Solutions for Homework Set I

Stochastic Processes(Stat 531/Math 547)

1. (Durrett Exercise 1.2)

Proof : By definition of conditional expectation, we know that

$$\int_G P(A|\mathcal{G})dP = \int_G E(1_A|\mathcal{G})dP = \int_G 1_A \, dP = P(A \cap G)$$

and

$$\int_\Omega P(A|\mathcal{G})dP = \int_\Omega E(1_A|\mathcal{G})dP = P(A).$$

Since by definition,

$$P(G|A) = \frac{P(A \cap G)}{P(A)}$$

we finish our proof.

2.

Assume that given p two flips are independent. Let

$$\Omega = \{H, T\} \times \{H, T\} \times [0, 1]$$

so that the coordinates of $\omega \in \Omega$ may be interpreted as the first flip, the second flip and the value of p. Let $\mathcal{C}$ be the class of all the subsets $\{H, T\}$ and take

$$\mathcal{F} = \mathcal{C} \times \mathcal{C} \times \mathcal{B}([0, 1]) \, .$$

To make the flips conditionally independent Bernoulli (p) given p, we define P as follows. Let $\mathcal{W}$ be the class of sets that are the product of a singleton in $\{H, T\}$, another singleton in $\{H, T\}$ and an interval in $[0, 1]$. We define P on W by

$$\begin{aligned}
P(X_1 = H, X_2 = H, p \in A) &= \int_A x^2 \, dx \\
P(X_1 = H, X_2 = T, p \in A) &= \int_A x(1-x) \, dx \\
P(X_1 = T, X_2 = H, p \in A) &= \int_A x(1-x) \, dx \\
P(X_1 = T, X_2 = T, p \in A) &= \int_A (1-x)^2 \, dx \, .
\end{aligned}$$

The class $\mathcal{W}$ is a π-system so this definition extends uniquely to $\sigma(\mathcal{W}) = \mathcal{F}$. We may now compute:

$$\begin{aligned}
P(X_2 = H | X_1 = H) &= \frac{\int_0^1 x^2 \, dx}{\int_0^1 x \, dx} \\
&= \frac{1/3}{1/2} \\
&= \frac{2}{3} \, .
\end{aligned}$$

3.

By Durrett Example 1.5 in Chapter 3 (page 179), we can get

$$P(S_\tau|S_0 = k) = \frac{-(-M-k)}{(M-k)-(-M-k)} = \frac{M+k}{2M}$$

Now

$$P(S_\tau|X_1 = 1, S_0 = k) = P(S_\tau|S_0 = k+1) = \frac{M+k+1}{2M}$$

Then, using Bayes rule

$$P(X_1 = 1|S_\tau = M, S_0 = k) = \frac{P(X_1 = 1)P(S_\tau|X_1 = 1, S_0 = k)}{P(S_\tau|S_0 = k)} = \frac{M+k+1}{2(M+k)}$$

Take $M \to \infty$, $\lim\limits_{M\to\infty} \frac{M+k+1}{2(M+k)} = \frac{1}{2}$.

For the second part of the question, using the results from Example 1.5, we get

$$P(S_\rho|S_0 = k) = \frac{-(-k)}{(M-k)-(-k)} = \frac{k}{M}$$

So similarly, $P(S_\rho|X_1 = 1, S_0 = k) = \frac{k+1}{M}$

$$P(X_1 = 1|S_\rho = M, S_0 = k) = \frac{k+1}{2k}$$

which is a constant w.r.t. M.

4.

a)

$$\begin{aligned}
& E(Var(X|\mathcal{F})) + Var(E(X|\mathcal{F})) \\
= \; & E[E(X^2|\mathcal{F}) - (E(X|\mathcal{F}))^2] + E[E(X|\mathcal{F})]^2 - (E[E(X|\mathcal{F})])^2 \\
= \; & E(X^2) - E[E(X|\mathcal{F}))^2] + E[E(X|\mathcal{F})]^2 - [E(X)]^2 \\
= \; & E(X^2) - [E(X)]^2 \\
= \; & Var(X)
\end{aligned}$$

b)

Let $\Omega = \{\textit{all the subjects}\}$, and let $\mathcal{F}$ be the power set of Ω. A reasonable probability measure can be defined on $\mathcal{F}$. Let X be a random variable, and $X(\omega) =$ *the amount of blood protein* for the subject ω. Let Y be a random variable and $Y(\omega) =$ *the score of task performance* for the subject ω. Define $\mathcal{F}_0$ to be a sub-sigma field, and $\mathcal{F}_0 = \sigma(X)$, which is the information contained in X.

According to part a), the variance of Y can be decomposed to the variation because of known information, $Var(E(Y|\mathcal{F}_0)]$, and the expected variation not explained by the known information, $E[Var(Y|\mathcal{F}_0)]$.

Therefore, saying "three quarters of the variation in performance may be explained by the level of blood protein A" is equivalent to saying $Var(E(Y|\mathcal{F}_0)) = \frac{3}{4}Var(Y)$.
c)
Assume X and Y follows a joint normal distribution $N((\mu_1, \mu_2), \sigma_1, \sigma_2, \rho)$. Then

$$E[Y|X] = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1)$$

Therefore,

$$Var(E[Y|X]) = \rho^2\frac{\sigma_2^2}{\sigma_1^2}\sigma_1^2 = \rho^2\sigma_2^2$$

Then, the portion of varianc of performance explained by the level of blood protein A is $Var(E[Y|X])/\sigma_2^2 = \rho^2$. Therefore, we can use $\hat{\rho}^2$ as an estimate, where

$$\hat{\rho} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2\sum(y_i - \bar{y})^2}}$$

5. (Durrett Exercise 1.10)
Proof : First since $E(Var(X|N)) = E(N\sigma^2) = \sigma^2 EN$ and $Var(E(X|N)) = Var(N\mu) = \mu^2 Var(N)$. By using the result in Exercise 1.9, we finish the proof.

6.
Solution : Let $\Omega = 1, \cdots, N$ be the finite population of people. Suppose $\Omega_1, \cdots, \Omega_k$ is a finite partition of Ω into disjoint sets, which represents the household. Let $F = 2^\Omega$, $a_i = |\Omega_i|$, then $\sum_{i=1}^k a_i = N$. For $x \in \Omega$, $\alpha(x) = \frac{1}{N}$, $\beta(x) = \frac{1}{k}\sum_{i=1}^k \frac{I_{[x\in\Omega_i]}}{a_i}$. Thus,

$$\frac{d\alpha}{d\beta}(x) = \frac{1}{\frac{N}{k}\sum_{i=1}^k \frac{I_{[x\in\Omega_i]}}{a_i}}$$

Another way to say this is that $d\alpha/d\beta$ is the household size divided by the average household size. The reason why they re-weight is as follows. They are able to sample according to β because they have a list of phone numbers and can sample uniformly. They actually wish to sample uniformly among individuals, that is, according to α. They can't re-weight the whole sample because they don't know household sizes until they interview someone in the house. So instead, they sample by β, ask what the household size is, and then give the respondent the weight $d\alpha/d\beta$.