

Analytic Combinatorics in Several Variables

Lecture notes for Math 581, Spring 2006, revised Spring 2009

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Contents

I	Combinatorial enumeration	1
1	Introduction	3
1.1	Arrays of numbers	3
1.2	Generating functions and asymptotics	5
1.3	New multivariate methods	6
1.4	Outline of the remaining chapters	8
2	Generating functions	11
2.1	Formal power series	11
2.2	Rational operations on generating functions	12
2.3	Algebraic generating functions	20
2.4	D-finite generating functions	28
2.5	Exponentiation: set partitions	34
3	Univariate asymptotics	39
3.1	Rational functions: an explicit formula	39
3.2	Saddle point methods	42
3.3	Circle methods	47
3.4	Transfer theorems	49
II	Mathematical background	55
4	Manifolds	57
4.1	Differential forms in \mathbb{R}^n	58

4.2	Differential forms in \mathbb{C}^n	62
4.3	Algebraic topology	64
5	Saddle integrals in one variable	73
5.1	Real integrands	75
5.2	Complex phase	79
5.3	Saddle points and Watson's Lemma	83
5.4	Analytic versus C^∞ category	85
6	Saddle integrals in more than one variable	93
6.1	Standard phase	95
6.2	Real part of phase has a strict minimum	97
6.3	Localization	100
6.4	Examples and extensions	106
7	Techniques of symbolic computation via Gröbner bases	109
7.1	Solving systems of polynomial equations	109
7.2	Examples of Gröbner basis computation	115
7.3	D-modules: computing with D-finite functions	119
8	Cones, Laurent series and amoebas	123
8.1	Cones and dual cones	124
8.2	Laurent series	125
8.3	Amoebas	129
III	Multivariate enumeration	133
9	Morse theoretic decomposition	135
9.1	Morse theory redux	135
9.2	Description of the quasi-local cycles	140
9.3	The critical point equations	143
9.4	Minimal points	146

10 Smooth point asymptotics	149
10.1 Residue integral: an explicit construction	150
10.2 The residue form: A more natural, less explicit construction	153
10.3 Formulae in quadratically nondegenerate cases	158
10.4 Quadratically degenerate cases	168
10.5 Limit theorems	169
10.6 Non-minimal smooth points: a solution in dimension two	169
11 Multiple point asymptotics	171
12 Cone point asymptotics	173
13 Scaling windows	175
14 Worked examples	177
IV Appendices	179
15 Stratification	181
15.1 Singular locus and Whitney stratifications	181
15.2 Extending $h_{\mathfrak{f}}$ to a proper Morse function	182
15.3 Compactification	182
16 Morse theory	185
16.1 Smooth Morse theory	186
16.2 Stratified Morse theory	193
16.3 Stratified Morse theory for nonproper functions	196
16.4 Integrals in relative homology	201

