My version of the proof of Theorem (3.8) of Chapter 1

For any positive real M, define a cutoff function ψ_M as follows.

$$\psi_M(x) = \begin{cases} 1 & |x| < M \\ M + 1 - |x| & M \le |x| < M + 1 \\ 0 & |x| \ge M + 1 \end{cases}$$

By linearity of expectation, it suffices to show that

$$\mathbb{E}h(X_n)\psi_M(X_n) \rightarrow \mathbb{E}h(X)\psi_M(X_n)\,, \tag{1}$$

$$|\mathbb{E}h(X_n)(1 - \psi_M(X_n))|, |\mathbb{E}h(X)(1 - \psi_M(X))| < K\phi(M)\,. \tag{2}$$

where $\phi(M) := \sup_{x:|x|\ge M} |h(x)/g(x)| \to 0$ as $M \to \infty$. Continuity of h and ψ_M imply that $h(X_n)\psi_M(X_n) \to h(X)\psi_M(X)$ almost surely, hence the first equation follows from bounded convergence. For the second equation, first inequality,

$$\begin{aligned} |\mathbb{E}h(X_n)(1 - \psi_M(X_n))| &\le \mathbb{E}|h(X_n)(1 - \psi_M(X_n))| \\ &\le \phi(M)\mathbb{E}|g(X_n)(1 - \psi_M(X_n))| \\ &\le \phi(M)\mathbb{E}|g(X_n)| \\ &\le K\phi(M)\,. \end{aligned}$$

The last inequality follows from Fatou's lemma applied to the sequence $|h(X_n)(1-\psi_M(X_n))|$.
□