

Solutions for Homework Set II

Stat 530 version

1. We will assume an infinite deck. We will also assume the dealer always stays on a soft 17, meaning he has 17 and one of his aces counts as 11.

Define $\Omega_0 = \{A, 2, 3, \dots, 10, J, Q, K\}$ Define $\mathbb{F}_0 = 2^{\Omega_0}$ (that is, the power set). Let \mathbb{P}_0 be uniform on the 13 states.

Define

$$X_0(\omega_0) = \begin{cases} 1 & \omega_0 = A \\ 10 & \omega_0 \in \{10, J, Q, K\} \\ \omega_0 & \text{otherwise} \end{cases} .$$

Now define $(\Omega, \mathbb{F}, \mathbb{P}) = (\Omega_0, \mathbb{F}_0, \mathbb{P}_0)^{12}$, $X_i(\omega) = X_0(\omega_i)$ and $Y_n(\omega) = 1_{\{\cup_{i=1}^n X_i(\omega) = A\}}$

Thus, X_i tells us the value of the i^{th} card, while Y_n tells us whether we have an A (ace) in our first n cards.

Let $S_0 = 0$, $T_0 = 0$, $S_n(\omega) = S_{n-1}(\omega) + 1_{\{T_{n-1} < 17\}} \cdot X_n(\omega)$

Let $T_n(\omega) = S_n(\omega) + 10 \cdot 1_{\{7 \leq S_n \leq 11\}} \cdot Y_n(\omega)$

Our random variable T_n checks after each card, if changing an ace to 11 would cause the dealer to stand.

Also, the indicator variable in S_n tells us whether the dealer takes another card or not.

Note that our probability space is a 12-tuple. This is because the dealer can draw at most 12 cards before he either busts or must stay

Thus, the probability of the dealer busting is given by

$$\mathbb{P}(T_{12} > 21) .$$

2. Let X_i denotes the temperature of year i , **we assume** $X_{1925}, \dots, X_{2004}$ **are i.i.d from some continuous distribution.** The number of record-high years

$$N = 1 + I_{X_{1926} > X_{1925}} + \dots + I_{X_{2004} > \max\{X_{1925}, \dots, X_{2003}\}}$$

By symmetry, the probability is $1/n$ that the n^{th} temperature is the highest of the first n . Thus, $EN = 1 + 1/2 + 1/3 + \dots + 1/79 = 4.95$. So we expect to see about 5 record high years. [Note: now you can use souped up Borel-Cantelli to get a weak law showing you do in fact see roughly $\log n$ records in n years.]

Since X_i are i.i.d. and has continuous distribution, then each order of $\{X_{1925}, \dots, X_{2004}\}$ is equally likely. Thus the probability of that both 2003 and 2004 has a record high temperature is

$$\frac{78!}{80!} = \frac{1}{6320} = 0.00016.$$

3. First in this case our model will be $X_n = X_{n-1} + \epsilon_n$ where ϵ_n are i.i.d. random variable with some continuous distribution. Since there is no global warming trend under our model assumption, we can assume that ϵ_i has symmetric distribution with mean 0.

Let A_n denotes the probability that X_n is record high. We want to show that $P(A_n \cap A_{n+1}) \geq 1/n(n+1)$. But

$$P(A_n \cap A_{n+1}) = P(A_{n+1}|A_n) = \frac{1}{2}P(A_n)$$

So it suffices to show that $P(A_n) > 2/n(n+1)$.

We will do that by induction, suppose it is true for any $k < n$. Notice that

$$\begin{aligned} P(A_n) &> P(X_n > X_{n-1}, A_{n-1}) + P(X_n > X_{n-2}, A_{n-2}, X_{n-1} < X_{n-2}) \\ &+ P(X_n > X_{n-3}, A_{n-3}, X_{n-1} < X_{n-3}, X_{n-2} < X_{n-3}) \end{aligned}$$

It is easy to see that first term on right hand side of the inequality is greater than $1/n(n+1)$. For second term, notice that it is equal to

$$P(X_n > X_{n-2}, A_{n-2}) - P(X_n > X_{n-1} > X_{n-2}, A_{n-2}) = 1/4P(A_{n-2}) > 1/2n(n+1).$$

It can be shown similarly that ,

$$P(X_n > X_{n-3}, A_{n-3}, X_{n-1} < X_{n-3}, X_{n-2} < X_{n-3}) > 1/2n(n+1).$$

Therefore, $P(A_n) > 2/n(n+1)$. ■

[Note: there are other ways to prove this inequality. The intuitive point is that $P(A_n \cap A_{n+1}) = \frac{1}{2}P(A_n)$, rather than $\frac{1}{n+1}P(A_n)$ as in the previous model.]

4. The probability model will be X_1, \dots, X_n are identically distributed and pair wise independent with uniform distribution on $[0, 1]$. The model will also have the property that subcollections of these of size three will not be independent, which will cause triples (X_i, X_j, X_k) not to be uniform on the unit cube. To construct such variables, take Y and Z IID uniform on $[0, 1]$ and let $X_n := Y + nZ \pmod{1}$. To see that this yields pairwise independence, we give the proof that X_0 and X_j are independent, the case $i > 0$ being similar.

$$(X_0, X_j) = (Y, Z) \begin{bmatrix} 1 & 1 \\ 0 & j \end{bmatrix}.$$

The determinant is j but for each point (x, x') , the pre-image of the event $(X_0 = x, X_j = x')$ has j points in it, each with density $1/j$. Thus the densities sum to 1 for any x, x' .

By the L^2 weak law (Durrett page 35) we can conclude that $(X_1 + \dots + X_n)/n$ converges to $1/2$ in L^2 and in probability.

5. We begin by noting if $x \sim N(\mu, \Sigma)$, then $Ax \sim N(A\mu, A\Sigma A^T)$ for some matrix A .

In our case $\mu = 0$ and $\Sigma = I$ with I being the identity matrix.

A necessary and sufficient condition for independence of Gaussian random variables is that their covariances are zero. This is equivalent to the covariance matrix being diagonal.

Thus in our example, this means $AA^T = D$ for some diagonal matrix D . Moreover, to ensure we do not have a degenerate random variable, the diagonal elements should be strictly positive.

Thus our necessary and sufficient condition is that AA^T be a strictly positive diagonal matrix (i.e. positive definite diagonal matrix). This is equivalent to saying that the rows of A are orthogonal and none is zero.