Homework Set III

1. Let $\{X_n\}$ and $\{Y_n\}$ be sequences of random variables. Suppose that $X_n \to X$ in distribution and $Y_n \to Y$ in distribution for some random variables X and Y. Does it follow that $(X_n, Y_n) \to (X, Y)$ in distribution? Now suppose instead that $X_n \to X$ in probability and $Y_n \to Y$ in probability for some random variables X and Y. Does it follow that $(X_n, Y_n) \to (X, Y)$ in probability?

2. (a) Show that if $X_n \to X$ almost surely and $N(n) \to \infty$ almost surely, then $X_{N(n)} \to X$ almost surely.
   (b) Give an example where $X_n \to 0$ in probability, $N(n) \uparrow \infty$ almost surely, and $X_{N(n)} \to 1$ almost surely.

3. Durrett chapter 1 exercise 7.4 (an investment problem).

4. Durrett chapter 1 exercise 6.12 (if it must happen once, it happens often).

5. Suppose 100 IID geometric random variables of mean 2 happened to sum to 300.
   (a) Approximately how unlikely is this?
   (b) If you wrote down a histogram for the 100 values, approximately what would you expect to find?
   (c) What does this have to do with geometric random variables being an exponential family?

6. A pair of quick problems about characteristic functions:
   (a) Prove or find a counterexample: if ϕ is a characteristic function then there is another characteristic function ψ with $\psi^2 = \phi$.
   (b) Let N be a Poisson random variable with mean λ, whose characteristic function is given in Example 3.2. Let $X_1, X_2, ...$ be i.i.d. with characteristic function ϕ and independent of N. Compute the characteristic function of the random sum

$$S = \sum_{i=1}^{N} X_i.$$