Stat 531 / Math 547, Spring 2008, Homework Set IV

1. Compare two Brownian motions started at zero, one conditioned on the event $B_1 \geq 5$ and the other conditioned on $B_1 \leq -5$. If you observe them for an infinitesimal amount of time $[0, dt]$ can you guess which is which? Include in your answer an interpretation of the question "Can you guess which is which?". Now answer the same question if the two observations are instead a Brownian motion $\{B_t\}$ and a sped-up Brownian motion $\{B_{2t}\}$.

2. Durrett, Chapter 7, exercise 5.2 (probability of BM with drift hitting a level).

3. Let B_t be a Brownian motion started from 0, defined on the usual space probability space $(\Omega, \mathcal{F}, \mathbf{P}_0)$. let $Y_t = B_t - tB_1$ be the usual Brownian bridge and let $Z_t := Y_t - \int_0^1 Y_s \, ds$ be the bridge, translated so as to have average value zero.

   Prove or disprove: The Gaussian process $\{Z_t : 0 \leq t < 1\}$ is invariant under rotations ϕ_s the "circle" $\mathbf{R}/\mathbf{Z}$, where ϕ_s is the map $t \mapsto t + s \bmod 1$.

4. Let $\{X_t, Y_t, Z_t\}$ be independent Brownian motions. In other words, each is a continuous martingale and

$$\begin{aligned} \langle X \rangle_t = \langle Y \rangle_t = \langle Z \rangle_t &= t\,; \\ \langle X, Y \rangle_t = \langle X, Z \rangle_t = \langle Y, Z \rangle_t &= 0\,. \end{aligned}$$

   Compute the drift and infinitesimal variance for the diffusion

$$R_t := \sqrt{X_t^2 + Y_t^2 + Z_t^2}\,.$$

   The final answer is not too messy – you should get a Bessel process (with what parameter?) but you must simplify! [Regarding the construction: it's OK just to do it on $[0, \tau)$ where $\tau = \inf\{t : R_t = 0\}$.]