

## Solutions for Homework Set II

### Math 546 version

1. Durrett Chapter 1 Exercise 2.4

**Proof:** For (i) Since for any continuous function, and any open sets  $B \in R$ ,  $f^{-1}(B)$  is open, thus  $f^{-1}(B)$  now using Theorem 2.1 in Durrett finishes the proof.

For (ii) Let  $\mathcal{F}$  be the  $\sigma$ - algebra that makes all the continuous functions measurable, we want to show that it contains all the open sets. Let  $f$  be a projection function on the first coordinate, i.e.,  $f(x_1, \dots, x_n) = x_1$ . Obviously,  $f$  is continuous. Suppose that there is an open set  $B$  not in  $\mathcal{F}$ . Then since  $f$  is a projection function,  $f(B)$  is open, therefore in  $\beta(R)$ , contradiction, which proves our claim. Now by definition,  $\mathcal{R}^d$  is the smallest  $\sigma$ - algebra contains all open sets, therefore, the proof is complete.

2. Durrett Appendix Exercise 4.3

**Proof:**(i.) Since

$$\int |g - \varphi| d\mu \leq \int |g^+ - \varphi^+| d\mu + \int |g^- - \varphi^-| d\mu$$

it suffice to prove the result when  $g > 0$ . By Exercise 4.2 in appendix, we can pick  $n$  large enough so that if  $E_{n,m} = \{x : m/2^n \leq f(x) \leq (m+1)/2^n\}$  and  $h(x) = \sum_{m=1}^{\infty} (m/2^n) 1_{E_{n,m}}$  then  $\int (g - h) d\mu < \epsilon/2$ . Since

$$\sum_{n=1}^{\infty} \frac{m}{2^n} \mu(E_{n,m}) = \int h d\mu \leq \int g d\mu < \infty$$

we can pick  $M$  large enough so that  $\sum_{m>M} \frac{m}{2^n} \mu(E_{n,m}) < \epsilon/2$ . If we let  $\varphi = \sum_{m=1}^M \frac{m}{2^n} \mu(E_{n,m})$  then

$$\int |g - \varphi| d\mu = \int g - h d\mu + \int h - \varphi d\mu < \epsilon.$$

(ii.) Pick  $A_m$  that are finite unions of open intervals so that  $\mu(A_m \Delta E_{n,m}) \leq \epsilon M^{-2}$  and let

$$q(x) = \sum_{m=1}^M \frac{m}{2^n} 1_{A_m}$$

Now the sum above is  $= \sum_{j=1}^k c_j 1_{(a_{j-1}, a_j)}$  almost everywhere (i.e., except at the end points of the intervals) for some  $a_0 < a_1 < \dots, < a_k$  and  $c_j \in R$ .

$$\int |\varphi - q| d\mu \leq \sum_{m=1}^M \frac{m}{2^n} \mu(A_m \Delta E_{n,m}) \leq \frac{\epsilon}{2^n}.$$

(iii.) To do that, we can let the continuous function  $r_j$  be 0 on  $(a_{j-1}, a_j)^c$ ,  $c_j$  on  $[a_{j-1} + \delta_j, a_j - \delta_j]$ , and linear otherwise. If we let  $r = \sum_{j=1}^k r_j(x)$  then

$$\int |q(x) - r(x)| = \sum_{j=1}^k \delta_j c_j.$$

Pick  $\delta_j$  so that  $\delta_j c_j < \epsilon/k$  finishes the proof.

### 3. Durrett Appendix Exercise 2.1

(3.1) Durrett Appendix Exercise 2.1

Let  $\mathcal{C} = \{\{1, 2\}, \{1, 3\}\}$  and let  $\mu_t$  be the probability measure defined by  $P(1) = \frac{1}{4} - t$ ,  $P(2) = \frac{1}{4} + t$ ,  $P(3) = \frac{1}{4} + t$ ,  $P(4) = \frac{1}{4} - t$  for any  $t \in [0, 1/4]$ . All these measures agree on  $\mathcal{C}$ , but are different on  $\sigma(\mathcal{C})$ , since  $\sigma(\mathcal{C})$  is all subsets of  $\{1, 2, 3, 4\}$ .

(3.2) Durrett Chapter 1 Exercise 4.4

For any  $A_i \in \sigma(X_i), \forall i$ , since  $f(x) = g_1(x_1) \cdots g_n(x_n)$  and

$$\int_{\mathbb{R}} g_1(X_1) dx_1 \cdots \int_{\mathbb{R}} g_n(X_n) dx_n = 1$$

$$\begin{aligned} P(X_1 \in A_1, \dots, X_n \in A_n) &= \int_{A_1 \times \dots \times A_n} \frac{g_1(X_1)}{\int_{\mathbb{R}} g_1(X_1) dx_1} \cdots \frac{g_n(X_n)}{\int_{\mathbb{R}} g_n(X_n) dx_n} dx_1 \cdots dx_n \\ &= \int_{A_1} \frac{g_1(X_1)}{\int_{\mathbb{R}} g_1(X_1) dx_1} dx_1 \cdots \int_{A_n} \frac{g_n(X_n)}{\int_{\mathbb{R}} g_n(X_n) dx_n} dx_n \\ &= P(X_1 \in A_1) \cdots P(X_n \in A_n) \end{aligned}$$

Since  $\frac{g_i(X_i)}{\int_{\mathbb{R}} g_i(X_i) dx_i}$  is a density function of  $X_i$ . This finishes the proof.

### 4 Durrett Chapter 1 Exercise 5.3

(i) Since  $f(U_1), f(U_2), \dots$  are independent and have  $E|f(U_i)| < \infty$   
This follows from the weak law of large numbers.

(ii) By Chebychev,

$$P\left(|I_n - I| > \frac{a}{n^{\frac{1}{2}}}\right) \leq \frac{n}{a^2} \text{var}(I_n) = \frac{\text{var}(f)}{a^2} \quad \text{where} \quad \text{var}(f) = \int f^2 - \left(\int f\right)^2$$

Extra Credit

We assume  $|f| \leq L$  and that  $f$  has mean zero.

First we show that  $\mathbb{E}e^{\lambda X}$  is maximized by  $\frac{1}{2}\delta_{-L} + \frac{1}{2}\delta_L$ .

Notice that (assuming we can interchange the integral and derivative)

$$\frac{d}{dx}\mathbb{E}e^{\lambda x} = \int_{-L}^L \frac{d}{dx}e^{\lambda x}dF(x) = \lambda \int_{-L}^L e^{\lambda x}dF(x).$$

So when  $\lambda \neq 0$  this derivative is never zero because the integral is always positive.

Consequently, since we have a convex, continuous function on a compact set, it must attain a maximum on the boundary, i.e. on the set  $\{-L, L\}$

By the restriction that  $f$  has mean zero, that only leaves  $f(x) = \frac{1}{2}\delta_{-L} + \frac{1}{2}\delta_L$

For the trivial case  $\lambda = 0$  any distribution gives the max.

Now if we choose  $\lambda = \frac{a}{\sqrt{n}}$  we have

$$\mathbb{E}e^{\lambda x} = \frac{1}{2} \left( e^{\frac{aL}{\sqrt{n}}} + e^{-\frac{aL}{\sqrt{n}}} \right) = \cosh \frac{aL}{\sqrt{n}}$$

Now from Jensen's inequality  $|\mathbb{E}X| \leq (\mathbb{E}X^2)^{\frac{1}{2}}$  and the fact *cosh* is a convex function.

We have

$$\mathbb{E}e^{\lambda x} \leq e^{\frac{a^2 L^2}{2n}}$$

We would expect this bound to be better since we restricted ourselves to the interval  $[-L, L]$ . Furthermore, our bound in this problem explicitly depends on  $n$ , whereas in the previous problem our bound did not. Thus, I believe there are certain ranges where each one dominates the other.

Note we can't really compare unless we have some bound for the variance in question 4(b).

If we assume the same  $[-L, L]$  restriction, then  $\text{var}(f) \leq L^2$ .