcal{V}$. The flow X, run for a short time $\epsilon > 0$, induces a homotopy between σ and a chain σ' supported on $\mathcal{V}$ on which h is maximized precisely at its critical points. When there is precisely one critical point $\mathbf{z}_* \in \mathcal{V}_1$, then the chain σ' is a construction of the generator $\mathcal{C}_{\|}$ for the tangential homology class. More generally, if there is a finite set E of critical points in $\mathcal{V}_1$, then intersections of σ' with small neighborhoods of each are generators of local tangential homology groups whose products with a circle γ generate the groups $H_{d,\mathbf{z}}(\mathcal{M})$.

The following example illustrates the application of Theorem 10.13 to asymptotic estimation of $a_{\mathbf{r}}$.

Example 10.14 (Delannoy numbers continued). To apply (10.10), we first need to identify $\mathcal{C}_{\|}$. By definition, given $\overline{\mathbf{r}}$, $\mathcal{C}_{\|}$ is a local 1-manifold in $\mathcal{V}$ passing through a point $(x, y) \in \mathsf{crit}_{\overline{\mathbf{r}}}$ where the function $h_{\mathcal{V}}$ is critical. We know h will have index 1 at such a point (Proposition 9.9) and thus $\mathcal{C}_{\|}$ is a path element passing through such a point (x, y) in a direction in which h has a nondegenerate maximum. Let us now specialize to the positive real critical point

$$(x_0, y_0) := \left(\frac{\sqrt{r^2 + s^2} - s}{r}, \frac{\sqrt{r^2 + s^2} - r}{s} \right) .$$

I claim that $\mathcal{C}_{\|}$ passes through (x_0, y_0) in a purely imaginary direction. To see this, parametrize $\mathcal{V}$ by the y-coordinate and note that the function h is the real part of the complex analytic function $\tilde{h} := -r \log x - s \log y$ which is real for real arguments. Along the curve $\mathcal{V}$, $\tilde{h}''$ is positive real. Thus the direction of (quadratic) greatest increase of h is the real direction and the direction of greatest decrease is the imaginary direction. Letting $c := h(x_0, y_0) = -r \log x_0 - s \log y_0$ we conclude that

$$\begin{aligned}
\int_{\mathcal{C}} \omega &\sim_c 2\pi i \int_{\mathcal{C}_\|} \text{Res}\,(\omega)) \\
&\sim_c 2\pi i \int_{y_0 - \epsilon i}^{y_0 + \epsilon i} -x^{-1-r} y^{-1-s} \frac{dy}{1+y} \\
&= 2\pi i \int_{-\epsilon}^{\epsilon} \left(\frac{1 - y_0 - it}{1 + y_0 + it} \right)^{-1-r} (y_0 + it)^{-1-s} \frac{i\,dt}{1 + y_0 + it}
\end{aligned}$$

In order to show that it is an actual one-variable integral I have written it as explicitly as possible; again, once coordinates are introduced it is not obvious that $t = 0$ is a stationary phase point. Another way to express the same integral leaves the contour of integration real:

$$\int_{\mathcal{C}_\|} \text{Res}\,(\omega) \sim_c \int_{y_0 - \epsilon i}^{y_0 + \epsilon i} \exp\left[(r+s) h_{\hat{\mathbf{r}}}(x(y), y)\right] \frac{1}{x(y)\, y} \frac{dy}{1+y} \,. \tag{10.11}$$

10.3 Formulae in quadratically nondegenerate cases

When the function f in equation (10.5) is quadratically nondegenerate, the formula in Corollary 10.3 leads directly to an asymptotic estimate. To obtain the leading term, we apply the nondegenerate multivariate saddle point estimate from equation (6.4), with $A = \psi$, $\lambda = r_d$ and $\mathcal{H}$ the Hessian matrix for f (observing also that the d in the theorem is our $d - 1$). The $(d-1)/2$ powers of 2π together with the $1 - d$ powers in Corollary 10.3 give $(2\pi)^{(1-d)/2}$. The square root of the Hessian determinant of f, interpreted as the product of the principal square roots of the eigenvalues, remains in the denominator. To evaluate $A(\mathbf{0}) = \psi(\mathbf{0}) = \Psi(\mathbf{z}_*^\circ)$, we compute the residue of $F(\mathbf{z}_*^\circ, w)/w$ at $w = g(\mathbf{z}_*^\circ) = (z_*)_k$. Because $F/w = G/(Hw)$ has a simple pole at $\mathbf{z}_*$, and because H_k is nonvanishing there,

$$\text{Res}\left(\frac{G}{wH}(\mathbf{z}_*) \; ; \; w \right) = \frac{G(\mathbf{z}_*)}{(z_k)_* \, \partial H / \partial z_k(\mathbf{z}_*)} \,.$$

Putting these together yields Theorem 3.5 of [PW02].

Theorem 10.15 (one quadratically degenerate smooth point). *Suppose* $\mathbf{z}_*$ *is an isolated, strictly minimal, smooth, critical point for* $h_{\hat{\mathbf{r}}}$ *on* $\mathcal{V}$*. Let* f_* *denote the function defined by* (10.5) *and suppose that the Hessian matrix* $\mathcal{H}_*$ *for* f_* *is nonsingular at the origin. Then*

(i) for $\hat{\mathbf{r}}$ *in a neighborhood of* $\hat{\mathbf{r}}_*$*, there is a smoothly varying strictly smooth critical point* $\mathbf{z}(\hat{\mathbf{r}})$ *such that* f*, defined by* (10.5) *has a nonsingular Hessian matrix.*

(ii) There is an asymptotic series

$$a_{\mathbf{r}} \sim \mathbf{z}(\hat{\mathbf{r}})^{-\mathbf{r}} \sum_{\ell=0}^{\infty} C_\ell r_k^{(1-d-\ell)/2}$$

for which the leading term is given by

$$C_0 = (2\pi)^{(1-d)/2} (\det \mathcal{H}_*)^{-1/2} \frac{G(\mathbf{z})}{z_k H_k(\mathbf{z})} . \tag{10.12}$$

The expansion is uniform in a neighborhood of $\hat{\mathbf{r}}$. *This means that for every* $N \geq 1$, *the remainder term*

$$\left| a_{\mathbf{r}} - \mathbf{z}(\hat{\mathbf{r}})^{-\mathbf{r}} \sum_{\ell=0}^{N-1} C_\ell r_k^{(1-d-\ell)/2} \right| = O\left(\mathbf{z}(\hat{\mathbf{r}})^{-\mathbf{r}} r_k^{(1-d-N)/2} \right)$$

uniformly as $\mathbf{r}$ *varies so that* $\hat{\mathbf{r}}$ *stays in the prescribed neighborhood of* $\hat{\mathbf{r}}_*$.

Proof: All that remains to prove is part (i). The critical point equations (9.4) define an algebraic scheme in terms of the parameter $\hat{\mathbf{r}}$. By assumption, at $\hat{\mathbf{r}}_*$, the variety $\mathbf{z}(\hat{\mathbf{r}}_*)$ is zero-dimensional. The set of $\hat{\mathbf{r}}$ for which zero-dimensionality fails is closed, hence zero-dimensionality holds in a neighborhood of $\hat{\mathbf{r}}_*$. Also, we have hypothesized that $\mathbf{z}(\hat{\mathbf{r}}_*)$ is a simple root; multiplicity greater than 1 is also a closed property, whence $\mathbf{z}(\hat{\mathbf{r}})$ is simple in a neighborhood of $\hat{\mathbf{r}}_*$. Failure of strict minimality and vanishing of $\det \mathcal{H}$ are also closed, finishing the proof of (i) and establishing the theorem. □

Already in [PW02] it is observed that if the intersection $\mathcal{V}_1$ of $\mathcal{V}$ with the torus through $\mathbf{z}_*$ is a finite set, E, and if H_k is nonvanishing at all points of E, then an asymptotic series for $a_{\mathbf{r}}$ may be obtained by summing the formulae arising from each point of E. Using Theorem 10.13 and the subsequent discussion, we see that it is not necessary to assume anything about the non-critical points on $\mathcal{V}_1$ other than smoothness.

Corollary 10.16 (finitely many quadratically nondegenerate smooth points). *Let* $F = G/H$ *and let* B *be a component of* $\mathsf{amoeba}(H)^c$. *Suppose that* $\hat{\mathbf{r}}_* \cdot \mathbf{x}$ *is a proper direction with minimizing point* $\mathbf{x}_* \in \partial B$. *Suppose that the set* E_* *of critical points of* $h_{\hat{\mathbf{r}}_*}$ *on* $\mathcal{V}_1 := \mathcal{V} \cap T(\mathbf{x}_*)$ *is finite and nonempty and that* $\mathcal{V}$ *is smooth in a neighboorhood of* $T(\mathbf{x}_*)$. *For each* $\mathbf{z}_* \in E_*$, *let* f_* *denote the parametrizing function defined in* (10.5) *and suppose the corresponding Hessian matrix* $\mathcal{H}^*$ *is nonsingular for each of these. Then these hypotheses hold for* $\hat{\mathbf{r}}$ *in a neighborhood of* $\hat{\mathbf{r}}_*$, *each point of* E *varying analytically with* $\hat{\mathbf{r}}$; *denoting the real and imaginary parts of the logarithm of each* $\mathbf{z} \in E$ *by* $\mathbf{z} = \exp(\mathbf{x} + i\mathbf{y})$, *on this neighborhood there is a uniform asymptotic expansion*

$$a_{\mathbf{r}} \sim \sum_{\ell=0}^{\infty} \exp(-\mathbf{r} \cdot \mathbf{x}) \sum_{\mathbf{z} \in E(\hat{\mathbf{r}})} \exp(-i\mathbf{r} \cdot \mathbf{y})\, C_{\mathbf{z},\ell}\, r_{k(\ell)}^{(1-d-\ell)/2} . \tag{10.13}$$

This is an asymptotic expansion in the sense that the remainder term for the partial sum over $0 \leq \ell \leq N-1$ *is* $O(\exp(-\mathbf{r} \cdot \mathbf{x}) |\mathbf{r}|^{(1-d-N)/2})$; *in particular, it is not asserted that the* $\ell = N-1$ *term is nonzero, however infinitely many of the terms will be nonzero.*

Proof: Compute the integral around each point $\mathbf{z}_* \in E_*$ by residues, as before; summing these contributions and substituting $\mathbf{z}_*^{-\mathbf{r}} = \exp(-\mathbf{r} \cdot \mathbf{x}_*) \exp(-i\mathbf{r}_* \cdot \mathbf{y})$ as $\exp(\mathbf{x}_* + i\mathbf{y})$ varies over points of E_* then recovers (10.13) for $\hat{\mathbf{r}} = \hat{\mathbf{r}}_*$. The failure of each hypothesis occurs on a closed set of $\hat{\mathbf{r}}$, and the formula (10.13) veries smoothly in $\hat{\mathbf{r}}$ with remainder term uniformly bounded in a neighborhood of $\hat{\mathbf{r}}_*$, proving the corollary. $\square$

We turn now to some special cases in which explicit formulae may be written down. The first nontrivial case is a single strictly minimal smooth point in two variables. The following formula, first stated in [PW02, Theorem 3.1], gives the leading asymptotic in terms of the partial derivatives of the numerator and denominator of the given rational function. We replicate those computations here.

Theorem 10.17 (smooth, $d = 2$, nondegenerate). *Let $F = G/H$ be meromorphic and suppose that as $\hat{\mathbf{r}}$ varies in a neighborhood $\mathcal{N}$ of $\hat{\mathbf{r}}_*$ there is a smoothly varying, strictly minimal, smooth critical point $\mathbf{z}(\hat{\mathbf{r}}_*)$ in direction $\hat{\mathbf{r}}_*$. Denoting $\mathbf{z} = (x, y)$ and $\mathbf{r} = (r, s)$, we suppose also that neither G nor the expression*

$$Q(\mathbf{z}) := -y^2 H_y^2 x H_x - y H_y x^2 H_x - x^2 y^2 (H_y^2 H_{xx} + H_x^2 H_{yy} - 2 H_x H_y H_{xy}) \tag{10.14}$$

vanishes for any $\mathbf{z} = \mathbf{z}(\hat{\mathbf{r}})$ with $\hat{\mathbf{r}} \in \mathcal{N}$. Then as $|\mathbf{r}| \to \infty$, uniformly over $\hat{\mathbf{r}} \in \mathcal{N}$,

$$a_{\mathbf{r}} = \left(G + O\left(s^{-1/2}\right)\right) \frac{1}{\sqrt{2\pi}} x^{-r} y^{-s} \sqrt{\frac{-y H_y}{s\, Q}}$$

evaluated at $\mathbf{z} = \mathbf{z}(\hat{\mathbf{r}})$. The square root should be taken to be the $-yH_y$ times the principal square root of $(-yH_y)^3/Q)$.

Proof: Applying Theorem 10.15, with $d = 2$, and using the bivariate notation $\mathbf{z} = (x, y)$, we see that the quantity $(2\pi r_k)^{(1-d)/2} G(\mathbf{z})/z_k H_k$ is equal to $\dfrac{G}{2\pi\, s} x^{-r} y^{-s}/(yH_y)$ so we need only to show that the reciprocal of the determinant of the Hessian matrix for f_* at zero is equal to $(-yH_y)^3/Q$. Because $d - 1 = 1$, the matrix is one-dimensional, so we require

$$Q = (-yH_y)^3 \, f_*''(0)\,, \tag{10.15}$$

which will show as well that the nonvanishing of Q is equivalent to the nonvanishing of $f''(0)$. Going back to the definition of f_* in equation 10.5, we see that $x = \mathbf{z}_*^\circ e^{i\theta} = x_* e^{i\theta}$ so that $f_*(\theta) = \log g(x) + L$ where L is linear and $d/d\theta = ix(d/dx)$. Thus,

$$\begin{aligned} \frac{d^2}{d\theta^2} f_* &= ix \frac{d}{dx}\left(ix \frac{d}{dx} (\log g)\right) \\ &= -x \frac{d}{dx} \frac{x\, g'}{g} \\ &= -x \frac{g' + x g''}{g} + \frac{x^2 (g')^2}{g^2}\,. \end{aligned} \tag{10.16}$$

The derivatives of g may be computed by implicitly differentiating the equation $H(x, g(x)) = 0$. Differentiating twice,

$$\begin{aligned} H_x + H_y\, g' &= 0 \\ H_{xx} + 2g'\, H_{xy} + g''\, H_y + (g')^2\, H_{yy} &= 0\,. \end{aligned}$$

Solving the first equation for g' gives $g' = -H_x/H_y$, after which, solving the second equation for g'' gives

$$g'' = -\frac{1}{H_y}\left(H_{xx} - 2\frac{H_x}{H_y}H_{xy} + \frac{H_x^2}{H_y^2}H_{yy}\right) .$$

Plugging this into (10.16) gives

$$f_*''(0) = \frac{x\left(xH_x^2H_y + g(x)\cdot[H_xH_y^2 + xH_{xx}H_y^2 - 2xH_xH_yH_{xy} + xH_x^2H_{yy}]\right)}{H_y^3\, g(x)^2}$$

Multiplying by $(-yH_y)^3$ and evaluating at $\theta = 0$, $x = x_*$, $g(x) = y_*$ and comparing to the definition of Q in (10.14) establishes that $Q = (-yH_y)^3$ and finishes the proof. ☐

The same way Corollary 10.16 extends Theorem 10.15, we may extend this result to finitely many critical points and arbitrary $\mathcal{V}_1$.

Corollary 10.18. *Assume the hypotheses of Theorem 10.17 but replace the assumption of strict minimality by the assumption that* $\hat{\mathbf{r}}$ *is proper and there is a finite nonempty set* $\mathbf{W}$ *of critical points for* $h_{\hat{\mathbf{r}}}$ *in* $\mathcal{V}_1$*. Then*

$$a_{\mathbf{r}} = \sum_{(x,y)\in\mathbf{W}} \frac{G(x,y)}{\sqrt{2\pi}}\, x^{-r}y^{-s}\sqrt{\frac{-yH_y(x,y)}{s\,Q(x,y)}} + O\left(s^{-1}\left|x^{-r}y^{-s}\right|\right) .$$

☐

Example 10.19 (Binomial coefficients continued once more). Having the formula (10.14), we may ignore the beginnings of the saddle point computation in Examples 10.4 and 10.7 and plug $\mathbf{z}_* = (x_*, y_*) = \hat{\mathbf{r}} = (\frac{r}{r+s}, \frac{s}{r+s})$ from Example 9.13 directly into Theorem 10.17. With $G = 1$ and $yH_y = -y = -s/(r+s)$, we get $Q = xy(x+y)$ and evaluating at $(x,y) = (x_*, y_*)$ gives $x + y = 1$ and $Q = (rs)/(r+s)^2$. Then,

$$a_{\mathbf{r}} \sim \left(\frac{r+s}{r}\right)^r \left(\frac{r+s}{s}\right)^s \sqrt{\frac{2\pi(r+s)}{(2\pi r)(2\pi s)}} .$$

One may recognize the usual approximation via Stirling's formula to the binomial coefficient $\binom{r+s}{r,s}$.

Example 10.20 (Delannoy numbers continued YET once more). Again, we go back to Example 2.29. We recall from (9.6) that

$$(x_*, y_*) = \left(\frac{\sqrt{r^2+s^2}-s}{r}, \frac{\sqrt{r^2+s^2}-r}{s}\right) \quad \text{or} \quad \left(\frac{-\sqrt{r^2+s^2}-s}{r}, \frac{-\sqrt{r^2+s^2}-r}{s}\right) .$$

The first of these is a strictly minimal critical point. To avoid plugging this expressions with radicals into the long expression (10.14) for Q, which would lead to simplifications at which Maple will balk, we solve directly for $-yH_y/Q$ by adding this in to the critical point equations. Thus, after defining H, Hx, Hy, Hxx, Hxy, Hyy and Q, we compute

```
Groebner[Basis] ([s*x*(1+y)-r*y*(1+x),1-x-y-x*y, W^2*Q+y*Hy], plex(x,y,W));
```

The first element of this is the elmination polynomial for the quantity $W := \sqrt{-yH_y/Q}$. Maple tells us this is

$$-s^2r + \left(-4\,r^3 - 4\,r^2s - 4\,s^2r - 4\,s^3\right) W^2 + \left(4\,s^2r + 4\,r^3\right) W^4 .$$

After some simplification, denoting $\rho := \sqrt{r^2+s^2}$, this yields

$$a_{r,s} \sim \left(\frac{r}{\rho - s}\right)^r \left(\frac{s}{\rho - r}\right)^s \sqrt{\frac{1}{2\pi\rho}\,\frac{\sqrt{rs}}{r+s-\rho}} .$$

Gaussian curvature of a real hypersurface

Already in two variables, the explicit expression for Q is somewhat messy. Writing down this type of expression for more variables or for higher order terms in the expansion seems pointless. However, one can write down another version of the formula for C_0 in Theorem 10.15 in terms of the curvature of $\mathcal{V}$. This has the advantage of being coordinate-free; it also helps with conceptual understanding in cases such as Examples 10.26 and 10.27 below. To state this version, we need to review the definition of the Gaussian curvature of a smooth hypersurface, and to extend it to certain complex algebraic hypersurfaces.

For a smooth orientable hypersurface $\mathcal{V} \subset \mathbb{R}^{d+1}$, the Gauss map $\mathcal{G}$ sends each point $p \in \mathcal{V}$ to a consistent choice of normal vector. We may identify $\mathcal{G}(p)$ with an element of S^d. For a given patch $P \subset \mathcal{V}$ containing p, let $\mathcal{G}[P] := \cup_{q \in P} \mathcal{G}(q)$, and denote the area of a patch P in either $\mathcal{V}$ or S^d as $A[P]$. Then the **Gauss-Kronecker** curvature of $\mathcal{V}$ at p is defined as

$$\mathcal{K} := \lim_{P \to p} \frac{A(\mathcal{G}[P])}{A[P]} . \tag{10.17}$$

When d is odd, the antipodal map on S^d has determinant -1, whence the particular choice of unit normal will influence the sign $\mathcal{K}$, which is therefore only well defined up to sign. When d is even, we take the numerator to be negative if the map $\mathcal{G}$ is orientation reversing and we have a well defined signed quantity. Clearly, $\mathcal{K}$ is equal to the Jacobian of the Gauss map at the point p. For computational purposes, it is convenient to have a formula for the curvature of the graph of a function from $\mathbb{R}^d$ to $\mathbb{R}$.

A number of formulae are available in the literature. If Q is a homogeneous quadratic form, we let $||Q||$ denote the determinant of the Hessian matrix of Q; to avoid confusion, we point out that the diagonal elements a_{ii} of this matrix are twice the coefficient of x_i^2 in Q. The determinant will be the same when the coefficients of $||Q||$ may be computed with respect to any orthonormal basis. For our purposes the following formulae are the most useful, the first of which is proved in [BBBP08].

Proposition 10.21 ([BBBP08, Corollary 2.4]). *Let $\mathcal{P}$ be the tangent plane to $\mathcal{V}$ at p and let $\mathbf{v}$ be a unit normal. Suppose that $\mathcal{V}$ is the graph of a smooth function h over $\mathcal{P}$, that is,*

$$\mathcal{V} = \{p + \mathbf{u} + h(\mathbf{u})\mathbf{v} : \mathbf{u} \in U \subseteq \mathcal{P}\} .$$

Let Q be the quadratic part of h, that is, $h(\mathbf{u}) = Q(\mathbf{u}) + O(|\mathbf{u}|^3)$. Then the curvature of $\mathcal{V}$ at p is given by

$$\mathcal{K} = ||Q|| \, .$$

□

Corollary 10.22 (curvature of the zero set of a polynomial). *Suppose $\mathcal{V}$ is the set $\{\mathbf{x} : H(\mathbf{x}) = 0\}$ and suppose that p is a smooth point of $\mathcal{V}$, that is, $\nabla H(p) \neq \mathbf{0}$. Let ∇ and Q denote respectively the gradient and quadratic part of H at p. Let $Q_\perp$ denote the restriction of Q to the hyperplane $\nabla_\perp$ orthogonal to ∇. Then the curvature of $\mathcal{V}$ at p is given by*

$$\mathcal{K} = \frac{||Q_\perp||}{(\sum_{j=1}^d H_j(\mathbf{x})^2)^{d/2}} \, . \tag{10.18}$$

Proof: Replacing H by $|\nabla|^{-1} H$ leaves $\mathcal{V}$ unchanged and reduces to the case $|\nabla H(p)| = 1$; we therefore assume without loss of generality that $|\nabla| = 1$. Letting $\mathbf{u}_\perp + \lambda(\mathbf{u})\nabla$ denote the decomposition of a generic vector $\mathbf{u}$ into components in $\langle \nabla \rangle$ and $\nabla_\perp$, the Taylor expansion of H near p is

$$H(p + \mathbf{u}) = \nabla \cdot \mathbf{u} + Q_\perp(\mathbf{u}) + R$$

where $R = O(|\mathbf{u}_\perp|^3 + |\lambda(\mathbf{u})||\mathbf{u}_\perp|)$. Near the origin, we solve for λ to obtain a parametrization of $\mathcal{V}$ by $\nabla_\perp$:

$$\lambda(\mathbf{u}) = Q_\perp(\mathbf{u}) + O(|\mathbf{u}|^3) \, .$$

The result now follows from Proposition 10.21. □

Gaussian curvature at minimal points of complex hypersurfaces

Suppose now that H is a real polynomial in $d+1$ variables and that p is a minimal smooth point of the corresponding complex algebraic hypersurface. We are interested in the curvature at $\log p$ of the logarithmic image $\mathcal{V}_{\log} := \{\mathbf{x} : H \circ \exp(\mathbf{x}) = 0\}$ of $\mathcal{V}$. The formula (10.18) from Corollary 10.22 is well defined up to a factor of ± 1, unless the denominator vanishes, which certainly may happen in general. However, when p is minimal, that is, $p \in \partial B$ for a component B of $\mathsf{amoeba}(H)^c$, then the gradient of $H \circ \exp$ is a complex scalar multiple of a real vector. This prevents the denominator from vanishing. When $F = G/H$, we may multiply top and bottom by a unit complex number so that $\nabla_{\log} H$ is real; we then define the complex curvature by the same equation, (10.18), with the positive square root chosen. When d is odd, the sign of the curvature corresponds to the unit normal in the direction $\hat{\mathbf{r}}_*(p) := \nabla_{\log} H / |\nabla_{\log} H|$.

Theorem 10.23 (curvature version). *Suppose for each unit vector $\hat{\mathbf{r}}$ in a compact subset $\mathbf{K}$ of the positive unit sphere, there is an isolated, strictly minimal, smooth, critical point $\mathbf{z}_*(\hat{\mathbf{r}})$ for the function $h_{\hat{\mathbf{r}}}$ on $\mathcal{V}$. Let $\mathcal{K}(\hat{\mathbf{r}})$ denote the curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*(\hat{\mathbf{r}})$ and suppose this does not vanish for $\hat{\mathbf{r}} \in \mathbf{K}$. Then*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi |\mathbf{r}|} \right)^{d/2} \mathbf{z}_*^{-\mathbf{r}} \mathcal{K}^{-1/2} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|} + O\left(|\mathbf{r}|^{(-d+1)/2} \right)$$

uniformly as $|\mathbf{r}| \to \infty$ *with* $\hat{\mathbf{r}} \in \mathbf{K}$. *Here the* $-1/2$ *power is taken to be the reciprocal of the product of the principal square roots of the eigenvalues of* Q *in the* negative $\hat{\mathbf{r}}$ *direction.*

Proof: By Cauchy's integral formula, and Theorem 10.13 with $\omega = \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) d\mathbf{z})/\mathbf{z}$, we have

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_\sigma \operatorname{Res}(\omega)\,,$$

where σ is the intersection class of T on $\mathcal{V}$. Let $\mathbf{z} = \exp(\zeta)$ so $d\mathbf{z} = \mathbf{z} d\zeta$, so, pulling a factor of $\mathbf{z}^{-\mathbf{r}} = \exp(-\mathbf{r} \cdot \zeta)$ through the residue, we get

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\tilde{\sigma}} \exp(-\mathbf{r} \cdot \zeta) \operatorname{Res}(\tilde{\mathbf{F}}(\zeta)\, d\zeta)\,, \tag{10.19}$$

where $\tilde{\mathbf{F}} = F \circ \exp$ and $\tilde{\sigma} = \log \sigma$.

Let $\mathcal{P} := T_{\zeta_*} \tilde{\mathcal{V}} := T_{\log \mathbf{z}_*} \log[\mathcal{V}]$ be the logarithmic tangent space. Recall that the (complex) orthogonal complement to $\mathcal{P}$ is $\mathbb{C} \cdot \hat{\mathbf{r}}$. Near ζ_* we may parametrize $\tilde{\mathcal{V}}$ by $\mathcal{P}$: locally we have

$$\tilde{\mathcal{V}} = \{\zeta_* + \zeta_\| + h(\zeta_\|)\hat{\mathbf{r}} : \zeta_\| \in \mathcal{P}\}\,;$$

given ζ, we may project back to the tangent plane by $\zeta_\| := \mathbf{x} - [\hat{\mathbf{r}} \cdot (\zeta - \zeta^*)\hat{\mathbf{r}}]$. Pick an orthonormal basis $\mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(d+1)}$ for $\mathcal{P}$. We use these for local coordinates, writing the general point $\zeta \in \mathbb{C}^{d+1}$ in a neighborhood of ζ_* as

$$\zeta = \zeta_* + u_1 \hat{\mathbf{r}} + \sum_{j=2}^{d+1} u_j \mathbf{v}^{(j)}\,.$$

Computing the residue of $F(\zeta)\, d\zeta$ in these coordinates, using Proposition 10.6, gives

$$\operatorname{Res}(F(\zeta)\, d\zeta) = \frac{G \circ \exp}{\partial H \circ \exp / \partial u_1}\, du_2 \wedge \cdots \wedge du_{d+1}\,.$$

The partial derivative in the direction of the gradient is the magnitude of the gradient, therefore, evaluating at ζ_*,

$$\operatorname{Res}(F(\zeta)\, d\zeta)(\zeta_*) = \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|}\, dA \tag{10.20}$$

where $dA := d\mathbf{u}_\| = du_2 \wedge \cdots \wedge du_{d+1}$ is equal to the oriented holomorphic d-area form for $\tilde{\mathcal{V}}$ as it is immersed in $\mathbb{C}^{d+1}$.

To understand the other factor in the integrand of (10.19), denoting $\lambda := |\mathbf{r}|$ and $\phi(\zeta) := \hat{\mathbf{r}} \cdot \zeta$, equation (10.19) becomes

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\tilde{\sigma}} \exp(-\lambda \phi(\zeta)) \operatorname{Res}(\tilde{\mathbf{F}}(\zeta)\, d\zeta)\,. \tag{10.21}$$

Let Q denote the quadratic part of h. By Proposition 10.21 (or Corollary 10.22) and the subsequent discussion, we see that the curvature $\mathcal{K}$ of $\tilde{\mathcal{V}}$ at the point ζ_* with respect to the unit normal $\hat{\mathbf{r}}$ is given by $||Q||$. As we will see, the eigenvalues of Q do not have positive real parts, which corresopnds to the fact that we will not be integrating in the real direction, so we will need one more change of coordinates in order to apply results on multivariate, quadratically nondegenerate, stationary phase integrals.

Let us compute $\tilde{\sigma}$. There is a lot of freedom when choosing $\mathbf{x}'$ and $\mathbf{x}''$ in the construction of σ in Lemma 10.10. A convenient choice is to make the segment $\overline{\mathbf{x}'\mathbf{x}''}$ parallel to $\hat{\mathbf{r}}$. The real tangent space to $\log \mathbf{H}$ is then the sum of the imaginary $(d+1)$-space and the real 1-space in direction $\hat{\mathbf{r}}$. The tangent space to $\tilde{\mathcal{V}}$ is the sum of the real d-space orthogonal to $\hat{\mathbf{r}}$ and the imaginary d-space orthogonal to $\hat{\mathbf{r}}$. The tangent space to $\tilde{\sigma}$ is the intersection of these, which is the imaginary $(d-1)$-space orthogonal to $\hat{\mathbf{r}}$, which is just $\operatorname{Im}\mathcal{P}$. Represent $\tilde{\sigma}$ as a graph over $\operatorname{Im}\mathcal{P}$. Because $\tilde{\sigma}$ is contained in the linear space $\operatorname{Im}\mathcal{P} + \mathbb{C}\cdot\hat{\mathbf{r}}$, we see that locally there is a unique analytic function $\alpha : \operatorname{Im}\mathcal{P} \to \mathbb{C}\cdot\hat{\mathbf{r}}$ such that $\zeta + \alpha(\zeta) \in \tilde{\sigma}$. Comparing to the previous parameterization we see that $\alpha = h$. The quadratic part of α is therefore equal to Q. Our multivariate integral formulae are in terms of real parametrizations. We therefore reparametrize $\operatorname{Im}\mathcal{P}$ by $\zeta = i\mathbf{y}$ and $d\zeta = i^d\, d\mathbf{y}$. In these coordinates, locally

$$\tilde{\sigma} = \{i\mathbf{y} + h(i\mathbf{y}) : \mathbf{y} \in \operatorname{Re}\mathcal{P}\}\,. \tag{10.22}$$

Using $\hat{\mathbf{r}}\cdot\mathbf{y}_{||} = 0$ and $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$, we obtain

$$\begin{aligned}\phi(i\mathbf{y} + h(i\mathbf{y})) &= \phi(\zeta_*) + h(i\mathbf{y}) \\ &= \phi(\zeta_*) + Q(i\mathbf{y}) + O(|\mathbf{y}|^3) \\ &= \phi(\zeta_*) - Q(\mathbf{y}) + O(|\mathbf{y}|^3)\,.\end{aligned}$$

We know, by minimality that ϕ is a smooth phase function whose real part has a minimum on $\tilde{\sigma}$ at ζ_*, which is $\mathbf{y} = 0$ in the parametrization (10.22). Applying Theorem 6.16 to (10.21) using the evaluation (10.20) then gives

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi}\right)^{d/2} \mathbf{z}_*^{-\mathbf{r}} \left(\det(-Q)\right)^{-1/2} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|} + O\left(|\mathbf{r}|^{(-d+1)/2}\right)\,,$$

where the square root is taken to be the reciprocal of the product of the principal square roots of the eigenvalues of $-Q$ in the positive $\hat{\mathbf{r}}$-direction, all of which have nonnegative real parts. The eigenvalues of $-Q$ in direction $\hat{\mathbf{r}}$ are the same as the eigenvalues of Q in direction $-\hat{\mathbf{r}}$, which finishes the proof of the theorem. □

Again, we may expand without difficulty to include the case where there are finitely many critical points on a minimizing torus.

Corollary 10.24. *Let $F = G/H$ be a quotient of Laurent polynomials. Suppose that for each $\hat{\mathbf{r}}$ in a compact subset $\mathbf{K}$ of the positive unit sphere, the funtion $\hat{\mathbf{r}}\cdot\mathbf{x}$ is uniquely maximized at $\mathbf{x}_* \in \overline{B}$, a component of the complement of* amoeba(H)*, and that the set $\mathbf{W}$ of critical points for $\hat{\mathbf{r}}\cdot\mathbf{x}$ in $\mathcal{V}_1 := \mathcal{V}\cap T(\mathbf{x}_*)$ is finite and non-empty for all $\hat{\mathbf{r}} \in \mathbf{K}$. Let $\mathcal{K}(\mathbf{z})$ denote the curvature at $\log\mathbf{z}$ of $\log\mathcal{V}$*

and suppose this is nonvanishing for all $\mathbf{r} \in \mathbf{K}$*. For each* $\mathbf{z} \in \mathbf{W}(\hat{\mathbf{r}})$ *we denote* $\mathbf{z} = \exp(\mathbf{x}_* + i\mathbf{y})$*. Then*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi|\mathbf{r}|}\right)^{d/2} e^{-\mathbf{r}\cdot\mathbf{x}} \left[\sum_{\mathbf{z}\in\mathbf{W}(\hat{\mathbf{r}})} e^{-i\mathbf{r}\cdot\mathbf{y}} \frac{G(\mathbf{z})}{\nabla_{\log}H(\mathbf{z})} \mathcal{K}(\mathbf{z})^{-1/2} + O(|\mathbf{r}|^{-1/2})\right]$$

uniformly as $|\mathbf{r}| \to \infty$ *with* $\hat{\mathbf{r}} \in \mathbf{K}$ □

Example 10.25 (general quantum random walk). The quantum random walk (QRW), first introduced by [ADZ93], is a quantum version of a classical random walk on an integer lattice. A QRW in d dimensions is specified by the steps, which are a set of d-vectors of some finite cardinality k, and the quantum coin, which is a $k \times k$ unitary matrix. The quantities $p_n(r_1, \ldots, r_d)$ denote the resulting probability amplitudes for the particle to be found at site $\mathbf{r} \in \mathbb{Z}^d$ at time n. Let $\mathbf{x} \in \mathbb{C}^d$ and $y \in \mathbb{C}$, and $\mathbf{z} := (\mathbf{x}, y)$ and form a generating function

$$F(\mathbf{z}) := \sum_{(\mathbf{r},t)\in\mathbb{Z}^d\times\mathbb{Z}^+} p_t(\mathbf{r})\mathbf{x}^{\mathbf{r}} y^t \,.$$

This is shown in, e.g. [BP07, Proposition 3.1] to be a rational function with denominator $\det(I - yM(\mathbf{x})U)$ where U is the unitary coin and M is a diagonal matrix of monomials $\mathbf{x}^{\mathbf{v}}$ where $\mathbf{v} \in \mathbb{Z}^d$ are the steps of the QRW. A consequence of choosing a unitary coin is the torality property (Proposition 2.1 of [BBBP08]):

$$\mathbf{x} \in T \text{ and } (\mathbf{x}, y) \in \mathcal{V} \implies |y| = 1\,.$$

Consequently, the intersection $\mathcal{V}_1$ of $\mathcal{V}$ with the unit torus, T in $\mathbb{C}^{d+1}$, is a real d-manifold. When $\mathcal{V}_1$ is smooth, it follows that the logarithmic intersection chain $\tilde{\sigma}$ is i times a real hypersurface in the flat torus. Let $\mathbf{z}_* \in \mathcal{V}_1$ be a critical point in direction $\mathbf{r}_*$. The function Q is purely imaginary and we may write $Q = iQ_{\mathbb{R}}$ where $Q_{\mathbb{R}}$ is a real quadratic form with some signature which we denote $\tau(\mathbf{z}_*)$. Computing the argument of the expression $\mathcal{K}^{-1/2}$ as it is defined in Theorem 10.23, we see that the expression is equal to $|\mathcal{K}|^{-1/2}e^{-i\pi\tau(\mathbf{z}^*)/4}$. Letting $\mathbf{W}(\hat{\mathbf{r}})$ denote the set of critical points in direction $\hat{\mathbf{r}}$ that are in the unit torus, Corollary 10.24 then becomes

$$a_{\mathbf{r}} \sim \left(\frac{1}{2\pi|\mathbf{r}|}\right)^{d/2} \sum_{\mathbf{z}=\exp(i\mathbf{y})\in\mathbf{W}(\hat{\mathbf{r}})} e^{-i\mathbf{r}\cdot\mathbf{y}} \frac{G(\mathbf{z})}{|\nabla_{\log}H(\mathbf{z})|} |\mathcal{K}(\mathbf{z})|^{-1/2} e^{-i\pi\tau(\mathbf{z})/4} \tag{10.23}$$

whenever $\mathbf{W}(\hat{\mathbf{r}})$ is non-empty. We now examine some particular examples.

Example 10.26 (QRW in one dimension). The earliest and most often studied example is the one-dimensional Hadamard QRW, whose steps are 0 and 1 and whose coin-flip matrix is

$$U := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

The resulting generating function has denominator $H := 1 - (1-x)\frac{y}{\sqrt{2}} - xy^2$. Parametrizing the positive arc in the unit sphere by $\lambda := r/s$, it is shown in [BP07, Theorem 2.1] that the set of

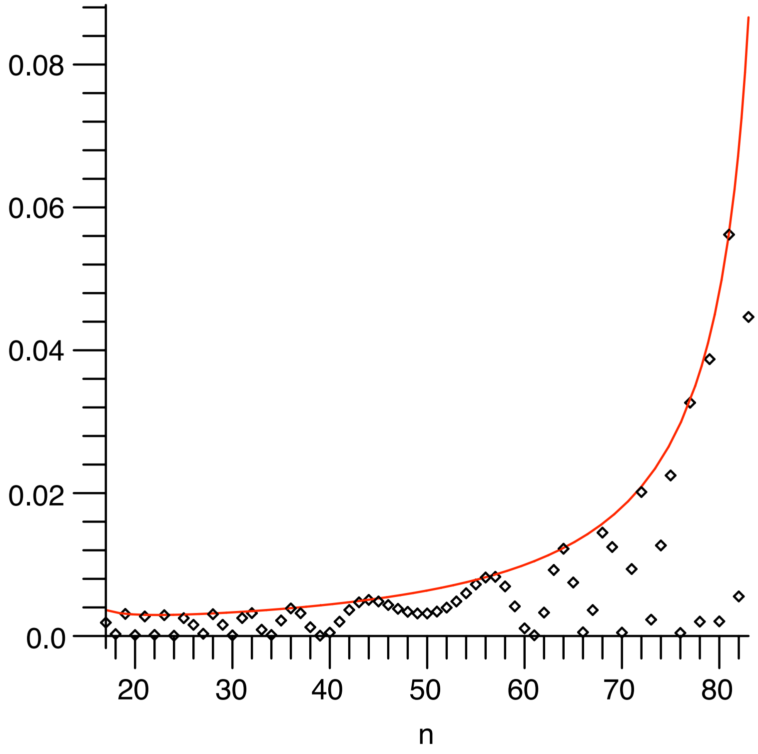


Figure 10.1: Time $t = 200$ probability values by location; the upper envelope is with the $\cos^2\theta$ phase term.

directions λ for which there is a critical point in the unit torus is the interval

$$J := \left[\frac{1-\sqrt{1/2}}{2}, \frac{1+\sqrt{1/2}}{2}\right].$$

For each λ in the interior of J, there are exactly two conjugate critical points $\mathbf{z} := (e^{i\alpha}, e^{i\beta})$ and $\overline{\mathbf{z}} := (e^{-i\alpha}, e^{-i\beta})$. The curvature $\mathcal{K}(\mathbf{z})$ is positive on the interior of J but goes to zero at the endpoints. Thus each point gives a contribution to $a_{\mathbf{r}}$ of magnitude $C|\mathbf{r}|^{-1/2}\mathcal{K}(\hat{\mathbf{r}})^{-1/2}$ where $\mathcal{K}$ is nonzero on the interior of J and goes to zero at the endpoints. The contributions from the two points are complex conjugates; the square of the modulus of their sum is the actual probability of finding the particle there; the modulus of the probability will be the above magnitude $C|\mathbf{r}|^{-1/2}\mathcal{K}(\hat{\mathbf{r}})^{-1/2}$, multiplied by $\cos^2\theta$ where $\theta = i\mathbf{r}\cdot\mathbf{y}$ is the phase at the critical point $\mathbf{z} = e^{i\mathbf{y}}$. This is rapidly oscillating with period $\Theta(1)$, giving rise to the probability plot shown in figure 10.1.

Example 10.27 (QRW in two dimensions). The probability plots of QRW in two dimensions are visually striking. To understand these better, we take another look at the formula (10.23). The Gauss map $\mathcal{G}$ on $\log \mathcal{V}_1$ maps any smooth point $\mathbf{y} \in \log \mathcal{V}_1$ to the unit vector $\hat{\mathbf{r}}$ such that $\mathbf{z} := \exp(i\mathbf{y}) \in \mathbf{W}(\hat{\mathbf{r}})$ is a critical point in direction $\hat{\mathbf{r}}$. The cone of directions of non-exponential decay of probabilities is the image of the logarithmic Gauss map. One way to plot this image is to parametrize it by $\log \mathcal{V}_1$, sampling points from $\log \mathcal{V}_1$ and, for each sampled point $\mathbf{z}$, plotting the image $\nabla_{\log} H(\mathbf{z})$ of the logarithmic Gauss map at $\mathbf{z}$. If we wish to see the magnitude of the probability there, ignoring the phase factor of $\cos^2 \theta$, we need to plot something proportional to $|\mathcal{K}^{-1}|$. This is the inverse of the Jacobian of the Gauss map, and is hence the ratio by which area is compactified under this map. We may imagine that we have built a model of $\log \mathcal{V}_1$ out of tinted plastic, and then projected it by the logarithmic Gauss map, so that the tint accumlates the most in places where the projection compactifies area the most. Figure 10.2 shows a number of examples of this in cases where $\mathcal{V}_1$ is smooth. Probabilities are shown at time 200, normalized in space by 1/100, with darker color indicative of greater magnitude. The region where any color is visible is the region of non-exponential decay. Wherever the logartithmic Gauss map folds the manifold $\mathcal{V}_1$ over itself, the Jacobian vanishes and we see a crease of infinite intensity. The smooth manifold $\log \mathcal{V}_1$ is in general a torus or union of tori, and one can try to envision how the intensity plots in figure 10.2 arise as projections of these tori.

10.4 Quadratically degenerate cases

Special case: d=2

`Say what happens exactly on the diagonal`

Airy limits in scaling windows

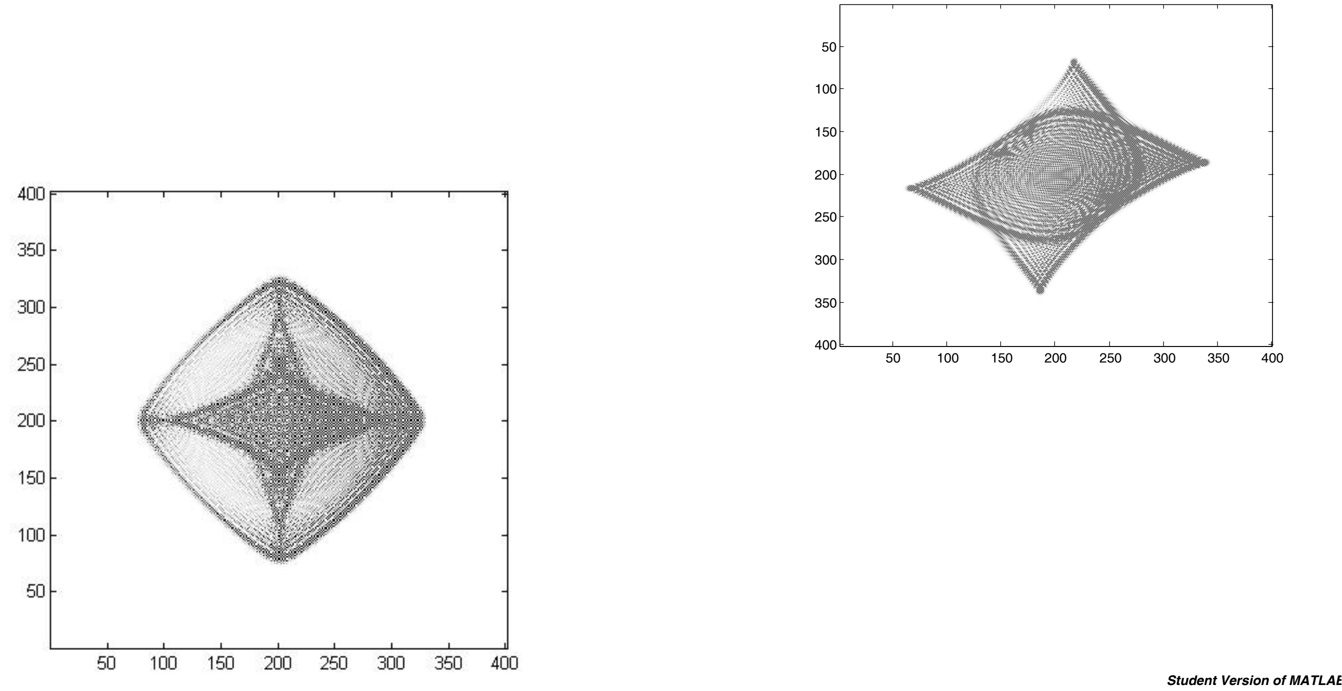
`Maybe Andy will write this!`

10.5 Limit theorems

`Take from my Math 546 lecture notes:`

`Local large deviation theorem;`

`Lattice local central limit theorem in the case of exponential decay.`



(a)

(b)

Figure 10.2: Time $t = 100$ color-intensity plots for two instances of two-dimensional QRW.

10.6 Non-minimal smooth points: a solution in dimension two

`Maybe Tim will write this!`

Notes

A precursor to the two derivations of the saddle-residue integrals in Section 10.1 were the multivariate asymptotic results of [BR83]. Breaking the symmetry among the coordinates, they wrote

$$F(\mathbf{z}) = \sum_{n=0}^{\infty} f_n(\mathbf{z}^\circ) z_d^n$$

and then used the fact that f_n is sometimes asymptotic to an n^{th} power $f_n \sim C \cdot g \cdot h^n$ to obtain Gaussian asymptotics when certain minimality conditions are satisfied near a smooth critical point. Their language is inherently one-dimensional, so geometric concepts such as smooth point did not arise explicitly. The results presented in this chapter were first obtained via coordinate methods in [PW02]. These methods are valid only when $\mathbf{z}$ is a minimal point. The residue version of these computations has not yet appeared in print, although it is present in several preprints, including [BBBP08] and [BP04]. Extending the validity of the coordinate version beyond the case of finite intersection of $\mathcal{V}$ with $T(\mathbf{x}_*)$ was done only in the preprint [BP08].

Exercises

Exercise 10.1. Let Res be the residue map on meromorphic forms with simple poles on a smooth variety V as defined in Proposition 10.5. Prove that Res is functorial, that is, it commutes with bi-holomorphic changes of coordinate.

Exercise 10.2. Prove an analogue of formula (10.8) for poles of higher order. First, show that the residue of $P\, d\mathbf{z}/z_1^k$ is given by

$$\text{Res}\left(\frac{P\, d\mathbf{z}}{z_1^k}; \mathbf{p}\right) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z_1}\right)^{k-1} P\, dz_2 \wedge \cdots \wedge dz_d\,.$$

Then copy the proof of Proposition 10.5 to extend this to residues of G/H^k, computing the formula explicitly for $k = 2$ and $k = 3$ (assuming that dH is nonvanishing).

Exercise 10.3. Let $F = 1/(1 - x - y)$ be the generating function for the binomial coefficients. In analogy with Example 10.14, find the residue integral $\int_\sigma \omega$ of Theorem 10.13 that estimates coefficients of the binomial generating function. Evaluate this asymptotically in order to obtain an asymptotic formula for a_{rs} and compare to what you obtain from Stirling's formula for $\binom{r+s}{r}$.