FINITE TRIPOD VARIANTS OF I/OM

ON IHARA'S QUESTION / ODA–MATSUMOTO CONJECTURE

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Abstract. In this note we introduce and prove a wide generalization and sharpening of Ihara's question / Oda-Matsumoto conjecture, for short I/OM. That leads to a quite concrete ***topological / combinatorial description*** of absolute Galois groups, in particular of $\mathrm{Gal}_{\mathbb{Q}} = \mathrm{Aut}(\overline{\mathbb{Q}})$, as envisioned by Grothendieck in his *Esquisse d'un Programme.*

1. Introduction/Motivation

A consequence of the results of this paper is a positive answer to a question by Ihara from the 1980's, which in the 1990's became a conjecture by Oda–Matsumoto, for short classical I/OM. In essence, the classical I/OM is about giving ***combinatorial / topological*** descriptions of the absolute Galois group of the rational numbers $\mathrm{Gal}_{\mathbb{Q}} = \mathrm{Aut}(\overline{\mathbb{Q}})$. Before giving the results in their full strength, let me briefly present the broader context in which the classical I/OM evolved as one of the main problems in Grothendieck's anabelian program, which itself grew out of [G1], [G2] (see [GGA]). To fix notation and context, let $\boldsymbol{\mathcal{G}}^{\mathrm{out}}$ be the category of profinite groups and outer homomorphisms. For geometrically integral $\mathbb{Q}$-varieties X, setting $\overline{X} := X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$, one has: First, viewing the algebraic fundamental group $\overline{\pi}_1(X) := \pi_1^{\mathrm{et}}(\overline{X}, *)$ of X as an object in $\boldsymbol{\mathcal{G}}^{\mathrm{out}}$ renders keeping track of base points irrelevant. Further, $X(\mathbb{C})$ endowed with the complex topology is a "nice" topological space, and $\overline{\pi}_1(X)$ is the profinite completion of the topological fundamental group $\pi_1^{\mathrm{top}}\big(X(\mathbb{C}), *\big)$; hence $\overline{\pi}_1(X)$ is in a precise sense an object of combinatorial / topological nature. Second, the canonical exact sequence of étale fundamental groups $1 \to \overline{\pi}_1(X) \to \pi_1^{\mathrm{et}}(X) \to \mathrm{Gal}_{\mathbb{Q}} \to 1$ gives rise to a *canonical representation* $\rho_X : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Out}\big(\overline{\pi}_1(X)\big) = \mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\mathrm{out}}}(\overline{\pi}_1(X))$, which is compatible with the canonical projections $\overline{\pi}(X) \to \overline{\pi}(Y)$ defined by morphisms $X \to Y$ of $\mathbb{Q}$-varieties. Hence if $\mathcal{V}$ is any category of geometrically integral $\mathbb{Q}$-varieties, its algebraic fundamental group functor $\overline{\pi}_{\mathcal{V}} : \mathcal{V} \to \boldsymbol{\mathcal{G}}^{\mathrm{out}}$, $X \mapsto \overline{\pi}_1(X)$ is well defined, and one gets a canonical representation:

$$\rho_{\mathcal{V}} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}), \quad \sigma \mapsto \big(\rho_X(\sigma)\big)_{X \in \mathcal{V}}$$

Thus the question of giving topological / combinatorial descriptions of $\mathrm{Gal}_{\mathbb{Q}}$ would follow from giving categories $\mathcal{V}$ of geometrically integral $\mathbb{Q}$-varieties for which $\rho_{\mathcal{V}}$ is an isomorphism.

Date: Dec 24, 2018.

Key words and phrases. Anabelian geometry, function fields, (generalized) [quasi] prime divisors, decomposition graphs, Hilbert decomposition theory, pro-ℓ Galois theory, algebraic/étale fundamental group

2010 MSC. Primary 11G99, 12F10, 12G99, 14A99.

Partially supported by the Simons Foundation Grant 342120. ***NOT funded by the NSF***: The ANT research directors (Douglas, Hodges, Libgober and Pollington), in their higher understanding of this matter, overruled the expert reviewers and panelists recommendation and decided not to fund this research.

Among other things, Grothendieck suggested to use subcategories $\mathcal{V} \subset \mathcal{T}$ of the *Teichmüller modular tower* $\mathcal{T}$ of all the moduli spaces $M_{g,n}$, and try to answer the two questions: First, for which categories $\mathcal{V}$ is the representation $\rho_{\mathcal{V}}$ injective. Second, describe the image $\mathrm{im}(\rho_{\mathcal{V}}) \subset \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$, and particular decide whether $\rho_{\mathcal{V}}$ is surjective.

There was and is an intensive and extensive effort to answer the questions above and related ones, starting with work by Deligne [De], Ihara [I1], see also [I2], [I3], Drinfeld [Dr], and subsequently by many others, e.g. [An], [F1], [F2], [H-Ma], [H-Sch], [H-Mz], [HLS], [I-M], [LNS1], [LNS2], [L-Sch], [Ma], [M-T], [Na], [N-Sch], [Sch], to mention a few. In particular, there is a canonical embedding of $\mathrm{Gal}_{\mathbb{Q}}$ in the *Grothendieck–Teichmüller group*

$$\mathrm{Gal}_{\mathbb{Q}} \hookrightarrow \widehat{GT} \subset \mathrm{Aut}(\widehat{F}_2),$$ [1]

as well as in its more sophisticated variants $\mathrm{Gal}_{\mathbb{Q}} \hookrightarrow \widehat{GT}_{\bullet}$. On the other hand, it turns out that all these more or less abstractly defined subgroups of $\mathrm{Aut}(\widehat{F}_2)$ are actually of the form

$$\widehat{GT}_{\bullet} = \mathrm{Aut}(\overline{\pi}_{\mathcal{V}_{\bullet}})$$

for properly chosen categories $\mathcal{V}_{\bullet}$ of geometrically integral $\mathbb{Q}$-varieties; e.g. $\widehat{GT} = \widehat{GT}_{\mathcal{V}_0}$, where $\mathcal{V}_0 := \{M_{0,4}, M_{0,5}\}$ is the full subcategory of $\mathcal{T}$ with objects $M_{0,4}$, $M_{0.5}$, cf. Harbater–Schneps [H-Sch]. On the other hand, the other categories $\mathcal{V}_{\bullet}$ under discussion, are not necessarily subcategories of $\mathcal{T}$.

Concerning concrete general results, the nature of $\rho_{\mathcal{V}}$ in the above cases and in general is only partially understood. First, concerning the *injectivity* of $\rho_{\mathcal{V}}$, Drinfel'd remarked that using Belyi's Theorem [Be] it follows that $\rho_{\mathcal{V}}$ is injective, provided $U_0 := M_{0,4} = \mathbb{P}^1\backslash\{0, 1, \infty\}$ lies in $\mathcal{V}$, and Voevodsky showed that the same is true if $X \in \mathcal{V}$, where $X := E\backslash\{*\}$ is the complement of a point in an elliptic curve E; Matsumoto [Ma] showed that the same holds if $X \in \mathcal{V}$ for some affine hyperbolic curve X, and finally, Hoshi–Mochizuki [H–Mz] proved that $\rho_{\mathcal{V}}$ is injective as soon as $\mathcal{V}$ contains any hyperbolic curve (complete or not). On the other hand, the question about describing non-tautologically the image $\mathrm{im}(\rho_{\mathcal{V}})$, in particular the question about the *surjectivity* of the representation $\rho_{\mathcal{V}}$, was/is less understood. Ihara asked in the 1980's whether $\rho_{\mathcal{V}}$ is an isomorphism, provided $\mathcal{V} = \mathfrak{Var}_{\mathbb{Q}}$; and Oda–Matsumoto conjectured (based on some motivic evidence) in the 1990's that Ihara's question has a positive answer. Let classical I/OM stand for Ihara's question / Oda–Matsumoto conjecture:

Classical I/OM. *Prove that if* $\mathcal{V} = \mathfrak{Var}_{\mathbb{Q}}$*, then* $\rho_{\mathcal{V}} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ *is an isomorphism.*

The author gave a proof (end of 1990's) of the above classical I/OM, and slightly later, André [An] showed that the p-adic tempered I/OM holds. [This variant of the I/OM is obtained by replacing $\mathbb{Q}$ by $\mathbb{Q}_p$ and $\overline{\pi}_1(X)$ by the tempered fundamental group $\pi_1^{\mathrm{temp}}(X)$, which carries more information than $\overline{\pi}_1(X)$.] Author's original proof of the classical I/OM was never published, because shortly later, he started developing a completely new approach to tackle I/OM types questions. That approach allows —among other things— to formulate and prove (*birational*) *pro-ℓ abelian-by-central* variants of I/OM, which are much stronger than and imply the classical I/OM, cf. [P5].[2] In a nutshell, the basic idea is as follows:

[1] Here, $\widehat{F}_2$ is the profinite free group on two generators.

[2] Among other things, the present note renders the "officially" unpublished [P5] obsolete.

Let k_0 be an arbitrary field, $k := \overline{k}_0$. For a category $\mathcal{V}$ of geometrically integral k_0-varieties, let $\mathcal{F}$ be the category of the functions fields $K := k(\overline{X})$, $X \in \mathcal{V}$ having as morphisms the k-embeddings $L := k(\overline{Y}) \hookrightarrow k(\overline{X}) =: K$ defined by the dominant $\mathcal{V}$-morphisms $X \to Y$. Let $\mathcal{C}$ be a Serre class of groups, e.g., finite (abelian-by-central) [ℓ-groups], unipotent/linear, etc., and $\overline{\pi}_1^{\mathcal{C}}(X)$ and $\mathrm{Gal}_K^{\mathcal{C}}$ be the corresponding completions of $\overline{\pi}_1(X)$, respectively Gal_K. Then paralleling the discussion above, one can formulate the pro-$\mathcal{C}$ I/OM for $\mathcal{V}$ and the pro-$\mathcal{C}$ I/OM for $\mathcal{F}$, where the latter should be rather called the birational pro-$\mathcal{C}$ I/OM for $\mathcal{V}$. Moreover, if $\mathcal{V}$ contains a basis of open neighborhoods of the generic point η_X for every $X \in \mathcal{V}$, e.g., $\mathcal{V} = \mathfrak{Var}_{k_0}$, then by taking limits one gets: Every automorphism $\Phi_{\mathcal{V}}^{\mathcal{C}} \in \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}^{\mathcal{C}})$ gives rise to an automorphism $\Phi_{\mathcal{F}}^{\mathcal{C}} \in \mathrm{Aut}(\mathrm{Gal}_{\mathcal{F}}^{\mathcal{C}})$, that is, to automorphisms $\Phi_K^{\mathcal{C}} \in \mathrm{Gal}_K^{\mathcal{C}}$, $K \in \mathcal{F}$, compatible with all $\mathcal{F}$-morphisms $L \hookrightarrow K$. And an easy verification shows that the pro-$\mathcal{C}$ I/OM for $\mathcal{V}$ follows from the birational pro-$\mathcal{C}$ I/OM for $\mathcal{V}$. In particular, for $\mathcal{C}$ the class of all the finite *abelian-by-central ℓ-groups,* $\ell \neq \mathrm{char}(k_0)$, one gets the (birational) pro-ℓ abelian-by-central I/OM for $\mathcal{V}$, as introduced [P5] and proved there for "sufficiently rich" categories $\mathcal{V}$ by using techniques developed to tackle the so called *Bogomolov's Program*; see [P3], [P4] for details about the latter. This also suggests that in the case of other classes $\mathcal{C}$, like the ones mentioned above, the corresponding pro-$\mathcal{C}$ I/OM type results might lead to Galois group like objects of interest in arithmetic/algebraic geometry. See Remark 2.10 for such an instance.

For the rest of the paper, we introduce notations as follows:

Notations 1.1. Let ℓ be a fixed prime number, and k_0 an arbitrary base field, $\mathrm{char}(k_0) \neq \ell$. For geometrically integral k_0-varieties X, let $\overline{\pi}_1(X) \twoheadrightarrow \Pi_X^{\mathrm{c}} \twoheadrightarrow \Pi_X$ be the pro-ℓ abelian-by-central, respectively pro-ℓ abelian (quotients of the) algebraic fundamental group of X. We notice the following:

1) First, the canonical projections $\overline{\pi}_1(X) \twoheadrightarrow \Pi_X^{\mathrm{c}} \twoheadrightarrow \Pi_X$ give rise canonically to projections $\mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\mathrm{out}}}\big(\overline{\pi}_1(X)\big) \to \mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\mathrm{out}}}(\Pi_X^{\mathrm{c}}) \to \mathrm{Aut}(\Pi_X)$, which usually are not injective or surjective.
2) One has $\Pi_X = \mathrm{H}^1_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell)^\vee$, and $\mathrm{Aut}^{\mathrm{c}}(\Pi_X) := \mathrm{im}\big(\mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\mathrm{out}}}(\Pi_X^{\mathrm{c}}) \to \mathrm{Aut}(\Pi_X)\big)/\mathbb{Z}_\ell^\times$ consists of the automorphisms compatible with $\cup : \mathrm{H}^1_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell) \times \mathrm{H}^1_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell) \to \mathrm{H}^2_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell)$.

Finally, for categories $\mathcal{V}$ of geometrically integral k_0-varieties, consider the corresponding quotients of $\overline{\pi}_{\mathcal{V}}$ and the resulting representation of Gal_{k_0} below

$$\overline{\pi}_{\mathcal{V}} \twoheadrightarrow \Pi_{\mathcal{V}}^{\mathrm{c}} \twoheadrightarrow \Pi_{\mathcal{V}}\,, \qquad \rho_{\mathcal{V}}^{\mathrm{c}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}) \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}}).$$

Notice that the classical I/OM is a rather theoretical question of foundational nature. On the other hand, by the discussion above, the (birational) pro-ℓ abelian-by-central I/OM for $\mathcal{V}$ is quite concrete and relates in down-to-earth terms to the étale ℓ-adic cohomology of the category $\mathcal{V}$. Further we notice that, strictly speaking, the "sufficiently rich" hypothesis under which the (birational) pro-ℓ abelian-by-central I/OM for $\mathcal{V}$ was proved in [P5] requires:

a) Every $Y \in \mathcal{V}$ is dominated by some $X \in \mathcal{V}$ satisfying: $\dim(X) > 1$ and $\mathcal{V}$ contains some basis $\mathcal{B}_X$ of Zariski open neighborhoods U_i of the generic point $\eta_X \in X$.
b) For X and $U_i \in \mathcal{B}_X$ as above, $\mathcal{V}$ contains (virtually) **all** dominant morphisms $U_i \to U_0$, where $U_0 = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ is the tripod.

Especially condition b) is quite restrictive and moves away from and beyond the Teichmüller tower type situation. This being said, the aim of this note is to

prove similar I/OM *type results but under much weaker hypotheses,* by weakening hypothesis b) to the extent that $\mathcal{V}$ contains the morphisms $U_i \to U_0$ defined by only **(finitely many)** rational maps $\varphi_t : X \dashrightarrow U_0$, given in advance, necessary to rigidify $\mathcal{V}$. This gives much more concrete descriptions of Gal_{k_0} for k_0 global and/or local fields, that might be used in studying representations of Gal_{k_0} and the (birational) Tate conjectures.

Example. *The birational Grothendieck–Teichmüller groups* $\widehat{GT}_{\mathrm{bir}}$ *and* $\widehat{GT}^{\mathrm{c}}_{\mathrm{bir}}$

Recall that $\mathcal{V}_0 := \{M_{0,4}, M_{0,5}\}$, and let $\varphi_i : M_{0,5} \to M_{0,4}$ be the morphisms of $\mathcal{V}_0$ defined by "forgetting the i^{th} marked point" for $1 \leqslant i \leqslant 5$. Further, one has $M_{0,4} = \mathbb{P}^1\backslash\{0, 1, \infty\} = U_0$, and $M_{0,5} = U_0 \times U_0 \backslash \Delta_{U_0}$ with Δ_{U_0} the image of the diagonal morphism $U_0 \hookrightarrow U_0 \times U_0$. Hence

$$M_{0,4} = \mathrm{Spec}\ \mathbb{Q}[t_0, \tfrac{1}{t_0}, \tfrac{1}{1-t_0}], \quad M_{0,5} = \mathrm{Spec}\,\mathbb{Q}[t_1, t_2, \tfrac{1}{t_1}, \tfrac{1}{t_2}, \tfrac{1}{1-t_1}, \tfrac{1}{1-t_2}, \tfrac{1}{t_1-t_2}],$$

and the projections $\varphi_i : M_{0,5} \to M_{0,4}$ are defined by the field embeddings $\mathbb{Q}(t_0) \hookrightarrow \mathbb{Q}(t_1, t_2)$, $t_0 \mapsto t \in \Theta_0$, where $\Theta_0 := \{t_1, t_2, t_1 - t_2, t', t''\}$ with $t', t'' \in \mathbb{Q}(t_1, t_2)$ explicitly computable.

We set $\Theta := \{t_1, t_2, t_2 - t_1\}$, and for an arbitrary but otherwise fixed basis $\mathcal{B} = \{U_i \mid i \in I\}$ (w.r.t. inclusion) for the complements of curves $C_i = V(f_i) \subset M_{0,5}$, consider the category:

$$\mathcal{V}_{0,\mathrm{bir}} := \mathcal{V}_{0,\Theta,\mathcal{B}}$$

with objects $\mathcal{B} \cup \{U_0\}$, and having as morphisms, first, the canonical inclusions $U_j \hookrightarrow U_i$ for $C_i \subset C_j$ and id_{U_0}, and second, the projections $\varphi_t : U_i \to U_0$ defined by $\mathbb{Q}(t_0) \hookrightarrow \mathbb{Q}(t_1, t_2)$, $t_0 \mapsto t \in \Theta$. Then in the above notation, one has the resulting canonical representations

$$(*) \qquad \rho_{\mathcal{V}_{0,\mathrm{bir}}} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}_{0,\mathrm{bir}}}) =: \widehat{GT}_{\mathrm{bir}}\,, \qquad \rho^{\mathrm{c}}_{\mathcal{V}_{0,\mathrm{bir}}} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_{0,\mathrm{bir}}}) =: \widehat{GT}^{\mathrm{c}}_{\mathrm{bir}}\,.$$

An easy verification shows that the category $\mathcal{V}_{0,\mathrm{bir}}$ satisfies Hypothesis (H), from Definition/Remark 2.2 below. Hence in this concrete situation, by Theorem 2.6, 1) and Theorem 2.7, 1) below, one gets the following far reaching generalization of the results from [P5]:

Theorem. *The canonical representations* $\rho_{\mathcal{V}_{0,\mathrm{bir}}}$ *and* $\rho^{\mathrm{c}}_{\mathcal{V}_{0,\mathrm{bir}}}$ *from* $(*)$ *above are isomorphisms.*

Acknowledgemnets: I would like to thank Ching-Li Chai, Franz Oort, Jakob Stix, Alexander Schmidt and Tamás Szamuely for technical discussions and help, Pierre Lochak for insisting that these facts should be thoroughly investigated, and many others who showed interest in this work: Yves André, Pierre Deligne, R. Hain, Y. Ihara, Minhyong Kim, M. Matsumoto, N. Nakamura, M. Saidi, A. Tamagawa for discussions on several occasions. Special thanks are due to the University of Heidelberg, University of Bonn, and there MPI Bonn, for the excellent working conditions during my visits there as visiting scientist. Last but not least, many thanks to the referee, for the careful reading of the manuscript and suggestions to improve the presentation.

2. Presentation of the results

As already mentioned, the results of this note refine and generalize the ones from [P5]. Essential technical steps and tools for the proofs developed here are new and go beyond what was done in loc.cit. The results presented here hold and will be proved over *arbitrary perfect base fields* k_0. In particular, the classical I/OM over $\mathbb{Q}$ is a consequence of the tame I/OM, as given below in Theorem 2.7. We begin by introducing/recalling notation and terminology.

Recall that $\boldsymbol{\mathcal{G}}^{\mathrm{out}}$ is the category of profinite groups and outer (continuous) group homomorphisms, i.e., for given $G, H \in \boldsymbol{\mathcal{G}}^{\mathrm{out}}$, an element of $\mathrm{Hom}_{\boldsymbol{\mathcal{G}}^{\mathrm{out}}}(G, H)$ is of the form $\mathrm{Inn}_H \circ f$ with $f : G \to H$ a continuous and Inn_H the inner automorphisms of H.

We set $k := \overline{k}_0$, and given a geometrically integral k_0-variey X and its base change $\overline{X} := X \times_{k_0} k$, we view the algebraic fundamental group $\overline{\pi}_1(X) := \pi_1^{\rm et}(\overline{X}, *)$ of X as an object in $\boldsymbol{\mathcal{G}}^{\rm out}$. Hence the ambiguity resulting from base points vanishes, and by mere definitions one has $\mathrm{Out}\big(\overline{\pi}_1(X)\big) = \mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\rm out}}(\overline{\pi}_1(X))$. Further, via the canonical exact sequence

$$1 \to \overline{\pi}_1(X) \to \pi_1^{\rm et}(X) \to \mathrm{Gal}_{k_0} \to 1$$

one gets a representation $\rho_X : \mathrm{Gal}_{k_0} \to \mathrm{Out}\big(\overline{\pi}_1(X)\big) = \mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\rm out}}(\overline{\pi}_1(X))$, and by the functoriality of the étale fundamental group, the collection of all the representations $(\rho_X)_X$, is compatible with the base changes of k_0-morphisms $f : X \to Y$ of geometrically integral k_0-varieties. In particular, for every category $\mathcal{V}$ of geometrically integral varieties over k_0, its algebraic fundamental group functor $\overline{\pi}_{\mathcal{V}} : \mathcal{V} \to \boldsymbol{\mathcal{G}}^{\rm out}$ gives rise to a representation

$$\rho_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}),$$

where $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ is the automorphism group of $\overline{\pi}_{\mathcal{V}}$. In down to earth terms, the elements $\Phi \in \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ are the families $\Phi = (\Phi_X)_{X\in\mathcal{V}}$, $\Phi_X \in \mathrm{Out}\big(\overline{\pi}_1(X)\big) = \mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\rm out}}(\overline{\pi}_1(X))$, which are compatible with $\overline{\pi}_1(f) : \overline{\pi}_1(X) \to \overline{\pi}_1(Y)$ for all $\mathcal{V}$-morphisms $f : X \to Y$.

Next we recall the pro-ℓ abelian-by-central I/OM from [P5] in detail. Let $\overline{\pi}_1 \to \Pi^{\rm c} \to \Pi$ be the pro-ℓ abelian-by-central and the pro-ℓ abelian quotients of $\overline{\pi}_1$ as introduced in Notations 1.1. Then by mere definitions, $\Pi_X^{\rm c} \to \Pi_X$ are the maximal pro-ℓ quotients of $\overline{\pi}_1(X)$ with Π_X abelian, and $\ker(\Pi_X^{\rm c} \to \Pi_X)$ in the center of $\Pi_X^{\rm c}$. Since the kernels in $\overline{\pi}_1(X) \to \Pi_X^{\rm c} \to \Pi_X$ are characteristic subgroups, there are canonical projections:

$$\mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\rm out}}(\overline{\pi}_1(X)) \to \mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\rm out}}(\Pi_X^{\rm c}) \to \mathrm{Aut}(\Pi_X).$$

Hence for every category $\mathcal{V}$ of geometrically integral k_0-varieties, the canonical morphisms of functors $\overline{\pi}_{\mathcal{V}} \to \Pi_{\mathcal{V}}^{\rm c} \to \Pi_{\mathcal{V}}$ give rise to homomorphisms $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}}) \to \mathrm{Aut}(\Pi_{\mathcal{V}}^{\rm c}) \to \mathrm{Aut}(\Pi_{\mathcal{V}})$. Further, $\mathbb{Z}_\ell^\times$ acts by multiplication on Π_X, and by general group theoretical non-sense, that action lifts to a $\mathbb{Z}_\ell^\times$-action on $\Pi_X^{\rm c}$. Hence we get naturally a representation:

$$\rho_{\mathcal{V}}^{\rm c} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\rm c}(\Pi_{\mathcal{V}}) := \mathrm{im}\big(\mathrm{Aut}(\Pi_{\mathcal{V}}^{\rm c}) \to \mathrm{Aut}(\Pi_{\mathcal{V}})\big)/\mathbb{Z}_\ell^\times.$$

Conjecture (pro-ℓ abelian-by-central I/OM over k_0). *Let $\mathcal{V} = \mathfrak{Var}_{k_0}$ be the category of geometrically integral k_0-varieties. Then $\rho_{\mathcal{V}}^{\rm c} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\rm c}(\Pi_{\mathcal{V}})$ is an isomorphism.*

We will prove more precise and much stronger assertions than the above pro-ℓ abelian-by-central I/OM over k_0. In order to present the results, we need some preparation as follows:

First, concerning general terminology, let $p_X : X \to S$, $p_Y : Y \to S$ be given S-schemes. When speaking about a morphism $f : X \to Y$, we mean a pair (f, f_S), where $f_S : S \to S$ is a scheme isomorphism such that $f_S \circ p_X = p_Y \circ f$. In particular, if $f_S = \mathrm{id}_S$, then f is actually an S-morphism. We denote by $\mathrm{Hom}_S(X, Y) \subseteq \mathrm{Hom}(X, Y)$ the corresponding spaces of S-morphisms, respectively morphisms from X to Y. Second, if $\mathrm{char}(S) = p > 0$, we tacitly assume that the schemes are perfect, i.e., the absolute Frobenius is an isomorphism of schemes, and identify two morphisms which differ by a Frobenius twist. To indicate this, we will use the notation $\mathrm{Hom}^{\rm i}(X, Y)$, and to reduce the amount of explanation, we will use this notation in the case $\mathrm{char}(S) = 0$ as well, where actually $\mathrm{Hom}^{\rm i}(X, Y) = \mathrm{Hom}(X, Y)$.

Let k_0 be a fixed *perfect field* with $\ell \neq \mathrm{char}(k_0)$. By the convention above, when speaking about a k_0-variety X, we will actually mean its perfect closure $X^{\rm i}$, and in particular, the function field $k_0(X)$ will be replaced by $k_0(X)^{\rm i} = k_0(X^{\rm i})$. Finally, up to Frobenius twists, a morphism between k_0-varieties X, Y is a morphism of schemes $f : X \to Y$ which induces a field isomorphism $f_{k_0} : k_0 \to k_0$ on k_0. Finally, up to Frobenius twists, the dominant rational maps $\varphi : X \dashrightarrow Y$ are in bijection with the field embeddings $k_0(Y)^{\mathrm{i}} \hookrightarrow k_0(X)^{\mathrm{i}}$ which map k_0 onto itself. In particular, up to Frobenius twists, the automorphisms $\phi : k_0(X)^{\mathrm{i}} \to k_0(X)^{\mathrm{i}}$ with $\phi(k_0) = k_0$ are in canonical bijection with the birational maps $\varphi : X \dashrightarrow X$, say $\phi \leftrightarrow \varphi$.

We also mention that in the case k_0 is replaced by its algebraic closure k, all the above facts hold, but something new specific to the situation happens. Namely, since $k \subset k(\overline{X})^{\mathrm{i}}$ is the unique maximal algebraically closed subfield, one has: Every field isomorphism ϕ of $k(\overline{X})$ maps k isomorphically onto itself, and therefore originates from a unique birational map $\varphi : \overline{X} \dashrightarrow \overline{X}$ up to Frobenius twists. But since ϕ does not necessarily map k_0 onto/into itself, φ is not necessarily induced by base change from a birational map $X \dashrightarrow X$.

Recall that $U_0 := \operatorname{Spec} k_0[t_0, 1/t_0, 1/(1-t_0)] = \mathbb{P}^1_{k_0} \backslash \{0, 1, \infty\}$ is the k_0-tripod (terminology by Hoshi–Mochizuki) with canonical parameter t_0 on $\mathbb{P}^1$, and that $\operatorname{Aut}^{\mathrm{i}}(U_0) = \operatorname{Aut}^{\mathrm{i}}(k_0) \times \mathfrak{S}_3$, and $\operatorname{Aut}^{\mathrm{i}}_{k_0}(U_0) = \mathfrak{S}_3$. Actually, setting $\mathfrak{U}_{t_0} := \{t_0, 1-t_0, 1/t_0, 1/(1-t_0), t_0/(t_0-1), (t_0-1)/t_0\}$, the representatives ϕ of elements in $\operatorname{Aut}^{\mathrm{i}}_{k_0}(U_0)$ are defined by $t_0 \mapsto t_\phi^{p^e}$ for some $e \in \mathbb{Z}$, $t_\phi \in \mathfrak{U}_{t_0}$.

Up to Frobenius twists, the rational dominant maps $\varphi_t : X \dashrightarrow U_0$ are in bijection with the field embeddings $\phi_t : k_0(t_0) \hookrightarrow k_0(X)$, $t_0 \mapsto t \in k_0(X)$ and $\phi_t(k_0) = k_0$, and φ_t is defined on all sufficiently small open subsets $U \subset X$.

Definition/Remark 2.1. In the above notations, for every open subset $U \subseteq X$ there exists a unique maximal open subset $U_{\max} \subseteq X$ such that $U \subseteq U_{\max}$ and the canonical projection $\Pi_U \to \Pi_{U_{\max}}$ is an isomorphism, or equivalently, $\ker(\mathrm{Gal}_K \to \Pi_U) = \ker(\mathrm{Gal}_K \to \Pi_{U_{\max}})$; in particular, $U_{\max}$ is uniquely determined by $\ker(\Pi_K \to \Pi_U)$. We say that U is maximal, if $U_{\max} = U$, and notice that $U \subset X$ is maximal if and only if $\overline{U} \subset \overline{X}$ is maximal. For a set of Zariski open subsets $\mathcal{B}$ of X, we denote $\mathcal{B}_{\max} := \{U_{\max} \,|\, U \in \mathcal{B}\}$, and notice that the base change $\overline{\mathcal{B}}_{\max}$ of $\mathcal{B}_{\max}$ under $k_0 \hookrightarrow k$ is precisely $\{V_{\max} \,|\, V \in \overline{\mathcal{B}}\}$. Further, if $\mathcal{B}$ is a basis of open neighborhoods of the generic point η_X, then so is $\mathcal{B}_{\max}$, thus so is $\overline{\mathcal{B}}_{\max}$ for $\eta_{\overline{X}}$. Finally, let $\varphi : \overline{X} \dashrightarrow \overline{X}$ be a birational map. Then φ is defined on $\overline{U}_{\max}$ for all sufficiently small open subsets $U \subset X$, and if φ is defined on some $\overline{U}_{\max}$, then $\varphi(\overline{U}_{\max}) = \varphi(\overline{U})_{\max}$.

Definition/Remark 2.2. In the above notations, let $\Theta \subset k_0(X)$ be a subset of non-constant functions, and $\varphi_t : X \dashrightarrow U_0$, $t_0 \mapsto t \in \Theta$, be the corresponding dominant rational k_0-maps.

1) For a *basis of neighborhoods* $\mathcal{B}$ of the generic point $\eta_X \in X$, consider the small category

$$\mathcal{V}_X := \mathcal{V}_{X,\Theta,\mathcal{B}}$$

with objects $\mathcal{B} \cup \{U_0\}$, and morphisms as follows: First, id_{U_0} and the canonical inclusions $U_j \hookrightarrow U_i$, and second, the restrictions $\varphi_{t,i} := \varphi_t|_{U_i}$, $t \in \Theta$, provided φ_t is defined on U_i.

2) Let $\varphi : \overline{X}^{\mathrm{i}} \dashrightarrow \overline{X}^{\mathrm{i}}$ be a birational map. We say that φ is $\mathcal{V}_X$-compatible, if φ satisfies:
   i) There exists $\varphi_0 \in \operatorname{Aut}^{\mathrm{i}}(\overline{U}_0)$ such that $\varphi_0 \circ \varphi_t = \varphi_t \circ \varphi$, $t \in \Theta$, as rational maps.
   ii) If $U \in \mathcal{B}$ and φ is defined on $\overline{U}_{\max}$, then $\varphi(\overline{U}_{\max}) = \overline{U}_{\max}$.

   The set of all the $\mathcal{V}_X$-compatible birational maps is a subgroup $\operatorname{Aut}^{\mathrm{i}}_{\mathcal{V}_X}(K) \leqslant \operatorname{Aut}^{\mathrm{i}}(K)$, and the image of the canonical embedding $\mathrm{Gal}_{k_0} \to \operatorname{Aut}(K^{\mathrm{i}})$ lies actually in $\operatorname{Aut}^{\mathrm{i}}_{\mathcal{V}_X}(K)$.

3) We say that $\mathcal{V}_X$ is rigid, if it satisfies the following equivalent conditions:
   i) The restriction of every $\phi \in \operatorname{Aut}^{\mathrm{i}}_{\mathcal{V}_X}(K)$ to $k_0(X)^{\mathrm{i}}$ is a power of Frobenius.

ii) The canonical embedding $\mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{i}}_{\mathcal{V}_X}(K)$ is surjective, thus $\mathrm{Aut}^{\mathrm{i}}_{\mathcal{V}_X}(K) = \mathrm{Gal}_{k_0}$.

Hypothesis (H): *The following are satisfied:* $\dim(X) > 1$, $k(X) = k(\Theta)$, $\mathcal{V}_X$ *is rigid.*

Remark 2.3. The following hold (the proofs being straightforward verifications):

1) The fact that $\mathcal{V}_X$ satisfies Hypothesis (H) above is somehow the generic case. Indeed:
   a) $\mathcal{V}_X$ is rigid, provided $K^{\mathrm{i}}|k_0$ is geometrically rigid, i.e., $\mathrm{Aut}_k(K^{\mathrm{i}}) = 1 = \mathrm{Aut}^{\mathrm{i}}(K)|_{k_0}$. In general, if $\varphi : \overline{X} \dashrightarrow \overline{X}$ is not a Frobenius twist, there are arbitrarily small open subsets $U \subset X$ with $\varphi(\overline{U}_{\max}) \neq \overline{U}_{\max}$. Hence if $\mathcal{B}$ is chosen randomly, then $\mathcal{V}_X$ is rigid.
   b) $K = k(\Theta)$, provided Θ contains a basis of a linear space $|L|$ for X which defines the birational class of X, i.e, the canonical rational map $X \dashrightarrow \mathbb{P}(|L|)$ is a birational map.
   c) Finally, if Θ is a *linear space* such that the canonical rational map $X \dashrightarrow \mathbb{P}(|\Theta|)$ is a birational map, then $K = k(\Theta)$ and $\mathcal{V}_X$ is rigid for all $\mathcal{B}$.
2) If $\mathcal{V}_X$ is rigid, then $\mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X})$ consists of all the systems of automorphisms $\big((\Phi_i)_i, \Phi_0\big)$ with $\Phi_i \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{U_i})$ and $\Phi_0 \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{U_0})$ such that for all $U_j \hookrightarrow U_i$ and $\varphi_{t,i} : U_i \to U_0$, $t \in \Theta$, the diagrams below are commutative:

$$(*) \qquad \begin{array}{ccc} \Pi_{U_j} & \xrightarrow{\Phi_j} & \Pi_{U_j} \\ \downarrow \text{can} & & \downarrow \text{can} \\ \Pi_{U_i} & \xrightarrow{\Phi_i} & \Pi_{U_i} \end{array} \qquad\qquad \begin{array}{ccc} \Pi_{U_i} & \xrightarrow{\Phi_i} & \Pi_{U_i} \\ \downarrow \Pi(\varphi_{t,i}) & & \downarrow \Pi(\varphi_{t,i}) \\ \Pi_{U_0} & \xrightarrow{\Phi_0} & \Pi_{U_0} \end{array}$$

Definition/Remark 2.4. In the above context, let $\big((\Phi_i)_i, \Phi_0\big) \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X})$ be given.

1) Since $\Pi^{\mathrm{c}}_K \to \Pi_K$ is the projective limit of the projective system $(\Pi^{\mathrm{c}}_{U_i} \to \Pi_{U_i})_{i \in I}$, it follows from $(*)$ above that $(\Phi_i : \Pi_{U_i} \to \Pi_{U_i})_i$ defines a *unique* $\Phi \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ satisfying:
   i) Let $\pi_t : \Pi_K \to \Pi_{U_0}$ be the canonical projections defined by $\imath_t : k_0(t_0) \hookrightarrow k_0(X)$, $t_0 \mapsto t \in \Theta$. Then $\pi_t \circ \Phi = \Phi_0 \circ \pi_t$, and in particular, $\Phi\big(\ker(\pi_t)\big) = \ker(\pi_t)$.
   ii) Let $p_{U_i} : \Pi_K \to \Pi_{U_i}$ be the canonical projection. Then $p_{U_i} \circ \Phi = \Phi_i \circ p_{U_i}$, and in particular, $\Phi\big(\ker(p_{U_i})\big) = \ker(p_{U_i})$.
2) Given an automorphism $\Phi \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$, we say that Φ is $\mathcal{V}_X$-compatible, if it satisfies conditions i), ii) above, and let $\mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)$ be the set of all such automorphisms of Π_K. An easy verification shows that one has canonical group embeddings:

$$\mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X}) \to \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K) \leqslant \mathrm{Aut}^{\mathrm{c}}(\Pi_K)\,.$$

3) Recalling the group of $\mathcal{V}_X$-compatible automorphisms $\mathrm{Aut}^{\mathrm{i}}_{\mathcal{V}_X}(K)$ as introduced in Definition/Remark 2.2, 2), one has that the canonical map $\mathrm{Aut}^{\mathrm{i}}(K) \to \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ arising from Galois Theory is *injective,* and gives rise by restriction to a *canonical embedding*:

$$\mathrm{Aut}^{\mathrm{i}}_{\mathcal{V}_X}(K) \hookrightarrow \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)\,.$$

The stronger/more precise form of the pro-ℓ abelian-by-central I/OM for $\mathcal{V}_X$ is as follows:

Conjecture [(Birational) pro-ℓ abelian-by-central I/OM for $\mathcal{V}_X$].
If $\mathcal{V}_X$ satisfies Hypothesis (H)*, then* $\mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X}) \to \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)$ *are isomorphisms.*

Definition 2.5. Let $\mathcal{V}$ be a category of geometrically integral k_0-varieties.

1) For $X, Y \in \mathcal{V}$, we say that X dominates Y, denoted $Y \prec X$, if there exists a dominant morphism $X \to Y$ which is a $\mathcal{V}$-morphism.

2) We say that $\mathcal{V}$ is connected, if for every X, Y in $\mathcal{V}$ there exist $X_0, \dots, X_{2m}$ in $\mathcal{V}$ such that $X_0 = X$, $X_{2m} = Y$, and for $0 \le i < m$ one has $X_{2i}, X_{2i+2} \prec X_{2i+1}$.
3) We say that $\mathcal{V}$ satisfies Hypothesis (H), if for every $X \in \mathcal{V}$ there exists some $\tilde{X} \in \mathcal{V}$ such that $\mathcal{V}$ contains a subcategory $\mathcal{V}_{\tilde{X}}$ satisfying Hypothesis (H), and there is some $U \in \mathcal{V}_{\tilde{X}}$ with $X \prec U$ and $\Pi_U \to \Pi_X$ surjective.

Theorem 2.6. *Let k_0 be a perfect field. In the above notations the following hold:*
1) *Suppose that the category $\mathcal{V}_X$ satisfies Hypothesis* (H). *Then the resulting canonical representations* $\mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X}) \to \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)$ *are isomorphisms.*
2) *Let $\mathcal{V}$ be a connected category satisfying Hypothesis* (H). *Then the canonical representation* $\rho^{\mathrm{c}}_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}})$ *is an isomorphism.*

An application of Theorem 2.6 is the following strengthening of the classical I/OM:

In the general context above, replace $\overline{\pi}_1, \mathrm{Gal}$ by their valuation tame quotients $\overline{\pi}^{\mathrm{t}}_1, \mathrm{Gal}^{\mathrm{t}}$. Then for every category of geometrically integral k_0-varieties $\mathcal{V}$ one gets a representation

$$\rho^{\mathrm{t}}_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}(\overline{\pi}^{\mathrm{t}}_{\mathcal{V}}).$$

Further, in the context of $\mathcal{V}_X$ above, every $\Phi^{\mathrm{t}} \in \mathrm{Aut}(\overline{\pi}^{\mathrm{t}}_{\mathcal{V}_X})$, say given by $\Phi_0, (\Phi^{\mathrm{t}}_i)_i$, defines an automorphism $\Phi^{\mathrm{t}} \in \mathrm{Out}(\mathrm{Gal}^{\mathrm{t}}_K)$, which is $\mathcal{V}_X$-compatible, i.e., maps $\ker\big(\mathrm{Gal}^{\mathrm{t}}_K \to \overline{\pi}^{\mathrm{t}}_1(U_i)\big)$ onto itself, thus induces isomorphisms $\Phi^{\mathrm{t}}_i : \overline{\pi}^{\mathrm{t}}_1(U_i) \to \overline{\pi}^{\mathrm{t}}_1(U_i)$, $U_i \in \mathcal{V}_X$, and $p^{\mathrm{t}}_t \circ \Phi^{\mathrm{t}} = \Phi_0 \circ p^{\mathrm{t}}_t$ for $p^{\mathrm{t}}_t : \mathrm{Gal}^{\mathrm{t}}_K \to \Pi_{U_0}$, $t \in \Theta$. Hence if $\mathrm{Aut}_{\mathcal{V}_X}(\mathrm{Gal}^{\mathrm{t}}_K) \leqslant \mathrm{Out}(\mathrm{Gal}^{\mathrm{t}}_K)$ denotes the subgroup of $\mathcal{V}_X$-compatible automorphisms, then one has a canonical embedding

$$\mathrm{Aut}(\overline{\pi}^{\mathrm{t}}_{\mathcal{V}_X}) \hookrightarrow \mathrm{Aut}_{\mathcal{V}_X}(\mathrm{Gal}^{\mathrm{t}}_K).$$

Theorem 2.7. *Let k_0 be a perfect field. In the above notations the following hold:*
1) *Suppose that the category $\mathcal{V}_X$ satisfies Hypothesis* (H). *Then the canonical representations* $\mathrm{Gal}_{k_0} \to \mathrm{Aut}(\overline{\pi}^{\mathrm{t}}_{\mathcal{V}_X}) \to \mathrm{Aut}_{\mathcal{V}_X}(\mathrm{Gal}^{\mathrm{t}}_K)$ *are isomorphisms.*
2) *Let $\mathcal{V}$ be a connected category satisfying Hypothesis* (H). *Then the canonical representation* $\rho^{\mathrm{t}}_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}(\overline{\pi}^{\mathrm{t}}_{\mathcal{V}})$ *is an isomorphism.*

The essential technical tool in the proof of the above results is Theorem 2.9 below, which is related to *Bogomolov's Program* as initiated in [Bo], see rather [P3], Introduction.

Definition/Remark 2.8. In the above notations, let $\Theta \subset K \backslash k$ be a non-empty set, endowed with a bijection $\theta : \Theta \to \Theta$, $t \mapsto u$, and for $t \in \Theta$, recall $\varphi_t : \overline{X} \dashrightarrow \overline{U}_0$ and the corresponding $\pi_t : \Pi_K \to \Pi_{U_0}$. We say that $\varphi : \overline{X} \dashrightarrow \overline{X}$, respectively $\Phi \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$, are weakly Θ-compatible, if for every $t \mapsto u$, there is $\varphi_0 \in \mathrm{Aut}(U_0)$, respectively $\Phi_0 \in \mathrm{Aut}(\Pi_{U_0})$, depending on t, u, such that $\varphi_t \circ \varphi_0 = \varphi_u \circ \varphi$, respectively $\Phi_0 \circ \pi_t = \pi_u \circ \Phi$.

Notice that φ and/or Φ being "weakly Θ-compatible" is in general much weaker than conditions i) from Definition/Remark 2.2, 2), respectively 2.4, 1) above. Further, the corresponding subsets $\mathrm{Aut}_{\Theta}(K^{\mathrm{i}}) \subset \mathrm{Aut}(K^{\mathrm{i}})$, $\mathrm{Aut}^{\mathrm{c}}_{\Theta}(\Pi_K) \subset \mathrm{Aut}(\Pi_K)$ are actually subgroups, and the canonical embedding $\mathrm{Aut}(K^{\mathrm{i}}) \to \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ defines an embedding $\mathrm{Aut}_{\Theta}(K^{\mathrm{i}}) \to \mathrm{Aut}^{\mathrm{c}}_{\Theta}(\Pi_K)$.

Theorem 2.9. *Let $K|k$ be a function field with* $\mathrm{td}(K|k) > 1$, *and $\Theta \subset K$ satisfy $K = k(\Theta)$. Then the canonical embedding* $\mathrm{Aut}_{\Theta}(K^{\mathrm{i}}) \to \mathrm{Aut}^{\mathrm{c}}_{\Theta}(\Pi_K)$ *is an isomorphism. Equivalently, for every* $\Phi \in \mathrm{Aut}^{\mathrm{c}}_{\Theta}(\Pi_K)$ *there exists* $\phi \in \mathrm{Aut}(K^{\mathrm{i}})$, *unique up to Frobenius twists and satisfying:*

i) *ϕ defines Φ, i.e., letting ϕ' be the prolongation of ϕ to the maximal pro-ℓ abelian extension $K'|K$, there exists $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\varepsilon \cdot \Phi(g) = \phi'^{-1} g\, \phi'$ for all $g \in \Pi_K$.*

ii) *For $t \mapsto u$ under $\theta : \Theta \to \Theta$, there exists $t_\phi \in \{t, 1-t, 1/t, 1/(1-t), t/(t-1), (t-1)/t\}$ and a power p^e, $e \in \mathbb{Z}$, of the characteristic exponent p of k, such that $\phi(u) = t_\phi^{p^e}$.*

It turns out that Theorem 2.6 follows relatively easily from Theorem 2.9, whereas Theorem 2.7 follows from Theorem 2.6 and some extra (partially quite technical) work. The techniques for the proof of Theorem 2.9 are the ones developed to tackle Bogomolov's Program, supplemented by some new ideas. Namely using Π_K^{c} endowed with $\pi_t : \Pi_K \to \Pi_{U_0}$, $t \in \Theta$, one has the following: First, Proposition 3.10 gives a recipe to recover the *divisorial subgroups* of Π_K, and based on that, Proposition 3.11 gives a group theoretical recipe to recover the *total decomposition graph* $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ of $K|k$, as introduced/defined in [P3], see section 3. Second, using the Construction 4.6, one gives in Proposition 4.7 a recipe to recover the *geometric rational quotients* of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$. Moreover, the group theoretical recipes to recover these objects are preserved under all the automorphisms $\Phi \in \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$ and/or $\Phi \in \mathrm{Aut}_\Theta(\mathrm{Gal}_K^{\mathrm{t}})$, see Propositions 4.7 and Lemma 7.6 below. Thus by the main result of [P3], Introduction, it follows that every $\Phi \in \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$ originates from geometry, i.e., there exists $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\varepsilon \cdot \Phi$ is defined by some automorphism $\phi : K^{\mathrm{i}}|k \to K^{\mathrm{i}}|k$, etc.

Remark 2.10. In very recent work, Topaz [To3] gives yet another refinement of the (birational) pro-ℓ abelian-by-central I/OM from [P5], in the spirit of the comments in the middle of the Introduction/Motivation above. He introduces, namely, and proves mod-ℓ abelian-by-central variants of I/OM as follows: First, consider the mod-ℓ abelian-by-central and mod-ℓ abelian, quotients $\overline{\pi}_1 \to \pi_1^c \to \pi_1^a$ of $\overline{\pi}_1$. Second, for $U_{\mathbf{a}} \subseteq U_0$ open, consider categories $\mathcal{U}_{\mathbf{a}}$ similar to the categories $\mathcal{V}_X$ above, but satisfying extra conditions, e.g., $\Theta = k_0(X) \backslash k_0$ consists of <u>all</u> the non-constant functions (as done in [P5] as well), and the dimension restriction $\dim(X) > 4$. Then $\mathrm{Aut}^{\mathrm{c}}(\pi^a_{\mathcal{U}_{\mathbf{a}}}) = \mathrm{Gal}_{k_0}$, thus giving a purely *combinatorial* description of Gal_{k_0}. Nevertheless, for the time being, it is unclear whether any of the mod-ℓ abelian-by-central variants of I/OM from [To3] holds for the "coarser" categories $\mathcal{V}_X$ and/or $\mathcal{V}$ which satisfy Hypothesis (H) as introduced above, e.g., Θ finite, and/or $1 < \dim(X) < 5$.

3. Recovering the total decomposition graph

A) *Recalling basics about (quasi) divisorial subgroups*

We begin by recalling a few basic definitions/notations from valuation theory, including Hilbert decomposition/ramification theory in pro-ℓ abelian field extensions, $\ell \neq$ char.

First, for an arbitrary field Ω containing μ_{ℓ^∞}, and a valuation $\mathfrak{v}$ of Ω, let $T_\mathfrak{v} \subset Z_\mathfrak{v}$ be the inertia/decomposition groups of $\mathfrak{v}$ in Π_Ω, and $\Omega^Z \subset \Omega^T$ be the corresponding fixed fields in the maximal pro-ℓ abelian extension $\Omega'|\Omega$. (Note that because Π_Ω is abelian, $T_\mathfrak{v} \subset Z_\mathfrak{v}$ depend on $\mathfrak{v}$ only, and not on the prolongation of $\mathfrak{v}$ to Ω' used to define them.) Further, let $U^1_\mathfrak{v} := 1 + \mathfrak{m}_\mathfrak{v} \subset \mathcal{O}_\mathfrak{v}^\times =: U_\mathfrak{v}$ be the principal $\mathfrak{v}$-units, respectively the $\mathfrak{v}$-units in $\Omega^\times$. Then by [P1], Fact 2.1, see also Topaz [To1], [To2], one has that $\Omega^Z \subseteq \Omega^{Z^1} := \Omega[\sqrt[\ell^\infty]{U^1_\mathfrak{v}}]$, and $\Omega^T \subseteq \Omega^{T^1} := \Omega[\sqrt[\ell^\infty]{U_\mathfrak{v}}]$. We denote $T^1_\mathfrak{v} := \mathrm{Gal}\big(\Omega'|\Omega^{T^1}\big) \subseteq T_\mathfrak{v}$, $Z^1_\mathfrak{v} := \mathrm{Gal}\big(\Omega'|\Omega^{Z^1}\big) \subseteq Z_\mathfrak{v}$ and call $T^1_\mathfrak{v} \subseteq Z^1_\mathfrak{v}$ the (minimized) inertia/decomposition groups of $\mathfrak{v}$. Since $\mathfrak{v}\Omega = \Omega^\times / U_\mathfrak{v}$ and $\Omega\mathfrak{v}^\times = U_\mathfrak{v}/U^1_\mathfrak{v}$, by Kummer theory and Pontrjagin duality, setting $\delta := \dim(\mathfrak{v}\Omega/\ell)$, one has:

$$(\dagger) \qquad T^1_\mathfrak{v} = \mathrm{Hom}_{\mathrm{cont}}\big(\mathfrak{v}\Omega, \mathbb{Z}_\ell(1)\big) \cong \mathbb{Z}_\ell^\delta, \quad \Pi^1_{\Omega\mathfrak{v}} := Z^1_\mathfrak{v}/T^1_\mathfrak{v} = \mathrm{Hom}_{\mathrm{cont}}\big(\Omega\mathfrak{v}^\times, \mathbb{Z}_\ell(1)\big).$$

We notice the following: First, if $\mathrm{char}(\Omega\mathfrak{v}) \neq \ell$, then by [P1], Fact 2.1, one has that $Z^1_\mathfrak{v} = Z_\mathfrak{v}$ and $\Pi^1_{\Omega\mathfrak{v}} = \Pi_{\Omega\mathfrak{v}}$. Second, if $\mathrm{char}(\Omega\mathfrak{v}) = \ell$, one has: Since $\ell \neq \mathrm{char}(\Omega)$, one must have $\mathrm{char}(\Omega) = 0$. Further, $T^1_\mathfrak{v} \subseteq Z^1_\mathfrak{v} \subseteq T_\mathfrak{v}$, thus $\Pi^1_{\Omega\mathfrak{v}} \subseteq T_\mathfrak{v}/T^1_\mathfrak{v}$ has trivial image in $\Pi_{\Omega\mathfrak{v}} = Z_\mathfrak{v}/T_\mathfrak{v}$, and the residue field of $\mathcal{O}^{z^1}_\mathfrak{v}$ contains $\sqrt[\ell^\infty]{\Omega\mathfrak{v}}$.

Second, let $\Omega|\kappa$ be a function field, say $\Omega = \kappa(Z)$ is the function field of some (geometrically) integral κ-variety Z. A defectless valuation, or a valuation without defect, of $\Omega|\kappa$ is any valuation $\mathfrak{v}$ of Ω which satisfies the Abhyankar equality

$$\mathrm{td}(\Omega|\kappa) = \mathrm{td}(\Omega\mathfrak{v}|\kappa\mathfrak{v}) + \mathrm{rr}(\mathfrak{v}\Omega/\mathfrak{v}\kappa),$$

where we denote $\mathrm{rr}(A) := \dim_\mathbb{Q} A \otimes \mathbb{Q}$ the rational rank of any abelian group A. Suppose that $\kappa = \overline{\kappa}$. Then given a defectless valuation $\mathfrak{v}$ of $\Omega|\kappa$, the following hold, see e.g., [Kh]:

a) $\mathfrak{v}\Omega/\mathfrak{v}\kappa$ is a finitely generated free abelian group, and $\Omega\mathfrak{v}|\kappa\mathfrak{v}$ is a function field.
b) The restriction $\mathfrak{v}_1 := \mathfrak{v}|_{\Omega_1}$ of $\mathfrak{v}$ to any function subfield $\Omega_1|\kappa \hookrightarrow \Omega|\kappa$ is defectless.

Coming back to the context from Introduction, recall that a prime divisor of $K|k$ is a discrete valuation v of K which is trivial on k and has a function field Kv as residue field satisfying $\mathrm{td}(Kv|k) = \mathrm{td}(K|k) - 1$. We call $T_v \subset Z_v$ a divisorial subgroup of Π_K. It turns out that knowing the divisorial subgroups $T_v \subset Z_v$ of Π_K is one of the *key technical ingredients* in reconstructing the function field $K|k$ from its pro-ℓ abelian-by-central Galois group Π^c_K.

Unfortunately, at the moment *there is no group theoretical recipe* to recover the divisorial groups $T_v \subset Z_v$ from the group theoretical information encoded in Π^c_K in the case of an arbitrary algebraically closed base field k. The best one can do so far in general is to recover the larger class of all the (minimized) quasi divisorial subgroups $T^1_\mathfrak{v} \subset Z^1_\mathfrak{v}$ of Π_K from the group theoretical information encoded in Π^c_K. The precise definitions and result are as follows: First, a valuation $\mathfrak{v}$ of $K|k$, which is not necessarily trivial on k, is called a quasi prime divisor of $K|k$ provided it satisfies the following:

i) $\mathfrak{v}K \neq \mathfrak{v}k$ and $\mathrm{td}(K\mathfrak{v}\,|\,k\mathfrak{v}) = \mathrm{td}(K|k) - 1$.
ii) No proper coarsening of v satisfies these properties.

Condition i) implies that $\mathfrak{v}$ is defectless on $K|k$, hence $\mathfrak{v}K/\mathfrak{v}k \cong \mathbb{Z}$, and $K\mathfrak{v}|k\mathfrak{v}$ is a function field. Second, a quasi prime divisor $\mathfrak{v}$ of $K|k$ is a prime divisor of $K|k$ iff $\mathfrak{v}$ is trivial on k.

Let $L|k \hookrightarrow K|k$ be a function subfield of $K|k$, and $\mathfrak{w} := \mathfrak{v}|_L$ for some quasi prime divisor $\mathfrak{v}$ of $K|k$. Since both the residual transcendence degree and the rational rank are additive in towers of function field extension, one has the following:

Remark 3.1. In the above notations, there are only two possibilities for $\mathfrak{v}$ and $\mathfrak{w}$, namely:

a) $\mathrm{rr}(\mathfrak{w}L/\mathfrak{w}k) = 1$, or equivalently, $\mathfrak{w}$ is a quasi prime divisor of $L|k$.
b) $\mathrm{td}(L|k) = \mathrm{td}(L\mathfrak{w}|k\mathfrak{w})$, thus $\mathfrak{w}$ is by definition a constant reduction (à la Deuring) of $L|k$.

The first point we want to make is that Galois theory encodes the nature of the above restriction $\mathfrak{w} = \mathfrak{v}|_L$ of $\mathfrak{v}$ to $L|k$ as follows:

Fact 3.2. *Let $p_L : \Pi_K \to \Pi_L$ be the canonical projection. Then the following hold:*

1) *p_L maps $T^1_\mathfrak{v} \subset Z^1_\mathfrak{v}$ into $T^1_\mathfrak{w} \subset Z^1_\mathfrak{w}$, and $p_L(Z^1_\mathfrak{v}) \subseteq Z^1_\mathfrak{w}$, $p_L(T^1_\mathfrak{v}) \subseteq T^1_\mathfrak{w}$ are open subgroups.*
2) *Therefore, $\mathfrak{w}$ is a quasi prime divisor of $L|k$ if $p_L(T^1_\mathfrak{v}) \neq 1$, respectively a constant reduction of $L|k$ if $p_L(T^1_\mathfrak{v}) = 1$.*

The second point we make is the result about recovering the quasi divisorial subgroups of Π_K from $\Pi^{\rm c}_K$ is as follows, see [P1] and Topaz [To1], [To2]:

Fact 3.3. *Let $\Pi^{\rm c}_K \to \Pi_K$ be the canonical projection, and for subgroups $G \subset \Pi_K$, let $G'' \subset \Pi^{\rm c}_K$ be their preimages in $\Pi^{\rm c}_K$. Then the following hold:*

1) *Let d be the maximal positive integer such that Π_K contains subgroups $\Delta \cong \mathbb{Z}_\ell^d$ with abelian preimage $\Delta'' \subset \Pi^{\rm c}_K$. Then $d = \mathrm{td}(K|k)$.*
2) *The minimized quasi divisorial subgroups of Π_K are precisely the pairs $T \subset Z$ which are maximal satisfying the following:*
   i) *Z contains subgroups $\Delta \cong \mathbb{Z}_\ell^d$ having an abelian preimage $\Delta'' \subset \Pi^{\rm c}_K$.*
   ii) *$T \cong \mathbb{Z}_\ell$, and its preimage $T'' \subset \Pi^{\rm c}_K$ is the center of $Z'' \subset \Pi^{\rm c}_K$.*

B) *Recovering the projection $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ from $\pi_t : \Pi_K \to \Pi_{U_0}$*

In the context and notations of Theorem 2.6, let $t \in K\backslash k$ be any non-constant function, and $\kappa_t \subset K$ be the relative algebraic closure of $k(t)$ in K. Then $\kappa_t|k(t)$ is a finite field extension, hence the projection $\Pi_{\kappa_t} \to \Pi_{U_0}$ defined by $t_0 \mapsto t$ has an open image. Therefore, since the canonical projection $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ is (by mere definitions) surjective, it follows that the canonical projection $\pi_t : \Pi_K \to \Pi_{U_0}$ defined by $k(t_0) \hookrightarrow K$, $t_0 \mapsto t$, has an open image in Π_{U_0}. Our aim is to show that there exist group theoretical recipes to recover the projection $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ from the given group theoretical projection $\pi_t : \Pi_K \to \Pi_{U_0}$.

Notations 3.4. In the above context, consider/define:

a) The set $\mathcal{Q}^0_t$ of all the quasi prime divisors $\mathfrak{v}$ of $K|k$ with $\pi_t(Z^1_{\mathfrak{v}}) \subset \Pi_{U_0}$ open.
b) The closed subgroup $\mathfrak{T}^0_t := \langle\, T^1_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{Q}^0_t \,\rangle \subset \Pi_K$ generated by the minimized inertia groups $T^1_{\mathfrak{v}}$, $\mathfrak{v} \in \mathcal{Q}^0_t$, and the resulting canonical projection $p^0_t : \Pi_K \to \Pi^0_t := \Pi_K/\mathfrak{T}^0_t$.
c) The set $\mathcal{Q}_t$ of all the quasi prime divisors $\mathfrak{v}$ of $K|k$ such that image $p^0_t(Z^1_{\mathfrak{v}}) \subset \Pi^0_t$ of $Z^1_{\mathfrak{v}}$ under $p^0_t : \Pi_K \to \Pi^0_t := \Pi_K/\mathfrak{T}^0_t$ is not a topologically finitely generated group.
d) The closed subgroup $\mathfrak{T}_t := \langle\, T^1_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{Q}_t \,\rangle \subset \Pi_K$ generated by the minimized inertia groups $T^1_{\mathfrak{v}}$, $\mathfrak{v} \in \mathcal{Q}_t$, and the resulting canonical projection $p_t : \Pi_K \to \Pi_t := \Pi_K/\mathfrak{T}_t$.

We proceed by shedding some light on the objects defined above.

Lemma 3.5. *For every quasi prime divisor $\mathfrak{v}$ of $K|k$ the following are equivalent:*

i) *$\mathfrak{v} \in \mathcal{Q}^0_t$.*
ii) *t is a $\mathfrak{v}$-unit and residually transcendental, that is, $\bar{t} \in K\mathfrak{v}$ is non-constant.*
iii) *The restriction of $\mathfrak{v}$ to $k(t)$ is the Gauss valuation defined by $v_k := \mathfrak{v}|_k$ and t. In particular, $\mathfrak{v}|_{\kappa_t}$ is a constant reduction of $\kappa_t|k$.*

Proof. First, the equivalence of ii), iii) follows by mere definitions. For the reverse implication ii), iii) $\Rightarrow$ i), we notice that $\Pi_{U_0} \cong \mathbb{Z}_\ell^2$ is noting but the Galois group of the maximal extension $\mathcal{K}_0|k(t_0)$ unramified outside $t_0 = 0, 1, \infty$. On the other hand, since $\bar{t}, \bar{t}-1 \in K\mathfrak{v}$ generate a $\mathbb{Z}$-submodule of rank two in $(K\mathfrak{v})^\times/(k\mathfrak{v})^\times$, it follows by mere definitions, that the image of $Z^1_{\mathfrak{v}}$ in Π_{U_0} under $t_0 \mapsto t$, is isomorphic to $\mathbb{Z}_\ell^2$ as well, thus open in $\Pi_{U_0} \cong \mathbb{Z}_\ell^2$.

Finally, for the implication i) $\Rightarrow$ ii), suppose that $\pi_t(Z^1_{\mathfrak{v}})$ is open in Π_{U_0}. Then by mere definitions, it follows that there exist $\mathfrak{v}$-units $\theta \in tk^\times$, $\eta \in (t-1)k^\times$, such that their images $\overline{\theta}$, $\overline{\eta}$ in $K\mathfrak{v}^\times$ generate a $\mathbb{Z}$-module of rank two in $(K\mathfrak{v})^\times/(k\mathfrak{v})^\times$. Hence setting $\theta = t/a$, $\eta = b(t-1)$

with $a, b \in k^\times$, we must have $t = a\theta$, and $\eta = ba\theta - b$. We claim that a, b are v_k-units. Indeed, one has the following case-by-case analysis:

- $v_k(b) < 0$. Since $\eta = b(a\theta - 1)$ is a $\mathfrak{v}$-unit, we must have $\mathfrak{v}(a\theta - 1) > 0$, hence $a\theta \in 1 + \mathfrak{m}_\mathfrak{v}$. Since θ is $\mathfrak{v}$-unit, so must be a, and $\overline{a}\,\overline{\theta} = 1$ in $K\mathfrak{v}$. Hence $\overline{\theta} = 1/\overline{a} \in k\mathfrak{v}$, contradiction!
- $v_k(b) > 0$. Then $\mathfrak{v}(a\theta - 1) < 0$, hence $\mathfrak{v}(a\theta) < 0$. Thus $\eta = (ba)\theta - b$ implies that ab is a v_k-unit, and $\overline{\eta}, \overline{\theta}$ have equal images in $(K\mathfrak{v})^\times/(k\mathfrak{v})^\times$, contradiction!
- $v_k(b) = 0$, i.e., b is a v_k-unit. Then $a\theta = b^{-1}\eta + 1$ is a $\mathfrak{v}$-unit (because the RHS is so), and since θ is a $\mathfrak{v}$-unit, so is a. Thus conclude that a, b both are v_k-units.

To proceed, notice that we have $t = \theta/a$. Since a is a v_k-unit, and θ is $\mathfrak{v}$-residually transcendental, it follows that t is a $\mathfrak{v}$-unit, and $\overline{t}$ is $\mathfrak{v}$-residually transcendental. □

For the next Lemma, we recall the following basic facts about the pro-ℓ abelian birational fundamental group $\Pi_{1,K|k}$ of $K|k$ (and correspondingly, for its subfield $\kappa_t|k$, etc.), see [P3], Appendix for further details. First, for every set of quasi prime divisors $\mathcal{Q}$ of $K|k$, let $\mathfrak{T}_\mathcal{Q} \subseteq \Pi_K$ be the closed subgroup generated by $(T^1_\mathfrak{v})_{\mathfrak{v}\in\mathcal{Q}}$. We set $\Pi_{1,\mathcal{Q}} := \Pi_K/\mathfrak{T}_\mathcal{Q}$ and call $\Pi_{1,\mathcal{Q}}$ the pro-ℓ abelian fundament group of $\mathcal{Q}$. The pro-ℓ abelian fundamental group of the set of all the prime divisors $\mathcal{D}_{K|k}$ of $K|k$ is called the (pro-ℓ abelian) birational fundamental group of $K|k$, and is denoted by $\Pi_{1,K|k}$. Notice/recall that $\Pi_{1,K|k}$ equals the abelian pro-ℓ (quotient of the) fundamental group Π_X of any complete regular model X of $K|k$ —if such models exist. In any case, Π_X is a quotient of $\Pi_{1,K|k}$ for every complete normal model X, and there always exist normal projective models X of $K|k$ such that $\Pi_{1,K|k} = \Pi_X$. In particular $\Pi_{1,K|k}$ is topologically finitely generated, or equivalently, it is a finite $\mathbb{Z}_\ell$-module.

Lemma 3.6. *In the above notations, set $\tilde{k} := \overline{\kappa_t}$, and $\tilde{K} := K\tilde{k}$. The following hold:*

1) $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ *factors through* $p^0_t : \Pi_K \to \Pi^0_t$, *say* $p_{\kappa_t} = q^0_t \circ p^0_t$ *with* $q^0_t : \Pi^0_t \to \Pi_{\kappa_t}$.
2) $\Delta^0_t := \ker(q^0_t) = \ker(\Pi^0_t \to \Pi_{\kappa_t})$ *is a quotient of* $\Pi_{1,\tilde{K}|\tilde{k}}$ *hence a finite* $\mathbb{Z}_\ell$*-module.*

Proof. To 1): First, recall that $\Pi_{\kappa_t} = \mathrm{Gal}\big(\kappa'_t\,|\,\kappa_t\big)$ is the Galois group of the maximal pro-ℓ abelian extension $\kappa'_t|\kappa_t$ of κ_t. Hence one has $\ker(p_{\kappa_t}) = \mathrm{Gal}\big(K'\,|\,K\kappa'_t\big)$. Second, by mere definitions, for every valuation $\mathfrak{v}$ of K one has that $T^1_\mathfrak{v} = \mathrm{Gal}\big(K'\,|\,K_{U_\mathfrak{v}}\big)$, where $U_\mathfrak{v}$ is the group of $\mathfrak{v}$-units, and $K_{U_\mathfrak{v}} := K[\sqrt[\ell^\infty]{U_\mathfrak{v}}\,]$. Third, the fact that $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ factors through $p^0_t : \Pi_K \to \Pi^0_t$ is equivalent to $\ker(p^0_t) \subset \ker(p_{\kappa_t})$. On the other hand, since $\ker(p^0_t)$ is generated by $T^1_\mathfrak{v}$, $\mathfrak{v} \in \mathcal{Q}^0_t$, one has: $\ker(p^0_t) \subset \ker(p_{\kappa_t})$ iff $T^1_\mathfrak{v} \subseteq \ker(p_{\kappa_t})$, $\mathfrak{v} \in \mathcal{Q}^0_t$. Switching to field extensions via the Galois correspondence, the inclusion $T^1_\mathfrak{v} \subseteq \ker(p_{\kappa_t})$ is equivalent to $K\kappa'_t \subset K_{U_\mathfrak{v}}$, $\mathfrak{v} \in \mathcal{Q}^0_t$, hence equivalent to $\kappa'_t \subset K_{U_\mathfrak{v}}$ for all $\mathfrak{v} \in \mathcal{Q}^0_t$. On the other hand, since k is algebraically closed, $k^\times$ is ℓ-divisible, hence $\sqrt[\ell^\infty]{U_\mathfrak{v}} = \sqrt[\ell^\infty]{k^\times \cdot U_\mathfrak{v}}$. Hence by Kummer theory and mere definition, the inclusion $\kappa'_t \subset K_{U_\mathfrak{v}}$ is equivalent to

$$(*) \qquad \kappa^\times_t \subseteq k^\times \cdot U_\mathfrak{v}\,.$$

To conclude, we notice that the above inclusion follows from the fact that $\mathfrak{v}|_{\kappa_t}$ is a constant reduction of $\kappa_t|k$. Indeed, since k is algebraically closed, hence $\mathfrak{v}k$ is divisible, it follows that $\mathfrak{v}(\kappa_t) = \mathfrak{v}(k)$. Hence every $u \in \kappa_t$ is of the form $u = au_1$ for some $a \in k$ with $\mathfrak{v}(a) = \mathfrak{v}(u)$, and $u_1 \in U_\mathfrak{v} \cap \kappa^\times_t$. This concludes the proof of assertion 1) of the Lemma.

To 2): First, we notice that a prime divisor v of $K|k$ lies in $\mathcal{Q}^0_t$ if and only if v is trivial on $k(t)$ if and only if v lies in $\mathcal{Q}_t$. Hence the set $\mathcal{D}_t$ of all such prime divisors v of $K|k$ is nothing but the set of prime divisors $\mathcal{D}_{K|\kappa_t}$ of the function field $K|\kappa_t$. Let $\mathcal{K}_t \subset K'$ be the maximal subextension in which all $v \in \mathcal{D}_t$ are unramified. Equivalently, if $\mathfrak{T}_{\mathcal{D}_t} \subset \Pi_K$ is the closed subgroup generated by $T^1_v = T_v$, $v \in \mathcal{D}_t$, it follows that $\mathrm{Gal}\big(\mathcal{K}_t\,|\,K\big) = \Pi_K/\mathfrak{T}_{\mathcal{D}_t}$. Clearly, $\kappa'_t \subset K'$, and recalling that $\tilde{k} := \overline{\kappa_t}$, one has that $\kappa'_t = \tilde{k} \cap K'$, and therefore, $\tilde{k}$ and K' are linearly disjoint over κ'_t. Hence one has an exact sequence of the form

$$(\dagger) \qquad 1 \to \mathrm{Gal}\big(\mathcal{K}_t\kappa'_t\,|\,K\kappa'_t\big) \hookrightarrow \mathrm{Gal}\big(\mathcal{K}_t\,|\,K\big) = \Pi^0_t \xrightarrow{q^0_t} \mathrm{Gal}\big(\kappa'_t\,|\,\kappa_t\big) = \Pi_{\kappa_t} \to 1,$$

in which $\Delta^0_t := \mathrm{Gal}\big(\mathcal{K}_t\,|\,K\kappa'_t\big) = \ker(\Pi^0_t \to \Pi_{\kappa_t})$ is the κ_t-geometric part of $\Pi^0_t = \mathrm{Gal}\big(\mathcal{K}_t\,|\,K\big)$.

Next recalling that $\tilde{K} := K\tilde{k}$, we set $\tilde{\mathcal{K}}_t := \mathcal{K}_t\tilde{k} \subset K'\tilde{k}$. Then since $\tilde{k}$ and K' are linearly disjoint over κ'_t, it follows that the canonical projection below is an isomorphism:

$$\tilde{\Delta}_t := \mathrm{Gal}\big(\tilde{\mathcal{K}}_t\,|\,\tilde{K}\big) \to \mathrm{Gal}\big(\mathcal{K}_t\kappa'_t\,|\,K\kappa'_t\big) = \Delta^0_t.$$

Let $\tilde{\mathcal{D}}_t$ be the set of all the prolongations $\tilde{v}\,|\,v$ of all the valuations $v \in \mathcal{D}_t$ to $\tilde{K}$. Since $\tilde{K}|\tilde{k}$ is the base change of $K|\kappa_t$ under the algebraic extension(s) $\kappa_t \hookrightarrow \kappa'_t \hookrightarrow \tilde{k}$, the following hold: First, since $\mathcal{D}_t$ is the set of all the prime divisors of $K|\kappa_t$, it follows that $\tilde{\mathcal{D}}_t$ equals the set $\mathcal{D}_{\tilde{K}|\tilde{k}}$ of all the prime divisors of $\tilde{K}|\tilde{k}$. Second, since each $v \in \mathcal{D}_t$ is unramified in $\mathcal{K}_t|K$, it follows that each prolongation $\tilde{v}$ of v to $\tilde{K} = K\tilde{k}$ is unramified in $\tilde{\mathcal{K}}_t|\tilde{K}$ (because the latter is the base change of $\mathcal{K}_t|K$ under $\kappa_t \hookrightarrow \tilde{k}$). Hence since $\tilde{\mathcal{D}}_t = \mathcal{D}_{\tilde{K}|\tilde{k}}$, we conclude that all the prime divisors of $\tilde{K}|\tilde{k}$ are unramified in $\tilde{\mathcal{K}}_t|\tilde{K}$. Therefore, $\tilde{\Delta}_t = \mathrm{Gal}\big(\tilde{\mathcal{K}}_t\,|\,\tilde{K}\big)$ is a quotient of $\Pi_{1,\tilde{K}|\tilde{k}}$, and so is its isomorphic quotient $\tilde{\Delta}_t \to \Delta^0_t = \ker(\Pi^0_t \to \Pi_{\kappa_t})$. $\square$

Lemma 3.7. *In the Notations 3.4, the following hold:*

1) *The set $\mathcal{Q}_t$ consists of all the quasi prime divisors $\mathfrak{v}$ of $K|k$ such that $\mathfrak{v}|_{\kappa_t}$ is a constant reduction of $\kappa_t|k$. Hence $\mathcal{Q}^0_t \subset \mathcal{Q}_t$, thus by mere definitions $\mathfrak{T}^0_t \le \mathfrak{T}_t$, and $p_t : \Pi_K \to \Pi_t$ factors through $p^0_t : \Pi_K \to \Pi^0_t$, say $p_t = \overline{q}_t \circ p^0_t$ for a unique $\overline{q}_t : \Pi^0_t \to \Pi_t$.*
2) *$p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ factors through $p_t : \Pi_K \to \Pi_t$, say $p_{\kappa_t} = q_t \circ p_t$ with $q_t : \Pi_t \to \Pi_{\kappa_t}$. Thus since $\Delta^0_t = \ker(q^0_t)$ is a quotient of $\Pi_{1,\tilde{K}|\tilde{k}}$, so is $\Delta_t := \ker(q_t) = \ker(q^0_t)/\ker(\overline{q}_t)$.*

Proof. To 1): First, if $\mathfrak{w} := \mathfrak{v}|_{\kappa_t}$ is a constant reduction of $\kappa_t|k$, then the minimized decomposition group $Z^1_{\mathfrak{w}} \subset \Pi_{\kappa_t}$ is not topologically finitely generated (by mere definitions). On the other hand, $p_{\kappa_t}(Z^1_{\mathfrak{v}}) \subseteq Z^1_{\mathfrak{w}}$ is an open subgroup, hence $p_{\kappa_t}(Z^1_{\mathfrak{v}}) \subset \Pi_{\kappa_t}$ is not topologically finitely generated. Thus finally, $p^0_t(Z^1_{\mathfrak{v}})$ is not finitely topologically generated either. Conversely, let $\mathfrak{v}$ be a quasi prime divisor such that $p^0_t(Z^1_{\mathfrak{v}})$ is not finitely topologically generated. By contradiction, suppose that $\mathfrak{w} := \mathfrak{v}|_{\kappa_t}$ is not a constant reduction. Then $\mathfrak{w}$ is a quasi prime divisor of $\kappa_t|k$, and therefore $Z^1_{\mathfrak{w}} = T^1_{\mathfrak{w}}$, because $\mathrm{td}(\kappa_t|k) = 1$. Since $T^1_{\mathfrak{w}} \cong \mathbb{Z}_\ell$, one finally has $Z^1_{\mathfrak{w}} \cong \mathbb{Z}_\ell$. Finally, recalling that $\ker(\Pi^0_t \to \Pi_{\kappa_t})$ is finitely generated, and that $p_{\kappa_t}(Z^1_{\mathfrak{v}}) \subset Z^1_{\mathfrak{w}}$ has finite index, it follows that $p^0_t(Z^1_{\mathfrak{v}}) \subset \Pi^0_t$ is topologically finitely generated, contradiction! The remaining assertions from assertion 1) of the Lemma are clear.

To 2): First, as in the proof of assertion 1) of the Lemma 3.6 above, especially the proof of the inclusion $(*)$, it follows that $T^1_{\mathfrak{v}} \subset \ker(p_{\kappa_t})$ for all $\mathfrak{v} \in \mathcal{Q}_t$. Hence $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ factors through $p_t : \Pi_K \to \Pi_t$, i.e., there exists $q_t : \Pi_t \to \Pi_{\kappa_t}$ such that $p_{\kappa_t} = q_t \circ p_t$. Second, $q^0_t : \Pi^0_t \to \Pi_{\kappa_t}$ factors through $q_t : \Pi_t \to \Pi_{\kappa_t}$, precisely, $q^0_t = \overline{q}_t \circ q_t$, with $\overline{q}_t : \Pi^0_t \to \Pi_t$ as introduced at 1). Therefore, $\ker(q_t) = \ker(q^0_t)/\ker(\overline{q}_t)$ is a quotient of $\ker(q^0_t)$, as claimed. $\square$

We next announce the group theoretical recipe to recover p_{κ_t} from π_t.

Proposition 3.8. *In the above notations, in order to simplify notations, for quasi divisorial subgroups $T^1_{\mathfrak{v}} \subset Z^1_{\mathfrak{v}}$ of Π_K, we set $\overline{T}^1_{\mathfrak{v}} := p_t(T^1_{\mathfrak{v}}), \overline{Z}^1_{\mathfrak{v}} := p_t(Z^1_{\mathfrak{v}}) \subset \Pi_t$. Then the following hold:*

1) *Δ_t is the unique $\mathbb{Z}_\ell$-submodule $\Delta \subset \Pi_t$ satisfying the following:*
   i) *For all $\mathfrak{v} \notin \mathcal{Q}_t$ one has that $\overline{Z}^1_{\mathfrak{v}} \subseteq \Delta \cdot \overline{T}^1_{\mathfrak{v}}$ and $\Delta \cap \overline{T}^1_{\mathfrak{v}} = 1$.*
   ii) *There exist $\mathfrak{v} \notin \mathcal{Q}_t$ such that $\Delta \subset \overline{Z}^1_{\mathfrak{v}}$, hence $\overline{Z}^1_{\mathfrak{v}} = \Delta \cdot \overline{T}^1_{\mathfrak{v}}$.*

($*$) *Therefore, the discussion above gives a group theoretical recipe to recover/reconstruct $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ from $\Pi^{\rm c}_K$ endowed with $\pi_t : \Pi_K \to \Pi_{U_0}$.*

2) *The above recipe to recover p_{κ_t} is invariant under Π_{U_0}-isomorphisms as follows: Let $L|l$ be a function field with l algebraically closed field, and for $u \in L\backslash l$, let $\kappa_u \subset L$ and $p_{\kappa_u} : \Pi_L \to \Pi_{\kappa_u}$, $\pi_u : \Pi_L \to \Pi_{U_0}$ defined by $t_0 \mapsto u \in L$, be correspondingly defined. Let $\Phi : \Pi_K \to \Pi_L$ be the abelianization of an isomorphism $\Phi^c : \Pi^{\rm c}_K \to \Pi^{\rm c}_L$ satisfying $\ker(\pi_u) = \Phi\big(\ker(\pi_t)\big)$. Then one has:*
   a) *Φ maps $(T^1_{\mathfrak{v}})_{\mathfrak{v}\in\mathcal{Q}^0_t}$, $(T^1_{\mathfrak{v}} \subset Z^1_{\mathfrak{v}})_{\mathfrak{v}\in\mathcal{Q}_t}$ isomorphically onto $(T^1_{\mathfrak{w}})_{\mathfrak{w}\in\mathcal{Q}^0_u}$, $(T^1_{\mathfrak{w}} \subset Z^1_{\mathfrak{w}})_{\mathfrak{w}\in\mathcal{Q}_u}$, respectively, thus gives rise to isomorphisms $\Phi^0_t : \Pi^0_t \to \Pi^0_u$, $\Phi_t : \Pi_t \to \Pi_u$ which map Δ^0_t, Δ_t isomorphically onto the corresponding Δ^0_u, Δ_u.*
   b) *Hence one has that $\ker(p_{\kappa_u}) = \Phi\big(\ker(p_{\kappa_t})\big)$. Moreover, the induced canonical isomorphism $\Phi_{t,u} : \Pi_{\kappa_t} \to \Pi_{\kappa_u}$ defined by Φ maps the quasi divisorial subgroups of Π_{κ_t} isomorphically onto the ones of Π_{κ_u}.*

Proof. To 1): We begin by showing that Δ_t satisfies the requirement 1), i) from Proposition 3.8. First, let $\mathfrak{v}$ be a quasi prime divisor of $K|k$ whose restriction to κ_t is not a constant reduction of $\kappa_t|k$. We claim that $\overline{Z}^1_{\mathfrak{v}} \subseteq \Delta_t \cdot \overline{T}^1_{\mathfrak{v}}$. Indeed, by Remark 3.1, it follows that $\mathfrak{v}(\kappa_t^\times)/\mathfrak{v}(k^\times) \cong \mathbb{Z}$. Further, by Fact 3.2 combined with the fact that $\mathrm{td}(\kappa_t|k) = 1$, it follows that $p_{\kappa_t}(T^1_{\mathfrak{v}}) = p_{\kappa_t}(Z^1_{\mathfrak{v}})$. Hence $Z^1_{\mathfrak{v}} \subseteq T^1_{\mathfrak{v}} \cdot \ker(p_{\kappa_t})$, thus $\overline{Z}^1_{\mathfrak{v}} \subseteq \overline{T}^1_{\mathfrak{v}} \cdot p_t\big(\ker(p_{\kappa_t})\big)$. Hence taking into account that $p_t\big(\ker(p_{\kappa_t})\big) = \ker(q_t) = \Delta_t$, we get $\overline{Z}^1_{\mathfrak{v}} \subseteq \overline{T}^1_{\mathfrak{v}} \cdot \Delta_t$. Next, since $T^1_{\mathfrak{v}} \cong \mathbb{Z}_\ell \cong p_{\kappa_t}(T^1_{\mathfrak{v}})$, it follows that p_{κ_t} maps $T^1_{\mathfrak{v}}$ isomorphically onto $p_{\kappa_t}(T^1_{\mathfrak{v}})$. Hence $p_{\kappa_t} = q_t \circ p_t$ implies that $p_t : T^1_{\mathfrak{v}} \to \overline{T}^1_{\mathfrak{v}}$ and $q_t : \overline{T}^1_{\mathfrak{v}} \to p_{\kappa_t}(T^1_{\mathfrak{v}})$ are isomorphisms as well. Finally, since $\Delta_t = \ker(\overline{p}_t)$, it follows that $\Delta_t \cap \overline{T}^1_{\mathfrak{v}} = 1$, as claimed.

We next prove that there exists a family $(v_i)_{i\in I}$ of prime divisors $v_i \notin \mathcal{Q}_t$ satisfying:

- $p_{\kappa_t}(T_{v_i}) \cap p_{\kappa_t}(T_{v_j}) = 1$ for all $i \neq j$.
- $p_{\kappa_t}(T_{v_i})$, $i \in I$, consists of almost all the divisorial subgroups of Π_{κ_t}.
- $\overline{Z}_{v_i} = \Delta_t \cdot \overline{T}_{v_i}$, $i \in I$. In particular, Δ_t and $\overline{Z}_{v_i}$ satisfy condition 1), ii) for all i.

The proof of this is not difficult, but a little bit involved, and we will do it a few steps:

a) First, recall that $q^0_t = q_t \circ \overline{q}_t$, and in the notations from (the proof of) Lemma 3.6, let $\mathcal{K} := \mathcal{K}_t^{\ker(\overline{q}_t)}$ be the corresponding fixed field in $\mathcal{K}_t$. Then $\mathrm{Gal}\big(\mathcal{K}\,|\,K\kappa'_t\big) = \Delta_t$, and one has an exact sequence of abelian groups

$$(\ddagger) \qquad 1 \to \mathrm{Gal}\big(\mathcal{K}\,|\,K\kappa'_t\big) = \Delta_t \hookrightarrow \mathrm{Gal}\big(\mathcal{K}\,|\,K\big) = \Pi_t \xrightarrow{p_{\kappa_t}} \mathrm{Gal}\big(\kappa'_t\,|\,\kappa_t\big) = \Pi_{\kappa_t} \to 1,$$

which is a quotient of the exact sequence (†) from the proof of Lemma 3.6, 2). Since Π_{κ_t} is a pro-ℓ abelian free group (being the ℓ-adic dual of $\kappa_t^\times/k^\times$), the exact sequence (‡) above is split. Hence there exists a $\mathbb{Z}/\ell$-elementary abelian extension $K_1|K$ with $\mathrm{Gal}\big(K_1\,|\,K\big) \cong \Delta_t/\ell$, and satisfying: K_1 and $K\kappa'_t$ are linearly disjoint over K, thus $K_1|K$ and $\tilde{K} = K\tilde{k}$ are linearly disjoint over K as well.[3] Hence $\tilde{K}_1 := K_1\tilde{k}$ is an abelian extension $\tilde{K}_1|\tilde{K}$ with Galois group Δ_t/ℓ, and recall that Δ_t/ℓ is finite, because Δ_t was a finite $\mathbb{Z}_\ell$-module.

b) Let X_t be the projective smooth k-curve with $k(X_t) = \kappa_t$, and consider a Δ_t/ℓ cover of (proper/normal) geometrically integral X_t-schemes $\mathcal{X}_1 \to \mathcal{X}$ with generic geometric fiber the field extension $K_1|K$. Then there exists an open subset $U_t \subset X_t$ such that for all $s \in U_t$, the fiber $\mathcal{X}_{1,s} \to \mathcal{X}_s$ at s is a Δ_t/ℓ-cover of (proper/normal) integral k-varieties. In particular, if $\mathcal{X}_1 \ni \eta_{1,s} \mapsto \eta_s \in \mathcal{X}$ are the generic points of $\mathcal{X}_{1,s} \to \mathcal{X}_s$, the corresponding extension of local rings $\mathcal{O}_{\eta_x} \hookrightarrow \mathcal{O}_{\eta_{1,s}}$ is an étale and totally inert extension of local rings, i.e., one has $[K_1 : K] = [\kappa(\eta_{1,s}) : \kappa(\eta_s)]$. On the other hand, $\mathcal{X}_{1,s} \subset \mathcal{X}_1$ and $\mathcal{X}_s \subset \mathcal{X}$ are Weil prime divisors. Thus the corresponding valuations v_1 of K_1, respectively v of K are prime divisors of $K_1|k$, respectively $K|k$, which satisfy: v_1 is the unique prolongation of v to K_1, and the residue field extension of $v_1|v$ is nothing but $\kappa(\eta_{1,s})|\kappa(\eta_s)$. Further, $v|_{\kappa_t} = v_s$ is the valuation of κ_t defined by $s \in X_t$. Hence if w is a prolongation of v to $K\kappa_t'$, then w is totally inert in $K_1\kappa_t'|K\kappa_t'$, and in particular, the decomposition group $\mathcal{Z}_{1,w}$ of w in $\mathrm{Gal}\big(K_1\kappa_t'|K\kappa_t'\big)$ is nothing but $\mathcal{Z}_{1,w} = \mathrm{Gal}\big(K_1\kappa_t'|K\kappa_t'\big)$.

c) Recall that $\mathcal{K}|K$ is the subextension of $K'|K$ with Galois group Π_t, or equivalently, the fixed field of $\mathfrak{T}_t$ in K', one has that $\Delta_t = \mathrm{Gal}\big(\mathcal{K}\kappa_t'|K\kappa_t'\big)$, and $K_1\kappa_t'|K\kappa_t'$ is the Galois subextension of $\mathcal{K}\kappa_t'|K\kappa_t'$ with Galois group $\mathrm{Gal}\big(K_1\kappa_t'|K\kappa_t'\big) = \Delta_t/\ell$. Hence the decomposition group $\mathcal{Z}_w$ of w in $\mathcal{K}\kappa_t'|K\kappa_t'$ satisfies: $\mathcal{Z}_w \hookrightarrow \Delta_t$ and $\mathcal{Z}_w \twoheadrightarrow \mathcal{Z}_{1,w} = \mathrm{Gal}\big(K_1\kappa_t'|K\kappa_t'\big) = \Delta_t/\ell$. Since Δ_t is a finite $\mathbb{Z}_\ell$-module by Lemma 3.7, 2), Nakayama Lemma implies $\mathcal{Z}_w = \Delta_t$.

Finally, by general decomposition theory, one has that: First, $\mathcal{Z}_w = Z_{t,v} \cap \Delta_t$, where $Z_{t,v}$ is the decomposition group of v in Π_t. Second, $Z_{t,v} = p_t(Z_v) = \overline{Z}_v$ is the image of $Z_v \subset \Pi_K$ under $p_t : \Pi_K \to \Pi_t$. Thus one has that

$$\Delta_t = \mathcal{Z}_w \subseteq \overline{Z}_v.$$

We thus conclude that for almost all closed points $s_i \in X_t$, the local ring $\mathcal{O}_{\mathcal{X},\eta_i}$ of the generic point η_i of $\mathcal{X}_{s_i}$ is a DVR whose valuation v_i satisfies the following:

a) $v_i(K) = v_{s_i}(\kappa_t)$, because the special fiber $\mathcal{X}_{s_i}$ is reduced. Hence $p_{\kappa_t}(T_{v_i}) = T_{v_{s_i}}$.

b) $\Delta_t \subset \overline{Z}_{v_i}$, hence $\overline{Z}_{v_i} = \Delta_t \cdot \overline{T}_{v_i}$.

Finally, to complete the proof of assertion 1) of the Proposition, we have to prove that Δ_t is the only closed subgroup of Π_t satisfying the conditions i), ii) from assertion 1). The proof of this assertion is easily to axiomatize as follows: Let $\Delta \subset \Pi_t$ be a further subgroup satisfying the conditions i), ii) from assertion 1). Since Δ satisfies ii), there exists $\mathfrak{v} \notin \mathcal{Q}_t$ such that $\Delta \cdot \overline{T}^1_{\mathfrak{v}} = \overline{Z}^1_{\mathfrak{v}}$, and since Δ_t satisfies i), it follows that $\overline{Z}^1_{\mathfrak{v}} \subseteq \Delta_t \cdot \overline{T}^1_{\mathfrak{v}}$, thus finally, $\Delta \subset \Delta_t \cdot \overline{T}^1_{\mathfrak{v}}$. Similarly, $\Delta_t \subset \Delta \cdot \overline{T}_{v_i}$ for all v_i. Finally, since $p_{\kappa_t}(T_{v_i}) \cap p_{\kappa_t}(T_{v_j}) = 1$ for $i \neq j$, we can choose v_i such that $p_{\kappa_t}(T_{\mathfrak{v}}) \cap p_{\kappa_t}(T_{v_j}) = 1$. Equivalently, we have $(\Delta_t \cdot \overline{T}^1_{\mathfrak{v}}) \cap (\Delta_t \cdot \overline{T}_{v_i}) = \Delta_t$.

Thus the equality $\Delta_t = \Delta$ will follows from the following quite general assertion:

Fact 3.9. *Let G be an arbitrary group, $T, T_1 \subset G$ be subgroups, and $\Delta, \Delta_1 \lhd G$ be normal subgroups satisfying: First, $\Delta \subset \Delta_1 T$, $\Delta_1 \subset \Delta T_1$, and second, $\Delta_1 \cap T_1 = 1 = \Delta \cap T$, $(\Delta_1 T_1) \cap (\Delta_1 T) = \Delta_1$. Then $\Delta = \Delta_1$.*

[3] Recall that $\tilde{k} := \overline{\kappa_t}$.

Proof. First, since $\Delta_1 \subset \Delta T_1$, every $\delta_1 \in \Delta_1$ is of the form $\delta_1 = \delta\tau_1$ with $\delta \in \Delta$, $\tau_1 \in T_1$. Second, $\Delta \subset \Delta_1 T$, implies that $\delta = \delta_1'\tau$ with $\delta_1' \in \Delta_1$, $\tau \in T$. Therefore, the following holds:

$$\delta_1 = \delta\tau_1 = \delta_1'\tau\tau_1, \quad \text{hence } \delta_1\tau_1^{-1} =: g := \delta_1'\tau.$$

Since $g = \delta_1\tau_1^{-1} \in \Delta_1 T_1$, $g = \delta_1'\tau \in \Delta_1 T$, and by hypothesis $(\Delta_1 T_1) \cap (\Delta_1 T) = \Delta_1$, it follows that $g \in \Delta_1$, thus concluding that $\tau, \tau_1 \in \Delta_1$. Since $\Delta_1 \cap T_1 = 1$, we get $\tau_1 = 1$, hence concluding that $\delta_1 = \delta\tau_1 = \delta \in \Delta$. And since $\delta_1 \in \Delta_1$ was arbitrary, we finally get $\Delta_1 \subseteq \Delta$. For the converse inclusion, let $\delta \in \Delta$ be arbitrary. Since $\Delta \subset \Delta_1 T$, one has $\delta = \delta_1\tau$ with $\delta_1 \in \Delta_1$, $\tau \in T$. Hence $\tau = \delta_1^{-1}\delta$, and $\delta_1 \in \Delta_1 \subseteq \Delta$ implies $\tau \in \Delta$, and therefore, $\tau = 1$ (because $\Delta \cap T = 1$). Hence finally $\delta_1 = \delta \in \Delta$, and since $\delta_1 \in \Delta_1$ was arbitrary, we get $\Delta_1 \subseteq \Delta$. Thus finally $\Delta = \Delta_1$, as claimed. □

To 2): The proof is an easy exercise of sorting through the proof of assertion 1), using the Φ maps the quasi divisorial subgroups of Π_K onto those of Π_L. □

C) *Recovering the divisorial groups in Π_K from Π_K^{c} endowed with π_t, $\pi_{t'}$ for $\kappa_t \neq \kappa_{t'}$*

In this subsection we give a group theoretical recipe which recovers the divisorial subgroups $T_v \subset Z_v$ of Π_K from Π_K^{c} endowed with two projections $\pi_t, \pi_{t'} : \Pi_K \to \Pi_{U_0}$ for $t, t' \in K$ such that $\kappa_t \neq \kappa_t'$ (that is, t, t' are algebraically independent over k).

First, by the discussion in the previous subsection, the projection $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ can be recovered/reconstructed by a group theoretical recipe from Π_K^{c} endowed with the projection $\pi_t : \Pi_K \to \Pi_{U_0}$ defined by $t_0 \mapsto t$. Further, for every quasi divisororial subgroup $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$ of Π_K, one has the following: $p_{\kappa_t}(T_{\mathfrak{v}}^1) \neq 1$ iff $\mathfrak{w} := \mathfrak{v}|_{\kappa_t}$ is a quasi prime divisor of $\kappa_t|k$, and if so, then by Fact 3.2, one has that $p_{\kappa_t}(T_{\mathfrak{v}}^1) = p_{\kappa_t}(Z_{\mathfrak{v}}^1) \subseteq T_{\mathfrak{w}}^1 = Z_{\mathfrak{w}}^1$ is open (and these groups are isomorphic to $\mathbb{Z}_\ell$). And $p_{\kappa_t}(T_{\mathfrak{v}}^1) = 1$ if and only if $p_{\kappa_t}(Z_{\mathfrak{v}}^1)$ has infinite $\mathbb{Z}_\ell$-rank, and if so, then $\mathfrak{w} := \mathfrak{v}|_{\kappa_t}$ is a constant reduction of $\kappa_t|k$ and $p_{\kappa_t}(Z_{\mathfrak{v}}^1) \subseteq Z_{\mathfrak{w}}^1$ is an open subgroup. Clearly, the same holds, correspondingly, about $p_{\kappa_{t'}}$.

Proposition 3.10. *In the above notations the following hold:*

1) *A quasi divisorial group $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$ of Π_K is divisorial, i.e., $\mathfrak{v}$ is a prime divisor of $K|k$ if and only if one of the following conditions is satisfied:*
   - i) *$p_{\kappa_t}(Z_{\mathfrak{v}}^1) \subseteq \Pi_{\kappa_t}$ is an open subgroup.*
   - ii) *$p_{\kappa_t}(Z_{\mathfrak{v}}^1) = p_{\kappa_t}(T_{\mathfrak{v}}^1)$ and there exists a quasi divisorial group $T_{\mathfrak{v}'}^1 \subset Z_{\mathfrak{v}'}^1$ of Π_K satisfying: First, $p_{\kappa_{t'}}(Z_{\mathfrak{v}'}^1) \subset \Pi_{\kappa_{t'}}$ is an open subgroup, and second, $p_{\kappa_t}(T_{\mathfrak{v}}^1) \cap p_{\kappa_t}(T_{\mathfrak{v}'}^1) \neq 1$.*
2) *The above recipe to recover the divisorial subgroups $T_v \subset Z_v$ of Π_K from Π_K^{c} endowed with π_t, $\pi_{t'}$ is invariant under Π_{U_0}-isomorphisms as follows: Let $L|l$ be a function field with l algebraically closed field, and $\pi_u, \pi_{u'} : \Pi_L \to \Pi_{U_0}$ be the projections defined by $t_0 \mapsto u$, respectively $t_0 \mapsto u'$ for some $u, u' \in L\backslash l$. Let $\Phi : \Pi_K \to \Pi_L$ be the abelianization of an isomorphism $\Phi^c : \Pi_K^{c} \to \Pi_L^{c}$ satisfying $\Phi\big(\ker(\pi_t)\big) = \ker(\pi_u)$, $\Phi\big(\ker(\pi_{t'})\big) = \ker(\pi_{u'})$. Then Φ maps the divisorial groups $T_v \subset Z_v$ of Π_K isomorphically onto the divisorial groups $T_w \subset Z_w$ of Π_L.*

Proof. To 1): For the implication "$\Rightarrow$" let $T_v \subset Z_v$ be a divisorial group of Π_K, and set $w := v|_{\kappa_t}$. First, if w is trivial, then $Z_w = \Pi_{\kappa_t}$. On the other hand, by Fact 3.2, one has that $p_{\kappa_t}(Z_v)$ is open in $Z_w = \Pi_{\kappa_t}$ is open. Hence the first condition from assertion 1) is satisfied. Second, if w is non-trivial, then $w|_k = v|_k$ being trivial, implies that w is a prime divisor of $\kappa_t|k$. Recalling that $\kappa_t \neq \kappa_{t'}$, that is, t, t' are algebraically independent, there exists a transcendence basis $\mathcal{T} = (t_2, \dots, t_d)$ of $K|\kappa_t$ with $t_2 = t'$. Let $w_{\mathcal{T}}$ be the Gauss valuation of $K|\kappa_t$, and v' be any prolongation of $w_{\mathcal{T}}$ to K. Then since $w_{\mathcal{T}}$ is trivial on $k(t')$, it follows that v' is trivial on $\kappa_{t'}$. Further, by mere definitions, $v'|_{\kappa_t} = w = v|_{\kappa_t}$. Therefore we get: $p_{\kappa_t}(T_v), p_{\kappa_t}(T_{v'}) \subseteq T_w$ are open, thus $p_{\kappa_t}(T_v) \cap p_{\kappa_t}(T_{v'}) \neq 1$, and $p_{\kappa_t}(T_v) = p_{\kappa_t}(Z_v)$, $p_{\kappa_t}(T_{v'}) = p_{\kappa_t}(Z_{v'})$. Second, $p_{\kappa_{t'}}(Z_{v'}) \subset \Pi_{\kappa_{t'}}$ is open, because $v'|_{\kappa_{t'}}$ is trivial. Hence the second condition of assertion 1) is satisfied.

For the converse implication $\Leftarrow$, let $T^1_{\mathfrak{v}} \subset Z^1_{\mathfrak{v}}$ be a quasi divisorial group in Π_K satisfying the hypotheses from assertion 1). Set $\mathfrak{w} := \mathfrak{v}|_{\kappa_t}$ and recall that $p_{\kappa_t}(Z^1_{\mathfrak{v}}) \subseteq Z^1_{\mathfrak{w}}$ is an open subgroup. Fist, suppose that $p_{\kappa_t}(Z^1_{\mathfrak{v}}) \subset \Pi_{\kappa_t}$ is open. Then by the discussion above, $Z^1_{\mathfrak{w}}$ is open in Π_{κ_t}. Since $\kappa_t|k$ is a function field, every non-trivial valuation w of κ_t has $Z_w \subset \Pi_{\kappa_t}$ of infinite index. Hence we conclude that $\mathfrak{w}$ must be the trivial valuation, hence $\mathfrak{w}|_k = \mathfrak{v}|_k$ is trivial on k. Thus finally, $\mathfrak{v}$ is a prime divisor of $K|k$. Second, suppose that $p_{\kappa_t}(Z^1_{\mathfrak{v}}) = p_{\kappa_t}(T^1_{\mathfrak{v}})$, and $p_{\kappa_t}(T^1_{\mathfrak{v}}) = p_{\kappa_t}(T^1_{\mathfrak{v}'})$ for some quasi prime divisorial group $T^1_{\mathfrak{v}'} \subset Z^1_{\mathfrak{v}'}$ with $p_{\kappa_{t'}}(Z^1_{\mathfrak{v}'}) \subset \Pi_{\kappa_{t'}}$ open. By the discussion above with respect to $p_{\kappa_{t'}}$ and $T^1_{\mathfrak{v}'} \subset Z^1_{\mathfrak{v}'}$, it follows that $\mathfrak{v}'$ is actually a prime divisor of $K|k$, and therefore, $w' := \mathfrak{v}'|_{\kappa_t}$ is a prime divisor of $\kappa_t|k$, and $p_{\kappa_t}(T^1_{\mathfrak{v}'}) \subset T^1_{w'} \cong \mathbb{Z}_\ell$ is an open subgroup. Hence since $p_{\kappa_t}(T^1_{\mathfrak{v}}) \subset T^1_{\mathfrak{w}}$ is open as well (by the discussion before the Proposition 3.10), the fact that $p_{\kappa_{t'}}(T^1_{\mathfrak{v}}) \cap p_{\kappa_{t'}}(T^1_{\mathfrak{v}'})$ is non-trivial, implies finally that $T^1_{\mathfrak{w}} \cap p_{\kappa_t}(T^1_{\mathfrak{v}'}) \neq 1$. On the other hand, since the inertia groups of distinct quasi prime divisors in Π_{κ_t} have trivial intersection, we conclude that $\mathfrak{w} = w'$, thus $\mathfrak{w}$ is a prime divisor of $\kappa_t|k$. Hence $\mathfrak{w}|_k$ is trivial, and since $\mathfrak{v}|_k = \mathfrak{w}|_k$ is trivial, it follows that $\mathfrak{v}|_k$ is trivial. Therefore, $\mathfrak{v}$ is a prime divisor of $K|k$.

To 2): This is an easy exercise involving sorting through the proof of assertion 1). □

D) *Recovering the total decomposition graph from Π^{c}_K endowed with π_t, $\pi_{t'}$ for $\kappa_t \neq \kappa_{t'}$*

We begin by recalling some facts from [P4], especially Proposition 3.5 of loc.cit. Recall that a valuation $\tilde{v}$ of $K|k$ is called a **prime r-divisor** if $\tilde{v}$ is the valuation theoretical composition $\tilde{v} = v_r \circ \dots \circ v_1$, where v_1 is a prime divisor of K, and inductively, v_{i+1} is a prime divisor of the residue function field $K\tilde{v}_i|k$, where $\tilde{v}_i := v_i \circ \dots \circ v_1$. By definition, the trivial valuation will be considered a generalized prime divisor of rank zero. We also notice that $r \leq \mathrm{td}(K|k)$, and that in the above notations, one has $\tilde{v}_r(K^\times) \cong \mathbb{Z}^r$ lexicographically.

Since generalized prime divisors $\tilde{v}$ are trivial on k, hence $\mathrm{char}(K\tilde{v}) \neq \ell$, one has $T^1_{\tilde{v}} = T_{\tilde{v}}$, and $Z^1_{\tilde{v}} = Z_v$. A **flag of generalized divisorial subgroups** of Π_K consists of the sequences of the decomposition/inertia groups $Z_{\tilde{v}_1} \geq \dots \geq Z_{\tilde{v}_r}$, $T_{\tilde{v}_1} \leq \dots \leq T_{\tilde{v}_r}$ defined by a flag of generalized (quasi) prime divisors $\tilde{v}_1 \leq \dots \leq \tilde{v}_r$, where $\tilde{v}_s$ is a prime s-divisor, $1 \leq s \leq r$.

The **total prime divisor graph $\mathcal{D}^{\mathrm{tot}}_K$** and its Galois theoretical counterpart, the **total decomposition graph $\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$**, were introduced in [P3]. First, $\mathcal{D}^{\mathrm{tot}}_K$ is defined as follows:

a) The vertices of $\mathcal{D}^{\mathrm{tot}}_K$ are the residue fields $K\tilde{v}$ of all the generalized prime divisors $\tilde{v}$ of $K|k$ viewed as distinct function fields.

b) For a prime r-divisor $\tilde{v}$ and a prime s-divisor $\tilde{w}$, there are edges from $K\tilde{v}$ to $K\tilde{w}$ only if $\tilde{v} \leq \tilde{w}$ and $s \leq r+1$, and if so, the edges are:
   i) If $\tilde{v} = \tilde{w}$, then the trivial valuation is the only edge from $K\tilde{v} = K\tilde{w}$ to itself.
   ii) If $\tilde{v} < \tilde{w}$, then the prime divisor $\tilde{w}/\tilde{v}$ of $K\tilde{v}|k$ is the only edge from $K\tilde{v}$ to $K\tilde{w}$.

Via the Galois correspondence and the Hilbert decomposition theory for valuations, the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ of $K|k$ is in bijection with $\mathcal{D}_K^{\rm tot}$, and is defined as follows:

a) The vertices of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ are the residue Galois groups $\Pi_{K\tilde{v}}$ of all the generalized prime divisors $\tilde{v}$ of $K|k$ viewed as distinct groups.

b) The unique edge from $\Pi_{K\tilde{v}}$ to $\Pi_{K\tilde{w}}$, if it exists, is endowed with the divisorial subgroup $T_{\tilde{w}/\tilde{v}} \subset Z_{\tilde{w}/\tilde{v}}$ of $\Pi_{K\tilde{v}}$. Note that if $\tilde{w} = \tilde{v}$, then the groups are $\{1\} = T_{\tilde{w}/\tilde{v}} \subset Z_{\tilde{w}/\tilde{v}} = \Pi_{K\tilde{v}}$.

Finally, notice that by [P3], Section 2, it follows that knowing $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ is equivalent to knowing the flags of generalized divisorial groups $Z_{\tilde{v}_1} \geq \cdots \geq Z_{\tilde{v}_r}$, $T_{\tilde{v}_1} \leq \cdots \leq T_{\tilde{v}_r}$ of Π_K.

We next show that $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ can be recovered by a group theoretical recipe from $\Pi_K^{\rm c}$ endowed with any two projections $\pi_t, \pi_{t'} : \Pi_K \to \Pi_{U_0}$ satisfying $\kappa_t \neq \kappa_{t'}$. First, recall that by Proposition 3.10, there exists a group theoretical recipe which recovers all the divisorial groups $T_v \subset Z_v$, $v \in \mathcal{D}_{K|k}$, from $\Pi_K^{\rm c}$ endowed with with any two projections $\pi_t, \pi_{t'} : \Pi_K \to \Pi_{U_0}$ satisfying $\kappa_t \neq \kappa_{t'}$. Hence given $\kappa_t \neq \kappa_{t'}$ and $\Pi_K^{\rm c}$ endowed with $\pi_t, \pi_{t'} : \Pi_K \to \Pi_{U_0}$, via that group theoretical recipe, one recovers the set $\mathfrak{Inr.div}(K|k) := \cup_{v\in\mathcal{D}_{K|k}} T_v$ of all the divisorial inertia elements in Π_K. Further, if $\Phi \in {\rm Aut}^{\rm c}(\Pi_K)$ is compatible with $\pi_t, \pi_{t'}$, i.e., satisfies $\Phi\big(\ker(\pi_t)\big) = \ker(\pi_t)$, $\Phi\big(\ker(\pi_{t'})\big) = \ker(\pi_{t'})$, then by Proposition 3.10, it follows that Φ maps $\{T_v \,|\, v \in \mathcal{D}_{K|k}\}$ onto itself, hence $\Phi\big(\mathfrak{Inr.div}(K|k)\big) = \mathfrak{Inr.div}(K|k)$. Further, by Theorems A and B from [P2], Introduction, it follows that the topological closure of $\mathfrak{Inr.div}(K|k)$ in Π_K is precisely the set of the inertia elements $\mathfrak{Inr}_k(K)$ at all the k-valuations of K. Thus $\kappa_t \neq \kappa_{t'}$ and $\Phi \in {\rm Aut}^{\rm c}(\Pi_K)$ compatible with $\pi_t, \pi_{t'}$ implies that $\Phi\big(\mathfrak{Inr}_k(K)\big) = \mathfrak{Inr}_k(K)$.

Proposition 3.11. *In the above notations, let $d = {\rm td}(K|k)$. Then the following hold:*

1) *Let $\Pi_K^{\rm c} \to \Pi_K$ be the canonical projection, and for subgroups $G \subset \Pi_K$, let $G'' \subset \Pi_K^{\rm c}$ be their preimages in $\Pi_K^{\rm c}$. Then a flag of closed subgroups $Z_1 \geq \cdots \geq Z_r$, $T_1 \leq \cdots \leq T_r$ of Π_K is a flag of generalized divisorial subgroups if and and only if Z_r contains a subgroup $\Delta \cong \mathbb{Z}_\ell^d$ with Δ'' abelian, the T_s, Z_s are maximal subgroups satisfying:*
   i) *$T_s \leq Z_s$, $T_s \cong \mathbb{Z}_\ell^s$ and $T_s'' \subset \Pi_K^{\rm c}$ is the center of $Z_s'' \subset \Pi_K^{\rm c}$, $s = 1, \dots, r$.*
   ii) *$T_s \subset \mathfrak{Inr}_k(K)$, $s = 1, \dots, r$.*

   In particular, this gives a group theoretical recipe which recovers $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ from $\Pi_K^{\rm c}$ endowed with the set of k-inertia elements $\mathfrak{Inr}_k(K)$.

2) *Moreover, the recipe under discussion is invariant under isomorphisms as follows: Let $L|l$ be a function field with l algebraically closed field, and $\Phi : \Pi_K \to \Pi_L$ be the abelianization of an isomorphism $\Phi^c : \Pi_K^{\rm c} \to \Pi_L^{\rm c}$ satisfying $\Phi\big(\mathfrak{Inr}_k(K)\big) = \mathfrak{Inr}_l(L)$. Then Φ maps the set of all the flags of generalized divisorial subgroups of Π_K bijectively onto those of Π_L, and therefore defines an automorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\mathcal{D}_L^{\rm tot}}$.*

Proof. The proof of Proposition 3.11 above is virtually identical with the one of [P4], Proposition 3.5, but using $\mathfrak{Inr.div}(K) \subset \mathfrak{Inr}_k(K)$ instead of $\mathfrak{Inr.q.div}(K) \subset \mathfrak{Inr.tm}(K)$.

To 1): First, T_r consists of inertia elements from $\mathfrak{Inr}_k(K)$, hence of *tame inertia elements,* and by hypothesis, T_r is commuting liftable. For $\sigma \in T \subset \mathfrak{Inr}_k(K)$, let v be any valuation of K trivial on k such that $\sigma \in T_v$. Then if v_σ is the canonical valuation of σ, as defined in Fact 3.3 from [P1], one has $\sigma \in T_{v_\sigma}$ and $v_\sigma \leq v$. Thus v_σ is trivial on k as well. Therefore, the valuation $\tilde{v} := \sup_{\sigma\in T_d} v_\sigma$ defined in Proposition 3.4 of loc.cit., satisfies $T_r \subset T_{\tilde{v}}$, and $\tilde{v}$ is trivial on k as well. Second, notice that $\Delta := T_r \cong \mathbb{Z}_\ell^d$ is contained Z_r, thus in every

Z_s, and its preimage $\Delta'' = T''_r$ in Π^{c}_K is commutative. Hence by [P4], Proposition 3.5, it follows that there exists a flag of generalized quasi prime divisors $\mathfrak{v}_1 \leq \cdots \leq \mathfrak{v}_r$ such that $T_s = T^1_{\mathfrak{v}_s} \subset Z^1_{\mathfrak{v}_s} = Z_s$ for $s = 1, \dots, r$. Finally, since $T_r = T_{\mathfrak{v}_r} \subseteq T_{\tilde{v}}$, it follows that $\mathfrak{v}_r \leq \tilde{v}$, thus $\mathfrak{v}_1 \leq \cdots \leq \mathfrak{v}_r \leq \tilde{v}$. Therefore, since $\tilde{v}$ is trivial on k, so are all the $\mathfrak{v}_s$, i.e., they are actually generalized prime divisors. The last assertion from 1) is clear.

To 2): Since Φ maps $\mathfrak{Inr}_k(K)$ homeomorphically onto $\mathfrak{Inr}_l(L)$, the arguments from the proof of assertion 1) show that Φ maps (flags of) generalized divisorial groups of Π_K isomorphically onto (flags of) generalized divisorial groups in Π_L. □

4. Recovering the rational quotients

A) *Generalities about* 1*-dimensional quotients*

Let $K|k$ be a function field with k algebraically closed, and $\mathcal{D}_{K|k}$ be the set of prime divisors of $K|k$. Recall that for every $u \in K\backslash k$, we denote by $\kappa_u \hookrightarrow K$ the relative algebraic closure of $k(u)$ in K, and notice that $\kappa_u \subset K$ strictly iff $\mathrm{td}(K|k) > 1$. Further define/consider:

$$\mathcal{D}^1_{\kappa_u} := \{ v \in \mathcal{D}_{K|k} \,|\, v \text{ non-trivial on } \kappa_u \}$$

The inclusion $\kappa_u \hookrightarrow K$ gives rise to a quotient $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$, which we call a geometric 1-dimensional quotient of Π_K. And we say that $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ is a rational quotient of Π_K if κ_u is a rational function field, i.e., of the form $\kappa_u = k(x)$ for some $x \in K$.

Recall that we identify (once and for all) the Tate module $\mathbb{Z}_\ell(1)$ of $\mathbb{G}_{m,K}$ with $\mathbb{Z}_\ell$, and do the same compatibly for all subfields of K. Hence by Kummer theory, $\mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell)$ is identified with the ℓ-adic completion $\widehat{K}$ of the multiplicative group $K^\times$;[4] and considering ℓ-adic duals, one has: Giving the projection $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ is equivalent to giving its ℓ-adic dual $\imath_{\kappa_u}$, which is the embedding of the ℓ-adic completions:

$$(*) \qquad \imath_{\kappa_u} : \widehat{\kappa}_u = \mathrm{Hom}_{\mathrm{cont}}(\Pi_{\kappa_u}, \mathbb{Z}_\ell) \hookrightarrow \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell) = \widehat{K}.$$

For every prime divisor v of $K|k$, we denote by $\jmath^v : \widehat{K} \to \mathrm{Hom}_{\mathrm{cont}}(T_v, \mathbb{Z}_\ell)$ the ℓ-adic dual of the embedding $T_v \hookrightarrow \Pi_K$, and notice that this $\jmath^v$ is nothing but the ℓ-adic completion of $v : K^\times \to \mathbb{Z}$ (after identifying $v(K^\times) = K^\times/U_v$ with $\mathbb{Z}$, $U_v \subset K^\times$ is the group of v-units). Further, the ℓ-adic completion $\widehat{U}_v$ of U_v is precisely $\widehat{U}_v = \ker(\jmath^v)$. Finally, let $\jmath_v : \widehat{U}_v \to \widehat{Kv}$ be the ℓ-adic dual of $\Pi_{Kv} = Z_v/T_v \hookrightarrow \Pi_K/T_v$, and notice that $\ker(\jmath_v) = \widehat{U^1_v}$ is precisely the ℓ-adic completion of the group of principal v-units $U^1_v \subset U_v$ of v.

Recall that a set of prime divisors D is called geometric, if there exists a normal model X of $K|k$, such that D coincides with the set of Weil prime divisors D_X of X, and notice that one can choose X to quasi-projective and normal. Further, the family of all the geometric sets of prime divisors is closed under finite intersections and unions, and any two geometric sets of prime divisors are almost equal. We define

$$\widehat{K}_{\mathrm{fin}} := \cup_D \, \{ \boldsymbol{x} \in \widehat{K} \mid \jmath^v(\boldsymbol{x}) = 0 \text{ for all but finitely many } v \in D \}, \quad D \text{ geometric}.$$

Notice that $\widehat{K}_{\mathrm{fin}}$ is a birational invariant of $K|k$, and for every geometric set of prime divisors D and every $\boldsymbol{x} \in \widehat{K}_{\mathrm{fin}}$, the set of all the $v \in D_X$ such that $\jmath^v(\boldsymbol{x}) \neq 0$ is finite. Further, if $\jmath_K : K^\times \to \widehat{K}$ is the ℓ-adic completion morphism, then $\jmath_K(K^\times) \subset \widehat{K}_{\mathrm{fin}}$, hence $\widehat{K}_{\mathrm{fin}}$ is ℓ-adically dense in $\widehat{K}$. Clearly, the same is true correspondingly for the function subfields

[4] In order to simplify notations, we denote the ℓ-adic completion of $K^\times$ simply by $\widehat{K}$, an not by $\widehat{K^\times}$.

$\kappa_u|k$, and under the embedding $\widehat{\kappa}_u \hookrightarrow \widehat{K}$ one has $\widehat{\kappa}_{u,\mathrm{fin}} = \widehat{\kappa}_u \cap \widehat{K}_{\mathrm{fin}}$. And since $\widehat{K}_{\mathrm{fin}} \subset \widehat{K}$ and $\widehat{\kappa}_{u,\mathrm{fin}} \subset \widehat{\kappa}_u$ are ℓ-adically dense subgroups, and $\widehat{\kappa}_u \hookrightarrow \widehat{K}$ is a topological embedding, it follows that the image of $\widehat{\kappa}_u \hookrightarrow \widehat{K}$ is the closure of the image of $\widehat{\kappa}_{u,\mathrm{fin}} \hookrightarrow \widehat{K}_{\mathrm{fin}}$ inside $\widehat{K}$.

Lemma 4.1. *In the above notations, the following hold:*

$$(*) \qquad \widehat{\kappa}_{u,\mathrm{fin}} = \{\boldsymbol{x} \in \widehat{K}_{\mathrm{fin}} \mid \forall\ v \in \mathcal{D}^1_{\kappa_u} \text{ one has: If } \boldsymbol{x} \in \widehat{U}_v, \text{ then } \jmath_v(\boldsymbol{x}) = 1\}$$

Proof. To the inclusion "$\subseteq$": Recall that $v \in \mathcal{D}^1_{\kappa_u}$ if and only if v is non-trivial on κ_u. Hence for all $v \in \mathcal{D}^1_{\kappa_u}$ one has: Since $\kappa_u|k$ is a function field in one variable, it follows that the residue field is $\kappa_u v = k$, thus $(U_v \cap \kappa_u)v = k$, and therefore $\jmath_v$ is trivial on the ℓ-adic completion of $U_v \cap \kappa_u$, which is $\widehat{U}_v \cap \widehat{\kappa}_u$. Thus the inclusion "$\subseteq$" follows.

For the reverse inclusion "$\supseteq$" one has: Let $\boldsymbol{x} \in \widehat{K}_{\mathrm{fin}} \setminus \widehat{\kappa}_{u,\mathrm{fin}}$ and let Δ be the $\mathbb{Z}_\ell$-submodule generated by $\boldsymbol{x}$, hence $\Delta \subset \widehat{K}_{\mathrm{fin}}$. Since one has $\widehat{K}_{\mathrm{fin}}/\widehat{\kappa}_{u,\mathrm{fin}} \hookrightarrow \widehat{K}/\widehat{\kappa}_u$, and the latter $\mathbb{Z}_\ell$-module is torsion free, it follows that $\widehat{K}_{\mathrm{fin}}/\widehat{\kappa}_{u,\mathrm{fin}}$ is torsion free too, thus $\Delta \cap \widehat{\kappa}_{u,\mathrm{fin}}$ is trivial. But then by [P3], Proposition 40, 3), it follows that for "many" valuations $v \in \mathcal{D}^1_{\kappa_u}$ one has: $\Delta \subset \widehat{U}_v$ and $\jmath_v$ maps Δ injectively into $\widehat{Kv}$ and therefore, $\jmath_v(\boldsymbol{x}) \neq 1$, etc. □

B) *Divisorial lattices and* 1*-dimensional quotients*

For readers sake, we first recall a few basic facts from [P3] as systemized in [P4]. Recall that a set D of prime divisors of $K|k$ is called geometric, if there is a quasi-projective normal model X of $K|k$ such that $D = D_X$ is the set of Weil prime divisors of X. If so, then $U_D := \Gamma(X, \mathcal{O}_X)^\times$ depends on D only (and not on X), and the canonical exact sequence

$$1 \to U_D \to K^\times \to \mathrm{Div}(D) \to \mathfrak{Cl}(D) \to 0$$

gives rise to its ℓ-adic completion

$$(*)_D \qquad 0 \to \mathbb{T}_{D,\ell} \to \widehat{K} \to \widehat{\mathrm{Div}}(D) \to \widehat{\mathfrak{Cl}}(D) \to 0,$$

where $\mathbb{T}_{D,\ell}$ is the ℓ-adic Tate module of the Weil divisor class group $\mathfrak{Cl}(D)$ of X.[5] Finally, let $\mathrm{Div}^0(D) \subset \mathrm{Div}(D)$ be the preimage of the maximal ℓ-divisible subgroup $\mathfrak{Cl}^0(D) \subset \mathfrak{Cl}(D)$.[6]

Fact 4.2. Recall that a subgraph $\mathcal{G} \subset \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$ is called geometric, if for every vertex $\Pi_{K\tilde{v}}$ of $\mathcal{G}$ with $\tilde{v}$ a prime r-divisor, $r < \mathrm{td}(K|k)$, the set $D_{\tilde{v}}$ of all the edges originating from $\Pi_{K\tilde{v}}$ is a geometric set. The following hold, see [P3] for details:

I) There are group theoretical recipes which recover from $\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$ the following:

- the geometric sets of prime divisors D, and the geometric decomposition graphs $\mathcal{G}$.
- the complete regular like geometric sets D, as introduced in [P3], Definition/Remark 21, and the complete regular like geometric decomposition graphs, see [P3], Proposition 22.

If D is complete regular like, the group theoretical recipes recover:

- the exact sequence $(*)_D$ above, see [P3], Proposition 23.
- $\mathrm{Div}^0(D)_{(\ell)} \subset \widehat{\mathrm{Div}}(D)$ up to multiplication by ℓ-adic units $\varepsilon \in \mathbb{Z}_\ell^\times$.[7]

[5] It turns out that the above exact sequences depend on $D = D_X$ only, and not on the concrete normal quasi-projective X with $D = D_X$.

[6] By the structure of $\mathfrak{Cl}(X)$, see [P3], Appendix, there is a *unique* maximal divisible subgroup in $\mathfrak{Cl}(X)$.

[7] Recall that for every abelian group A we denote $A_{(\ell)} := A \otimes \mathbb{Z}_{(\ell)}$.

- If D is complete regular like, then $\widehat{U}_K := \mathbb{T}_{D,\ell}$ and $\mathrm{Div}^0(D)_{(\ell)}$ are *birational invariants* of $K|k$, thus so is the preimage $\mathcal{L}_K \subset \widehat{K}$ of $\mathrm{Div}^0(D)_{(\ell)}$ under $\widehat{K} \to \widehat{\mathrm{Div}}(D)$. Hence $\mathcal{L}_K \subset \widehat{K}$ can be recovered from $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ up to multiplication by ℓ-adic units $\varepsilon \in \mathbb{Z}_\ell^\times$.

- Finally, $\jmath_K(K^\times) \subset \mathcal{L}_K$ and $\jmath_K(K^\times) \cap \widehat{U}_K = 1$, and if $\jmath_K(K^\times) \subset \varepsilon \cdot \mathcal{L}_K$, then $\varepsilon \in \mathbb{Z}_{(\ell)}^\times$.

(∗) We call $\mathcal{L}_K$ the canonical $\widehat{U}_K$-divisorial lattice of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$.

II) By [P3], Proposition 39, the recipes to recover the above objects from $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ are invariant under isomorphisms of total decomposition graphs as follows: Let $L|l$ be a further function field with l algebraically closed, and $\mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}}$ be its total decomposition graph. Let $\Phi : \Pi_K \to \Pi_L$ be an isomorphism which maps $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ isomorphically onto $\mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}}$, and $\hat{\phi} : \widehat{L} \to \widehat{K}$ be the Kummer isomorphism of Φ, i.e., the ℓ-adic dual of Φ. Then:
   - Φ maps the (complete regular like) geometric decomposition graphs of $K|k$ isomorphically onto the such ones of $L|l$.
   - One has $\hat{\phi}(\widehat{U}_L) = \widehat{U}_K$, and there exist $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\hat{\phi}(\mathcal{L}_L) = \varepsilon \cdot \mathcal{L}_K$, and ε is unique up to multiplication by elements $\eta \in \mathbb{Z}_{(\ell)}^\times$.

- In particular, if one replaces Φ by its multiple $\Phi_\varepsilon := \varepsilon \cdot \Phi$, then the Kummer isomorphism $\hat{\phi}_\varepsilon$ of Φ_ε satisfies $\hat{\phi}_\varepsilon(\mathcal{L}_L) = \mathcal{L}_K$.

  Language. We say that Φ is adjusted, if $\hat{\phi}(\mathcal{L}_L) = \mathcal{L}_K$.

Remark 4.3. In the above notation, the following hold:

1) For $u \in K^\times$ let $\mathsf{u} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(u)$ and $\boldsymbol{u} \in \mathbb{Z}_\ell \cdot \mathsf{u}$ be non-trivial (equivalently, u is non-constant and $\mathsf{u} = \alpha \cdot \jmath_K(u)$, $\boldsymbol{u} = \beta \cdot \mathsf{u}$ with $\alpha, \beta \neq 0$). Then for every prime divisor v of $K|k$ the following hold: $u \in U_v$ iff $\mathsf{u} \in \widehat{U}_v$ iff $\boldsymbol{u} \in \widehat{U}_v$. And if so, then $\jmath_v\big(\jmath_K(u)\big) \neq 1$ iff $\jmath_v(\mathsf{u}) \neq 1$ iff $\jmath_v(\boldsymbol{u}) \neq 1$ in $\widehat{Kv}$. Therefore, the following sets of prime divisors are equal:
   a) $\mathcal{D}_u := \{v \mid u \in U_v \text{ and } \jmath_v\big(\jmath_K(u)\big) \neq 1\}$
   b) $\mathcal{D}_{\mathsf{u}} := \{v \mid \mathsf{u} \in \widehat{U}_v \text{ and } \jmath_v(\mathsf{u}) \neq 1\}$
   c) $\mathcal{D}_{\boldsymbol{u}} := \{v \mid \boldsymbol{u} \in \widehat{U}_v \text{ and } \jmath_v(\boldsymbol{u}) \neq 1\}$

   and $\mathcal{D}_u = \mathcal{D}_{\mathsf{u}} = \mathcal{D}_{\boldsymbol{u}} = \mathcal{D}_{\kappa_u} := \{v \mid v \text{ is trivial on } \kappa_u\} = \mathcal{D}_{K|k} \backslash \mathcal{D}^1_{\kappa_u}$. Hence one has:

(†) Given any of the following: $u \in K\backslash k$ and $\mathcal{D}_u$ as at a); and/or $\mathsf{u} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(u)$ and $\mathcal{D}_{\mathsf{u}}$ as at b); and/or $\boldsymbol{u} \in \mathbb{Z}_\ell \cdot \mathsf{u}$ and $\mathcal{D}_{\boldsymbol{u}}$ as at c), enables one to recover the set $\mathcal{D}^1_{\kappa_u} \subset \mathcal{D}_{K|k}$.

(‡) Hence using Lemma 4.1 one can first recover $\widehat{\kappa}_{u,\mathrm{fin}} \hookrightarrow \widehat{K}_{\mathrm{fin}}$ and second, taking ℓ-adic duals, one finally recovers the projection of Galois groups $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$.

2) Let $u \in K$ be a non-constant function. Then $\kappa_u|k$ is a function field in one variable, hence it has a unique complete normal model $X_u \to k$, which is a projective smooth curve over k. Then $X_u(k)$ is in a canonical bijection with the set of prime divisors of $\kappa_u|k$, say $X_u(k) \ni a \leftrightarrow v_a \in \mathcal{D}_{\kappa_u|k}$, and we denote the total prime divisor graph of $\kappa_u|k$ simply by $\mathcal{G}_{\kappa_u}$. The canonical divisorial lattice $\mathcal{L}_{\kappa_u}$ corresponds to the canonical system of inertia generators $(\tau_a)_{a\in X_u(k)}$ of the inertia groups T_{v_a}, $a \in X_u(k)$. This system of generators satisfies the unique relation $\Pi_a \tau_a = 1$ in Π_{κ_u}. In particular, if $(\tau'_a)_a$ is another system of inertia generators satisfying $\Pi_a \tau'_a = 1$, then there exists a unique

ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\tau'_a = \tau_a^\varepsilon$ for all $a \in X_u(k)$. If so, the divisorial lattice corresponding to $(\tau'_a)_a$ is nothing but $\varepsilon^{-1}\cdot\mathcal{L}_{\kappa_t}$. See [P3], Sections 4.2, 5.2 for details.

- In particular, let X_t, X_u be complete smooth k-curves, and $\kappa_t := k(X_t)$, $\kappa_u := k(X_u)$. Then for an isomorphism $\Phi_{t,u} : \Pi_{\kappa_t} \to \Pi_{\kappa_u}$, the following are equivalent:
  i) $\Phi_{t,u}$ maps $\{T_{v_b} \,|\, b \in X_t(k)\}$ bijectively onto $\{T_{v_a} \,|\, a \in X_u(k)\}$.
  ii) There is a bijection $X_t(k) \to X_u(k)$, $b \mapsto a$, such that $\Phi_{t,u}(T_{v_b}) = T_{v_a}$.
  iii) If $(\tau_b)_{b\in X_t(k)}$ and $(\tau_a)_{a\in X_u(k)}$ are the canonical inertia generators, then there exists $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\Phi_{t,u}(\tau_b) = (\tau_a^\varepsilon)$ for all $b \mapsto a$.
  iv) $\Phi_{t,u}$ defines an isomorphism of decomposition graphs $\Phi_{t,u} : \mathcal{G}_{\kappa_t} \to \mathcal{G}_{\kappa_u}$.

3) In the above notations, one can recover $\mathcal{G}_{\kappa_u}$ from $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ endowed with the 1-dimension quotient $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ as follows: Let v be a prime divisor of $K|k$. Then $v \in \mathcal{D}^1_{\kappa_u}$ iff v is non-trivial on κ_u iff $p_{\kappa_u}(T_v) \subset \Pi_{\kappa_u}$ is non-trivial. And if so, and v_a is the restriction of v to κ_u, then $p_{\kappa_u}(T_v) \subseteq T_{v_a}$ is an open subgroup, and moreover, T_{v_a} is a maximal pro-cyclic subgroup of Π_{κ_u}, and the maximal one containing $p_{\kappa_u}(T_v)$. Conversely, for every prime divisor v_a of $\kappa_u|k$ there exists some prime divisor $v \in \mathcal{D}^1_{\kappa_u}$ which restricts to v_a, thus $p_{\kappa_u}(T_v) \subseteq T_{v_a}$ is non-trivial. Since for all prime divisors v_a of $\kappa_u|k$ one has $T_{v_a} = Z_{v_a}$, it follows that the above procedure recovers $\mathcal{G}_{\kappa_u}$ from $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ endowed with the group theoretical 1-dimension quotient $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$.

   Moreover, the above procedure does not only recover $\mathcal{G}_{\kappa_u}$, but it recovers as well the morphism of total decomposition groups $p_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\kappa_u}$ defined by $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$.

(∗) Since $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ originates from the embedding of function fields $\kappa_u|k \hookrightarrow K|k$, the induced morphism of total decomposition graphs $p_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\kappa_u}$ is divisorial in the sense of [P3], Definition/Remark 31 and Proposition 40. And the Kummer morphism $\imath_{\kappa_u} : \widehat{\kappa}_u \to \widehat{K}$ maps $\widehat{U}_{\kappa_u} \subset \mathcal{L}_{\kappa_u}$ injectively into $\widehat{U}_K \subset \mathcal{L}_K$. Thus by loc.cit., 5) and the discussion above, it follows that $\mathcal{L}_K$ is the unique divisorial $\widehat{U}_K$-lattice for $K|k$ with $\imath_{\kappa_u}(\mathcal{L}_{\kappa_u}) \subset \mathcal{L}_K$, i.e., if $\mathcal{L}'_K = \epsilon\cdot\mathcal{L}_K$ and $\imath_{\kappa_u}(\mathcal{L}_{\kappa_u}) \subset \mathcal{L}'_K$, then $\mathcal{L}'_K = \mathcal{L}_K$, hence $\epsilon \in \mathbb{Z}_{(\ell)}^\times$. Moreover, $\mathcal{L}_K/\mathcal{L}_{\kappa_u} \subset \widehat{K}/\widehat{\kappa}_u$ are torsion free (because $\Pi_K \to \Pi_{\kappa_u}$ is surjective).

4) Finally, let $\Phi \in \mathrm{Aut}^{\rm c}(\Pi_K)$ define an isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\mathcal{D}_K^{\rm tot}}$. Suppose that $t, u \in K$ are non-constant functions such that $p_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$, $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ satisfy $\Phi\big(\ker(p_{\kappa_t})\big) = \ker(p_{\kappa_u})$, or equivalently, there exists $\Phi_{t,u} : \Pi_{\kappa_t} \to \Pi_{\kappa_u}$ satisfying $\Phi_{t,u}\circ p_{\kappa_t} = p_{\kappa_u}\circ\Phi$. (Note that $\Phi_{t,u}$ is actually unique.) By the discussion at items 2), 3) above, $\Phi_{t,u}$ maps the set of divisorial groups $T_{v_a} = Z_{v_a}$, $a \in X_t(k)$, of Π_{κ_t} isomorphically onto the the divisorial groups $T_{v_b} = Z_{v_b}$, $b \in X_u(k)$. Hence the isomorphism $\Phi_{t,u}$ defined by Φ satisfies the equivalent condition i)–iv) from item 2) above.

5) The fact that a 1-dimensional quotient $p_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\kappa_u}$ is an abstract rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ in the sense of [P3], section 5, B), and/or [P4], Definition 5.2, is equivalent to the fact that κ_u is a rational function field by [P3], Proposition 41, i.e., $\kappa_u = k(x)$ for some $x \in K$. On the other hand, $\kappa_u|k$ is a rational function field iff $X_u = \mathbb{P}^1_k$ iff the inertia groups $(T_{v_a})_{v_a}$ generate Π_{κ_u} iff $\widehat{U}_{\kappa_u} = 1$. These equivalent conditions are encoded in $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ endowed with $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ and are equivalent to the fact that the canonical divisorial $\widehat{U}_{\kappa_u}$-lattice $\mathcal{L}_{\kappa_u}$ for $\kappa_u|k$ is nothing but $\mathcal{L}_{\kappa_u} = \jmath_{\kappa_u}({\kappa_u}^\times)_{(\ell)}$.

C) *The behavior of* $\jmath_K\big(\langle\Theta, 1-\Theta\rangle\big)_{(\ell)}$ *under weakly* Θ*-compatible automorphisms*

Recall the automorphism group $\mathrm{Aut}^{\mathrm{c}}_{\Theta}(\Pi_K)$ as introduced in Definition/Remark 2.8. In this subsection we show that $\Sigma_\Theta := \jmath_K\big(\langle\Theta, 1-\Theta\rangle\big)_{(\ell)}$ can be reconstructed from Π^{c}_K endowed with π_t, $t\in\Theta$, by a group theoretical recipe invariant under all $\Phi\in\mathrm{Aut}^{\mathrm{c}}_{\Theta}(\Pi_K)$.

Remark 4.4. In the notation/context from Definition/Remark 2.8, let $|\Theta|>1$. Then given Π^{c}_K endowed with the projections $p_t:\Pi_K\to\Pi_{U_0}$, $t\in\Theta$, and $\Phi\in\mathrm{Aut}^{\mathrm{c}}_{\Theta}(\Pi_K)$, one has:

1) By Proposition 3.8, all $p_{\mathcal{K}_t}:\Pi_K\to\Pi_{\mathcal{K}_t}$ can be recovered/reconstructed from π_t, $t\in\Theta$. Further, $p_u\circ\Phi=\Phi_0\circ p_t$ for some $t,u\in\Theta$ and $\Phi_0\in\mathrm{Aut}(\Pi_{U_0})$ iff $\Phi\big(\ker(p_{\mathcal{K}_t})\big)=\ker(p_{\mathcal{K}_u})$.
2) By Propositions 3.10 and 3.11, one can recover the total decomposition graph of $\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$ of $K|k$. Further, Φ defines an automorphism of decomposition graphs $\Phi:\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}\to\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$.
3) By Fact 4.2, II), after replacing Φ by a properly chosen ℓ-adic multiple, we can suppose that Φ is adjusted, hence its Kummer isomorphism $\hat\phi$ satisfies $\hat\phi(\mathcal{L}_K)=\mathcal{L}_K$.
4) By Remark 4.3, 2)–5) above, the group theoretical isomorphism $\Phi_{t,u}:\Pi_{\mathcal{K}_t}\to\Pi_{\mathcal{K}_u}$ satisfying $\Phi_{t,u}\circ p_{\mathcal{K}_t}=p_{\mathcal{K}_u}\circ\Phi$ defines an isomorphism $\Phi_{t,u}:\mathcal{G}_{\mathcal{K}_t}\to\mathcal{G}_{\mathcal{K}_u}$.
5) Therefore, one can recover the Kummer homomorphisms $\imath_{\mathcal{K}_t}:\widehat{\mathcal{K}_t}\hookrightarrow\widehat{K}$ of $p_{\mathcal{K}_t}$ for $t\in\Theta$, and if $\hat\phi_{t,u}:\widehat{\mathcal{K}_u}\to\widehat{\mathcal{K}_t}$ is the Kummer morphism of $\Phi_{t,u}$, one has $\hat\phi\circ\imath_{\mathcal{K}_u}=\imath_{\mathcal{K}_t}\circ\hat\phi_{t,u}$.
6) **Claim.** *In the above context, suppose that* $\hat\phi(\mathcal{L}_K)=\mathcal{L}_K$. *Then* $\hat\phi_{t,u}(\mathcal{L}_{\mathcal{K}_u})=\mathcal{L}_{\mathcal{K}_t}$.

Proof of the Claim. By Remark 4.3, 3) above, applied to $\Phi_{t,u}:\mathcal{G}_{\mathcal{K}_t}\to\mathcal{G}_{\mathcal{K}_u}$, it follows that $\hat\phi_{t,u}(\mathcal{L}_{\mathcal{K}_u})=\varepsilon\cdot\mathcal{L}_{\mathcal{K}_t}$ for some $\varepsilon\in\mathbb{Z}_\ell^\times$. Now since $\imath_{\mathcal{K}_t}(\mathcal{L}_{\mathcal{K}_t})\subset\mathcal{L}_K$, one must have

$$\varepsilon\cdot\mathcal{L}_K\supset\varepsilon\cdot\imath_{\mathcal{K}_t}(\mathcal{L}_{\mathcal{K}_t})=\imath_{\mathcal{K}_t}(\varepsilon\cdot\mathcal{L}_{\mathcal{K}_t})=\imath_{\mathcal{K}_t}\big(\hat\phi_{t,u}(\mathcal{L}_{\mathcal{K}_u})\big)=\hat\phi\big(\imath_{\mathcal{K}_u}(\mathcal{L}_{\mathcal{K}_u})\big)\subset\mathcal{L}_K,$$

hence $(\varepsilon\cdot\mathcal{L}_K)\cap\mathcal{L}_K\neq 1$. Therefore, $\varepsilon\in\mathbb{Z}^\times_{(\ell)}$, and $\hat\phi_{t,u}(\mathcal{L}_{\mathcal{K}_u})=\varepsilon\cdot\mathcal{L}_{\mathcal{K}_t}=\mathcal{L}_{\mathcal{K}_t}$, as claimed.

Proposition 4.5. *In the above context, for every non-constant* $t\in K$, *let* $\langle t,1-t\rangle\subset K^\times$ *be the subgroup generated by* $t, 1-t$, *and set* $\Sigma_t:=\jmath_K(\langle t,1-t\rangle)_{(\ell)}\subset\mathcal{L}_K$. *Further set* $\Sigma_\Theta:=\langle\,\Sigma_t\mid t\in\Theta\,\rangle$. *Let* $\Phi\in\mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ *be adjusted, i.e.,* $\hat\phi(\mathcal{L}_K)=\mathcal{L}_K$. *Then one has:*

1) *If* $t,u\in K^\times$ *are such that* $\Phi\big(\ker(\pi_t)\big)=\ker(\pi_u)$, *then* $\hat\phi(\Sigma_u)=\Sigma_t$.
2) *In particular, if* Φ *is weakly* Θ*-compatible, then* $\hat\phi(\Sigma)=\Sigma$.

Proof. Let $T_0,T_1,T_\infty\subset\Pi_{U_0}$ be the inertia groups above the points $t_0=0, t_0=1, t_0=\infty$, correspondingly. Then T_0,T_1,T_∞ are the only maximal cyclic subgroups of Π_{U_0} containing the non-trivial images $\pi_t(T_v)$, $v\in\mathcal{D}_{K|k}$. Further, these inertia groups have canonical generators $\tau_0,\tau_1,\tau_\infty$, respectively, satisfying the unique relation $\tau_0\tau_1\tau_\infty=1$. Hence if Φ_0 is an automorphism of Π_{U_0} which maps $\{T_0,T_1,T_\infty\}$ onto itself, there exist a unique $\varepsilon_0\in\mathbb{Z}_\ell^\times$ and a permutation $\binom{0\ 1\ \infty}{\alpha\ \beta\ \gamma}$ such that $\Phi_0(\tau_0)=\tau_\alpha^{\varepsilon_0}$, etc. The ℓ-adic dual of Π_{U_0} is the $\mathbb{Z}_\ell$-module $\widehat{\mathcal{L}}_{U_0}$ with generators l_0,l_1,l_∞ defined by $l_i(\tau_j)=\delta_{ij}$ for $i,j\in\{0,1,\infty\}$, thus satisfying the only relation $l_0+l_1+l_\infty=0$. And after identifying the Tate module $\mathbb{Z}_\ell(1)$ with $\mathbb{Z}_\ell$, and setting $\Sigma_{t_0}:=\langle t_0,1-t_0\rangle_{(\ell)}$, via Kummer Theory there is a canonical isomorphism

$$(\dagger)\qquad \widehat\Sigma_{t_0}\to\widehat{\mathcal{L}}_{U_0},\quad t_0\mapsto l_0-l_\infty,\ 1-t_0\mapsto l_1-l_\infty$$

By mere definition, the ℓ-adic dual $\hat\phi_0:\widehat{\mathcal{L}}_{U_0}\to\widehat{\mathcal{L}}_{U_0}$ of Φ_0 is given by $(l_0,l_1,l_\infty)\mapsto\varepsilon_0\cdot(l_\alpha,l_\beta,l_\gamma)$. To give $\hat\phi_0$ in terms of $\widehat\Sigma_{t_0}$, recall that for every l_i-l_j, $i\neq j$, $i,j\in\{0,1,\infty\}$, there is a unique

$t_{i,j} \in \mathfrak{U}_{t_0}$ with $t_{i,j} \mapsto l_i - l_j$, where $\mathfrak{U}_{t_0} := \{t_0, 1-t_0, 1/t_0, 1/(1-t_0), t_0/(t_0-1), (t_0-1)/t_0\}$. Hence the above bijection $(l_0, l_1, l_\infty) \mapsto \varepsilon_0 \cdot (l_\alpha, l_\beta, l_\gamma)$ translates into:

$$(\ddagger) \qquad \hat{\phi}_0 : \widehat{\Sigma}_{t_0} \to \widehat{\Sigma}_{t_0}, \quad t_0 \mapsto t^{\varepsilon_0}_{\alpha,\gamma}, \ 1-t_0 \mapsto t^{\varepsilon_0}_{\beta,\gamma}.$$

To 1): For $\Phi \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$, let $t, u \in K\backslash k$ and $\Phi_0 \in \mathrm{Aut}(\Pi_{U_0})$ be such that $\Phi_0 \circ \pi_t = \pi_u \circ \Phi$. Then by Remark 4.4, Φ gives rise to an isomorphisms $\Phi_{t,u} : \Pi_{\kappa_t} \to \Pi_{\kappa_u}$ which is an isomorphism of decomposition graphs $\Phi_{t,u} : \mathcal{G}_{\kappa_t} \to \mathcal{G}_{\kappa_u}$. Further, by Remark 4.4, Claim, it follows that the Kummer isomorphism $\hat{\phi}_{t,u}$ of $\Phi_{t,u}$ satisfies: $\hat{\phi}_{t,u}(\mathcal{L}_{\kappa_u}) = \mathcal{L}_{\kappa_t}$. Hence taking into account the commutative diagram

$$\begin{array}{ccccc} \Pi_K & \to & \Pi_{\kappa_u} & \to & \Pi_{U_0} \\ \downarrow^{\Phi} & & \downarrow^{\Phi_{t,u}} & & \downarrow^{\Phi_0} \\ \Pi_K & \to & \Pi_{\kappa_t} & \to & \Pi_{U_0} \end{array}$$

it follows that its ℓ-adically dual diagram is:

$$(*) \qquad \begin{array}{ccccc} \mathcal{L}_K & \leftarrow & \mathcal{L}_{\kappa_u} & \leftarrow & \Sigma^{\varepsilon_0}_{t_0} \\ \uparrow \hat{\phi} & & \uparrow \hat{\phi}_{t,u} & & \uparrow \hat{\phi}_0 \\ \mathcal{L}_K & \leftarrow & \mathcal{L}_{\kappa_t} & \leftarrow & \Sigma_{t_0} \end{array}$$

Arguing as in the proof of the Claim from Remark 4.4, it follows that $\varepsilon_0 \cdot \mathcal{L}_{\kappa_u} \cap \mathcal{L}_{\kappa_u} \neq 1$, and therefore, $\varepsilon_0 \in \mathbb{Z}^\times_{(\ell)}$. Thus taking into account $(\ddagger)$, it follows that $\hat{\phi}_0(\Sigma_{t_0}) = \Sigma_{t_0}$.

Next recall the canonical projection $q_t : \Pi_{\kappa_t} \to \Pi_{U_0}$ defined by $t_0 \mapsto t \in \kappa_t$, which gives rise to the factorization $\pi_t = q_t \circ p_{\kappa_t}$. Then the ℓ-adic dual of q_t, i.e., its Kummer homomorphism, is an embedding $\widehat{\Sigma}_{t_0} \hookrightarrow \widehat{\kappa}_t$ whose restriction to Σ_{t_0} satisfies:

$$\Sigma_{t_0} \hookrightarrow \mathcal{L}_{\kappa_t}, \quad t_0 \mapsto \jmath_{\kappa_t}(t), \ 1-t_0 \mapsto \jmath_{\kappa_t}(1-t).$$

Hence $\Sigma_t := \jmath_{\kappa_t}(\langle t, 1-t\rangle)_{(\ell)} \subset \mathcal{L}_{\kappa_t}$ can be recovered from $q_t : \Pi_{\kappa_t} \to \Pi_{U_0}$, as being:

$$(*)_t \qquad \Sigma_t := \jmath_{\kappa_t}(\langle t, 1-t\rangle)_{(\ell)} = \mathrm{im}(\Sigma_{t_0} \hookrightarrow \mathcal{L}_{\kappa_t}).$$

The same holds correspondingly for $q_u : \Pi_{\kappa_u} \to \Pi_{U_0}$ defined by $t_0 \mapsto u \in \kappa_u$, and we finally get that $\Sigma_u := \jmath_{\kappa_t}(\langle u, 1-u\rangle)_{(\ell)} \subset \mathcal{L}_{\kappa_u}$ can be recovered from $q_u : \Pi_{\kappa_u} \to \Pi_{U_0}$, as being

$$(*)_u \qquad \Sigma_u := \jmath_{\kappa_u}(\langle u, 1-u\rangle)_{(\ell)} = \mathrm{im}(\Sigma_{t_0} \hookrightarrow \mathcal{L}_{\kappa_u}).$$

Hence by the commutativity of the diagram $(*)$ above, $\hat{\phi}(\Sigma_u) = \Sigma_t$.

To 2): Let $\theta : \Theta \to \Theta$ be the bijection defining Φ as weakly Θ-compatible, and recall that $\ker(\pi_u) = \Phi\big(\ker(\pi_t)\big)$ for $\theta(t) = u$. Thus Φ bring adjusted, it follows by assertion 1), that $\hat{\phi}(\Sigma_u) = \Sigma_t$ for $\theta(t) = u$. Proceed by taking into account that θ is a bijection, etc. □

D) *Recovering the rational projections*

Recall that the non-constant functions $x \in K$ such that $\kappa_x = k(x)$, for short general elements of K, are quite abundant in K. Indeed, by the discussion from [P3], Fact/Definition 43, one has: Let $x, t \in K$ be fixed algebraically independent functions over k, with x separable, e.g., general. For later use, we notice that the following hold:

a) $t_a := t + ax$ is a general element of K for almost all $a \in k$.

b) $t_{a',a} := t/(a'x + a)$ is a general element of K for all $a' \in k^\times$ and almost all $a \in k$.

c) $t_{a'',a',a} := (a''t + a'x + a + 1)/(t + a'x + a)$ is a general element of K for all $a'' \in k$ and almost all $a', a \in k$.

d) Moreover, setting $\alpha := a'' - 1$, an obvious direct computation shows the following:

$$t_{a'',a',a} = \frac{a'x + a + 1}{a'x + a} \cdot \frac{a'' t_{a',a+1} + 1}{t_{a',a} + 1}, \quad \alpha t + 1 = (ax' + a)(t_{a'',a',a} - 1)/(t_{a',a} + 1)\,.$$

e) Finally, suppose that t, t' are algebraically independent over k, and let $\langle t, t' \rangle \subset K^\times$ be the multiplicative subgroup generated by t, t'. Then if there is no prime number q such that both t and t' are q^{th} powers in K, then $\langle t, t' \rangle$ contains general elements. In general, if there are prime numbers q such that t, t' are q^{th} powers in K, consider the maximal number n such that t, t' are both n^{th} powers in K, say $t = t_0^n$, $t' = t_0'^n$. Then there is no prime number q such that both t_0 and t_0' are q^{th} powers in K. Hence $\langle t_0, t_0' \rangle$ contains general elements, say $x = t_0^r t_0'^s$ with $r, s \in \mathbb{Z}$. Hence setting $u := x^n$, one has:
   i) $u = t^r t'^s \in \langle t, t' \rangle$.
   ii) $\kappa_u = k(x)$ is a rational function field.

Recall that given any $\mathsf{u} \in \jmath_K(K^\times)_{(\ell)}$, say $\mathsf{u} = \theta \cdot \jmath_K(u)$ with $u \in K^\times$ non-constant and some non-zero $\theta \in \mathbb{Z}_{(\ell)}$, the Lemma 4.1 above gives a recipe to recover the 1-dimensional quotient $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$, and shows that p_{κ_u} does not depend on the specific u and/or u, but only on $\mathbb{Z}_{(\ell)} \cdot \mathsf{u}$. Further, by Remark 4.3 above, especially item 5), among the 1-dimensional projections p_{κ_u}, one can single out the *rational quotients,* i.e., satisfying $\kappa_u = k(x)$, $x \in K$.

Construction 4.6. In the context of Theorem 2.9, we will construct inductively a sequence $(\Sigma_n)_n$ of subsets and a sequence of $\mathbb{Z}_{(\ell)}$-submodules $(\mathcal{K}_n)_n$ of $\jmath_K(K^\times)_{(\ell)}$ as follows:

Step 1: Constructing Σ_1, $\mathcal{K}_1$:

By the hypothesis of Theorem 2.9, one has $K = k(\Theta)$. Hence since $\mathrm{td}(K|k) > 1$, there are algebraically independent $t, t' \in \Theta$. Thus by item e) above, there exists $u \in \langle \Theta, 1-\Theta \rangle$ such that $\kappa_u = k(x)$ for some $x \in K = k(X)$. In particular, the set Σ_1 of all the rational quotients p_{κ_u} with $u \in \langle \Theta, 1-\Theta \rangle$ is non-empty. This being said, we let $\mathcal{K}_1 \subset \mathcal{L}_K$ be the $\mathbb{Z}_{(\ell)}$-submodule generated by Σ_Θ and the images $\imath_{\kappa_u}(\mathcal{L}_{\kappa_u}) = \jmath_K(\kappa_u)_{(\ell)}$ for all the $p_{\kappa_u} \in \Sigma_1$.

Step $(n+1)$: Constructing Σ_{n+1}, $\mathcal{K}_{n+1}$:

Supposing that Σ_n and $\mathcal{K}_n \subseteq \jmath_K(K^\times)_{(\ell)} \subseteq \mathcal{L}_K$ are constructed, we proceed as follows: For $\mathsf{u} \in \mathcal{K}_n$, $\mathsf{u} \neq 1$, let $u \in K^\times$ be such that $\mathsf{u} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(u)$, and $p_{\kappa_u} \colon \Pi_K \to \Pi_{\kappa_u}$ be the 1-dimensional quotient defined by u as indicated at Remark 4.3, 1) above. By Remark 4.3, 5), we can recover the fact $\kappa_u|k$ is a rational function field from $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ endowed with p_{κ_u}. Define Σ_{n+1} to be the set of rational quotients $p_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$, $\mathsf{u} \in \mathcal{K}_n$. We also notice that by the discussion at Remark 4.3, 4) and 5), it follows that for every $p_{\kappa_u} \in \Sigma_{n+1}$ one has that $\mathcal{L}_{\kappa_u} = \jmath_{\kappa_u}({\kappa_u}^\times)_{(\ell)}$ is the unique divisorial lattice for $\kappa_u|k$ such that $\imath_{\kappa_u}(\mathcal{L}_{\kappa_u}) \subset \mathcal{L}_K$, and therefore $\imath_{\kappa_u}(\mathcal{L}_{\kappa_u}) \subset \jmath_K(K^\times)_{(\ell)} \subseteq \mathcal{L}_K$. Finally, let $\mathcal{K}_{n+1} \subseteq \mathcal{L}_K$ be the $\mathbb{Z}_{(\ell)}$-submodule generated by Σ_Θ and all the images $\imath_{\kappa_u}(\mathcal{L}_{\kappa_u})$, $p_{\kappa_u} \in \Sigma_{n+1}$, and notice that $\mathcal{K}_{n+1} \subseteq \jmath_K(K^\times)_{(\ell)}$. And obviously, $\Sigma_n \subseteq \Sigma_{n+1}$ and therefore, $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$ for all $n \geq 1$.

Proposition 4.7. *In the above notations, let $\Sigma := \cup_n \Sigma_n$, $\mathcal{K} := \cup_n \mathcal{K}_n$. Then one has:*

1) *$\mathcal{K} = \jmath_K(K^\times)_{(\ell)}$ and Σ is the set of all the rational quotients $p_{\kappa_x} : \Pi_K \to \Pi_{\kappa_x}$ of Π_K.*

2) *Let the Kummer morphism of $\Phi \in \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$ satisfy $\hat\phi(\mathcal{L}_K) = \mathcal{L}_K$. Then $\hat\phi(\mathcal{K}) = \mathcal{K}$, and for every $p_{\kappa_x} \in \Sigma$ there exists $p_{\kappa_y} \in \Sigma$ such that $\Phi\big(\ker(p_{\kappa_x})\big) = \ker(p_{\kappa_y})$.*

Proof. To 1): We first show that $\mathcal{K} = \jmath_K(K^\times)_{(\ell)}$. By hypothesis one has $K = k(\Theta)$, hence every element $t \in K^\times$ is a fraction t'/t'', where $t', t'' \neq 0$ are polynomials in the elements of Θ. In particular, it suffices to prove that all the polynomials $t' = \sum_i^n a_i M_i$, with $a_i \in k^\times$ and M_i monomials in the elements of Θ, lie in $\mathcal{K}$. For that, we make induction on n.

$\underline{n=1}$: Then $t' = a_1 M_1$ with $a_1 \in k^\times$ and M_1 a monomial in the elements of Θ. But then $\jmath_K(M_1) \in \Sigma_\Theta$, and since by definition one has $\Sigma_\Theta \subset \mathcal{K}_1 \subset \mathcal{K}$, we are done.

$\underline{n \Rightarrow (n+1)}$: Let $t' = \sum_i^{n+1} a_i M_i$, where $n \geq 1$, hence $n+1 \geq 2$, and $a_i \in k^\times$. Setting $b_i := a_i/a_{n+1}$ for $i = 1, \dots, n$ and $u' := \sum_{j=1}^n b_j M_j$, it follows by the induction hypothesis that $\jmath_K(u') \in \mathcal{K}$, hence there exist $m > 0$ such that $\jmath_K(u') \in \mathcal{K}_m$. Thus setting $t := u'/M_{n+1}$, one has that $\jmath_K(t) = \jmath_K(u')/\jmath_K(M_{n+1})$, hence $\jmath_K(t) \in \mathcal{K}_m$, and notice that $t' = a_{n+1}M_{n+1}(t+1)$. Hence in order to prove that $\jmath_K(t') \in \mathcal{K}$, it is sufficient to prove that $\jmath_K(t+1) \in \mathcal{K}$.

Claim. $\jmath_K(t+1) \in \mathcal{K}_{m+2}$.

Indeed, first recall that by the discussion at Step 1, there exists some $u \in \langle \Theta, 1-\Theta \rangle$ such that $\kappa_u = k(x)$ for some $x \in K$, i.e., p_{κ_u} is a rational quotient of Π_K. Hence by the definition of $\mathcal{K}_1$, one has $\jmath_K(\kappa_x^\times) \subset \mathcal{K}_1$, thus $\jmath_K(a'x + a) \in \mathcal{K}_1$ for all $a' \in k^\times$ and $a \in k$. Hence in the notations and by the discussion at the beginning of this subsection, one has:

- Since $\mathcal{K}_1 \subseteq \mathcal{K}_m$ and $\jmath_K(t) \in \mathcal{K}_m$, one has that $\jmath_K(t_{a',a}) \in \mathcal{K}_m$. Further, $t_{a',a}$ is a general element of K for all $a' \in k^\times$ and almost all $a \in k$, $a \notin a' \cdot \Sigma_{t,x}$ for some finite set $\Sigma_{t,x}$. Hence by mere definitions, one has that $p_{\kappa_{t_{a',a}}} \in \Sigma_{m+1}$, and therefore, $\jmath_K(\kappa^\times_{t_{a',a}}) \subset \mathcal{K}_{m+1}$.

  In particular, setting $b := a+1$, for almost all $a', a \in k$ one has that $a'x+a$, $a'x+b$, $t_{a',a}+1$ are general elements of K whose images under $\jmath_K$ lie in $\mathcal{K}_{m+1}$. And if $a'' := \alpha + 1 \in k^\times$, the element $a''t_{a',b} + 1$ is general as well, and its image under $\jmath_K$ lies in $\mathcal{K}_{m+1}$ as well.
- Second, taking into account the formula given above under d), it follows that for all $a, a'' \in k$ and $a' \in k^\times$, one has that $\jmath_K(t_{a'',a',a}) \in \mathcal{K}_{m+1}$. Further, for all $a'' \in k$ and and almost all $a, a' \in k$, it follows that $t_{a'',a',a}$ is a general element of K, hence by mere definitions one has $p_{\kappa_{t_{a'',a',a}}} \in \Sigma_{m+2}$. Thus concluding that $\jmath_K(\kappa^\times_{t_{a'',a',a}}) \subset \mathcal{K}_{m+2}$.

Hence we finally conclude that of all $\alpha \in k$ and almost all $a, a' \in k$, one has:

$$\alpha t + 1 = (a'x + a)(t_{a'',a',a} - 1)/(t_{a',a} + 1),$$

and therefore, $\jmath_K(\alpha t + 1) \in \mathcal{K}_{m+2}$ for all $\alpha \in k$. This concludes the proof of the Claim, thus of the fact that $\mathcal{K} = \jmath_K(K^\times)$. Finally, the assertion that $\cup_n \Sigma_n$ consists of all rational quotients of Π_K is more or less clear: Let namely $x \in K$ be such that $p_{\kappa_x} : \Pi_K \to \Pi_{\kappa_x}$ is a rational quotient. Since $\jmath_K(x) \in \mathcal{K}$, and $\mathcal{K} = \cup_n \mathcal{K}_n$, one has $\jmath_K(x) \in \mathcal{K}_n$ for n sufficiently large. But then by mere definitions, for every such n, one has that $p_{\kappa_x} \in \Sigma_{n+1}$.

To 2): We first claim that for every $n > 0$ and every $p_{\kappa_x} \in \Sigma_n$ there exists some $p_{\kappa_y} \in \Sigma_n$ such that $\Phi\big(\ker(p_{\kappa_x})\big) = \ker(p_{\kappa_y})$ and $\hat\phi(\mathcal{K}_n) = \mathcal{K}_n$. We prove this by induction on n:

$\underline{n=1}$: Recall that Σ_1 consists of all the rational quotients $p_{\kappa_x} : \Pi_K \to \Pi_{\kappa_x}$ of the form $k(x) = \kappa_u$, $u \in \Sigma_\Theta$. Notice that p_{κ_u} being a rational quotient of Π_K depends on $\mathbb{Z}_{(\ell)} \cdot \jmath_K(u)$ only, and not on the specific u. Now let $\Phi \in \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$ be adjusted, i.e., its Kummer isomorphisms $\hat\phi$ satisfies $\hat\phi(\mathcal{L}_K) = \mathcal{L}_K$. Then by Proposition 4.5, one has that $\hat\phi(\Sigma_\Theta) = \Sigma_\Theta$. For $p_{\kappa_u} \in \Sigma_1$, let $\hat\phi(u) = \alpha \cdot \jmath_K(t)$ for some $t \in \langle \Theta, 1-\Theta \rangle$ and $\alpha \in \mathbb{Z}_{(\ell)}$. Then $p_{\kappa_t} \circ \Phi = p_{\kappa_u}$ is a rational projection of Π_K, hence $p_{\kappa_t} \in \Sigma_1$, as claimed. In particular, for $\Phi \in \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$

and $\hat\phi$ are as above, one has $\hat\phi(\mathcal{L}_{\kappa_u}) = \mathcal{L}_{\kappa_t} \subset \mathcal{K}_1$. Since this is true for all $p_{\kappa_u} \in \Sigma_1$ and for Σ_Θ as well, we conclude that $\hat\phi(\mathcal{K}_1) = \mathcal{K}_1$.

$\underline{n \Rightarrow (n+1)}$: By the induction hypothesis, we have $\hat\phi(\mathcal{K}_n) = \mathcal{K}_n$, and by the construction of $\mathcal{K}_n$ one has $\mathcal{K}_n \subset \jmath_K(K^\times)_{(\ell)}$. Thus $\forall\, \mathsf{x} \in \mathcal{K}_n\ \exists\, \mathsf{y} \in \mathcal{K}_n$ such that $\hat\phi(\mathsf{x}) = \mathsf{y}$. Now if $x, y \in K^\times$ are such that $\mathsf{x} \in \mathbb{Z}_{(\ell)}\cdot\jmath_K(x)$ and $\mathsf{y} \in \mathbb{Z}_{(\ell)}\cdot\jmath_K(y)$, let p_{κ_x} and p_{κ_y} be the 1-dimensional quotients defined by κ_x and κ_y as indicated at Remark 4.3, 3). Then by loc.cit., 1), and reasoning as in the proof of Lemma 4.8, 1) below, it follows that $\{\, T_v \subset Z_v \mid v \in \mathcal{D}^1_x \,\}$ is mapped by Φ isomorphically onto the $\{\, T_w \subset Z_w \mid w \in \mathcal{D}^1_y \,\}$. But then by Lemma 4.8 below, it follows there is an isomorphism of profinite groups $\Phi_{x,y} : \Pi_{\kappa_x} \to \Pi_{\kappa_y}$ satisfying $\Phi_{x,y} \circ p_{\kappa_x} = p_{\kappa_y} \circ \Phi$ and defining an isomorphism of decomposition graphs $\Phi_{x,y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$, and the Kummer morphism $\hat\phi_{x,y} : \widehat{\kappa}_y \to \widehat{\kappa}_x$ of $\Phi_{x,y}$ satisfies $\hat\phi_{x,y}(\mathcal{L}_{\kappa_y}) = \mathcal{L}_{\kappa_x}$. In particular, p_{κ_x} is a rational quotient iff p_{κ_y} is so. Thus we conclude that for every rational quotient $p_{\kappa_x} \in \Sigma_{n+1}$, there exists $p_{\kappa_y} \in \Sigma_{n+1}$ and an isomorphism $\Phi_{x,y}$ as above such that $\Phi_{x,y} \circ p_{\kappa_x} = p_{\kappa_y} \circ \Phi$. Clearly, $\Sigma_{n+1} \to \Sigma_{n+1}$ via $p_{\kappa_x} \mapsto p_{\kappa_y}$, defines a bijection. And notice that the Kummer isomorphism $\hat\phi$ of Φ satisfies $\hat\phi\big(\imath_{\kappa_y}(\mathcal{L}_{\kappa_y})\big) = \imath_{\kappa_x}(\mathcal{L}_{\kappa_x})$. Thus since $\mathcal{K}_{n+1}$ is generated by Σ_Θ and all the $\jmath_K(\kappa_x^\times)$ with $p_{\kappa_x} \in \Sigma_{n+1}$, we get: $\hat\phi(\mathcal{K}_{n+1}) = \mathcal{K}_{n+1}$, hence $\mathcal{K}_{n+1}$ is invariant under $\hat\phi$. Finally, since $\mathcal{K} := \cup_n \mathcal{K}_n$ and $\Sigma := \cup_n \Sigma_n$, the above discussion concludes the proof. $\square$

Lemma 4.8. *In the above notations, let $\Phi \in \mathrm{Aut}^c_\Theta(\Pi_K)$ be such that its Kummer isomorphism $\hat\phi$ satisfies $\hat\phi(\mathcal{L}_K) = \mathcal{L}_K$. Then for $x, y \in K\backslash k$, one has:*

1) *The automorphism Φ maps the divisorial subgroups $T_v \subset Z_v$, $v \in \mathcal{D}^1_x$ isomorphically on the divisorial subgroups $T_w \subset Z_w$, $w \in \mathcal{D}^1_y$ if and only if $\Phi\big(\ker(p_{\kappa_x})\big) = \ker(p_{\kappa_y})$.*
2) *Let $\ker(p_{\kappa_y}) = \Phi\big(\ker(p_{\kappa_x})\big)$. Then the abstract isomorphism $\Phi_{x,y} : \Pi_{\kappa_x} \to \Pi_{\kappa_y}$ induced by Φ defines an isomorphism of decomposition graphs $\Phi_{x,y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ whose Kummer isomorphism $\hat\phi_{x,y} : \widehat{\kappa}_y \to \widehat{\kappa}_x$ satisfies $\hat\phi_{x,y}(\mathcal{L}_{\kappa_y}) = \mathcal{L}_{\kappa_x}$.*

Proof. For every divisorial subgroup $T_v \subset Z_v$ of Π_K and its image $\Phi(T_v) = T_w \subset Z_w = \Phi(T_v)$ under Φ, one has the following, see e.g., [P3], Remark 26:

a) $\hat\phi$ maps $\widehat{U}_w$ isomorphically onto $\widehat{U}_v$.

b) $\hat\phi$ maps $\ker(\jmath_w)$ isomorphically onto $\ker(\jmath_v)$.

In particular, $v \in \mathcal{D}^1_x$ iff $\mathsf{x} \in \widehat{U}_v$ and $\jmath_v(\mathsf{x}) \neq 1$ iff $\mathsf{y} \in \widehat{U}_w$ and $\jmath_w(\mathsf{y}) \neq 1$ iff $w \in \mathcal{D}^1_y$. Thus using a), b) above, by Lemma 4.1, we conclude that $\{\, T_v \subset Z_v \mid v \in \mathcal{D}^1_x \,\}$ is mapped isomorphically onto $\{\, T_w \subset Z_w \mid w \in \mathcal{D}^1_y \,\}$ iff $\hat\phi$ maps $\widehat{\kappa}_{y,\mathrm{fin}}$ isomorphically onto $\widehat{\kappa}_{x,\mathrm{fin}}$. On the other hand, hand, by taking ℓ-adic duals, we conclude that $\hat\phi$ maps $\widehat{\kappa}_{y,\mathrm{fin}}$ isomorphically onto $\widehat{\kappa}_{x,\mathrm{fin}}$ iff $\Phi\big(\ker(p_{\kappa_x})\big) = \ker(p_{\kappa_y})$. This concludes the proof of assertion 1).

To 2): By 1) above, Φ maps $\{\, T_v \subset Z_v \mid v \in \mathcal{D}^1_x \,\}$ isomorphically onto $\{\, T_w \subset Z_w \mid w \in \mathcal{D}^1_y \,\}$. Further, by Remark 4.3, 3), we have: Let $v \in \mathcal{D}^1_x$ be given, and v_α be the restriction of v on κ_x. Then $\Phi(T_v) = T_w$ for some prime divisor w of $K|k$ such that $p_{\kappa_y}(T_w) = \Phi_{x,y}\big(p_{\kappa_x}(T_v)\big)$, thus $p_{\kappa_y}(T_w)$ is non-trivial, because $p_{\kappa_x}(T_v)$ is so, and $\Phi_{x,y}$ is an isomorphism. Therefore, the restriction w_β of w to κ_y is non-trivial. Further, since $\Phi_{x,y}$ is an isomorphism, and T_{v_a} is the unique maximal pro-cyclic subgroup of Π_{κ_x} containing $p_{\kappa_x}(T_v)$, it follows that $\Phi_{x,y}(T_{v_a})$ is the unique maximal pro-cyclic subgroup of Π_{κ_y} which contains $p_{\kappa_y}(T_w)$. We conclude that $\Phi_{x,y}(T_{v_a}) = T_{w_b}$ for some $b \in X_y(k)$, where X_y is the projective smooth model of $\kappa_y|k$. Thus

$\Phi_{x,y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ is an (abstract) isomorphism of decomposition groups. Finally, one has $\imath_{\kappa_x}(\mathcal{L}_{\kappa_x}) \subset \mathcal{L}_K$ and $\imath_{\kappa_y}(\mathcal{L}_{\kappa_y}) \subset \mathcal{L}_K$. Since $\hat\phi \circ \imath_{\kappa_y} = \imath_{\kappa_x} \circ \hat\phi_{x,y}$, it follows that $\mathcal{L}'_{\kappa_x} := \hat\phi_{x,y}(\mathcal{L}_{\kappa_y})$ is a divisorial lattice for $\kappa_x|k$ such that

$$\imath_{\kappa_x}(\mathcal{L}'_{\kappa_x}) = \imath_{\kappa_x}\big(\hat\phi_{x,y}(\mathcal{L}_{\kappa_y})\big) = \hat\phi\big(\imath_{\kappa_y}(\mathcal{L}_{\kappa_y})\big) \subset \hat\phi(\mathcal{L}_K) = \mathcal{L}_K.$$

Thus $\imath_{\kappa_x}(\mathcal{L}'_{\kappa_x}) \subset \mathcal{L}_K$. Hence by the uniqueness of $\mathcal{L}_{\kappa_x}$ with the property that $\imath_{\kappa_x}(\mathcal{L}_{\kappa_x}) \subset \mathcal{L}_K$, it follows that $\mathcal{L}'_{\kappa_x} = \mathcal{L}_{\kappa_x}$. Thus we conclude that $\hat\phi_{x,y}(\mathcal{L}_{\kappa_y}) = \mathcal{L}'_{\kappa_x} = \mathcal{L}_{\kappa_x}$, as claimed. □

5. Concluding the proof of Theorem 2.9

The *injectivity* of $\mathrm{Aut}_\Theta(K^{\mathrm{i}}) \to \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$ follows from the one of $\mathrm{Aut}(K^{\mathrm{i}}) \to \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$, which is well known, and we will not repeat the quite standard arguments here.

For the *surjectivity* of $\mathrm{Aut}_\Theta(K^{\mathrm{i}}) \to \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$, let $\Phi \in \mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$ be given. Then by Proposition 4.7 above, Φ defines an isomorphism $\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K} \to \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$ which is compatible with all the rational quotients $\Phi_{\kappa_x} : \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K} \to \mathcal{G}_{\kappa_x}$. Hence by the Main Theorem from [P3], Introduction, it follows that there exists an isomorphism of fields $\phi : K^{\mathrm{i}} \to K^{\mathrm{i}}$ and some ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\varepsilon \cdot \Phi$ is defined by ϕ, i.e., if ϕ' is some prolongation of ϕ to K', then $\varepsilon \cdot \Phi(g) = \phi'^{-1} g\, \phi'$ for all $g \in \Pi_K$. This proves assertion i) of Theorem 2.9. For assertion ii), we notice that replacing Φ by $\Phi_\varepsilon := \varepsilon \cdot \Phi$,[8] we can suppose that actually $\varepsilon = 1$, and therefore, the Kummer isomorphism of Φ is simply the ℓ-adic completion of the multiplicative isomorphism $\phi : K^\times \to K^\times$. Then by the commutativity of the diagram $(*)$ from the proof of Proposition 4.5, one has that $\hat\phi(\mathcal{L}_{\kappa_u}) = \mathcal{L}_{\kappa_t}$, hence $\hat\phi(\widehat{\kappa}_u) = \widehat{\kappa}_t$. Thus since $\widehat{\kappa}_x \neq \widehat{\kappa}_y$ for $\kappa_x \neq \kappa_y$, we conclude that $\phi(\kappa^{\mathrm{i}}_u) = \kappa^{\mathrm{i}}_t$. Further, by loc. cit., one has that $\hat\phi(\Sigma_u) = \Sigma_t$.

For $t \in \Theta$ consider the corresponding $u \in \Theta$ and $\Phi_0 \in \mathrm{Aut}(\Pi_{U_0})$ such that $\Phi_0 \circ \pi_t = \pi_u \circ \Phi$. Then recalling the notations and facts from the first part of the proof of Proposition 4.5, especially the facts (†) and (‡), it follows that the ℓ-adic dual of Φ_0 is defined by

$$\hat\phi_0 : \widehat{\Sigma}_{t_0} \to \widehat{\Sigma}_{t_0}, \quad t_0 \mapsto t^{\varepsilon}_{\alpha,\gamma}, \;\; 1 - t_0 \mapsto t^{\varepsilon}_{\beta,\gamma},$$

where $t_{i,j} \in \mathfrak{U}_{t_0} := \{t_0, 1 - t_0, 1/t_0, 1/(1 - t_0), t_0/(t_0 - 1), (t_0 - 1)/t_0\}$ is the unique function with divisor $i - j$ for $i, j \in \{0, 1, \infty\}$, $i \neq j$. Further, the commutative diagram $(*)$ from the proof of Proposition 4.5, gives rise canonically to the commutative diagram below:

$$(*) \qquad \begin{array}{ccc} \Sigma_{t_0} & \to & \Sigma_u \\ \downarrow \hat\phi_0 & & \downarrow \hat\phi_{t,u} \\ \Sigma_{t_0} & \to & \Sigma_t \end{array} \qquad\qquad \begin{array}{ccc} t_0 & \mapsto & u \\ \downarrow & & \downarrow \\ \hat\phi_0(t_0) & \mapsto & \hat\phi_{t,u}(u) \end{array}$$

Finally, recalling that $\varepsilon \in \mathbb{Z}_{(\ell)}$, let us write $\varepsilon = m/n$ with $m, n \in \mathbb{Z}$, $n > 0$. Then the equalities $\hat\phi_0(t_0) = t^{\varepsilon}_{\alpha,\gamma}$, $\hat\phi_0(1 - t_0) = t^{\varepsilon}_{\beta,\gamma}$ are equivalent to $\hat\phi_0(t_0)^n = t^m_{\alpha,\gamma}$, $\hat\phi_0(1 - t_0)^n = t^m_{\beta,\gamma}$. Thus recalling that $\hat\phi_0$ and $\hat\phi_{t,u}$ are induced by the field isomorphism $\phi : K^{\mathrm{i}} \to K^{\mathrm{i}}$, it follows that there exists a field isomorphism $\phi_0 : k(t_0)^{\mathrm{i}} \to k(t_0)^{\mathrm{i}}$, and $a, b \in k^\times$ such that:

$$\phi_0(t_0)^n = a\, t^m_{\alpha,\gamma}, \quad \phi_0(1 - t_0)^n = b\, t^m_{\beta,\gamma},$$

where $t_{\alpha,\gamma}, t_{\beta,\gamma} \in \mathfrak{U}_{t_0}$ are as introduced above.

On the other hand, since $\phi_0 \in \mathrm{Aut}(k(t_0)^{\mathrm{i}})$, it follows that t_0, $t_{\alpha,\gamma}$, $t_{\beta,\gamma}$ are purely inseparable over the field $k(t_0) = k(t_{\alpha,\gamma}) = k(t_{\beta,\gamma})$. Hence we conclude that m, n are actually powers of

[8] Note that by mere definition, Φ and Φ_ε represent the same element in $\mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$.

the characteristic exponent p, thus $\varepsilon = m/n = p^e$ for some $e \in \mathbb{Z}$, and one has: $\phi_0(t_0) = a t_{\alpha,\gamma}^{p^e}$, $\phi_0(1-t_0) = b t_{\beta,\gamma}^{p^e}$, and since ϕ_0 is a field morphism, we finally get:

$$1 - a t_{\alpha,\gamma}^{p^e} = b t_{\beta,\gamma}^{p^e} \text{ for some } a, b \in k^\times$$

Recalling that $(t_{\alpha,\gamma}) = \alpha - \gamma$, $(t_{\beta,\gamma}) = \beta - \gamma$ are the divisors of $t_{\alpha,\gamma}$, respectively $t_{\beta,\gamma}$, and that the functions from $\mathfrak{U}_{t_0} = \{t_0, 1-t_0, 1/t_0, 1/(1-t_0), t_0/(t_0-1), (t_0-1)/t_0\}$ take only the values $0, 1, \infty$ on $\{\alpha, \beta, \gamma\} = \{0, 1, \infty\}$, one has:

a) Since $t_{\beta,\gamma}(\beta) = 0$, one has $1 - a\, t_{\alpha,\gamma}(\beta) = 0$, and therefore, $a = 1$.

b) Since $t_{\alpha,\gamma}(\alpha) = 0$, one has $1 = b\, t_{\beta,\gamma}(\alpha)$, and therefore, $b = 1$.

Thus going back to $\phi_{t,u} : \kappa_u{}^{\mathrm{i}} \to \kappa_t{}^{\mathrm{i}}$ via $t_0 \mapsto u$, we get:

$$\forall\, t \in \Theta\ \exists\, e \in \mathbb{Z}\ \exists\, u \in \Theta \text{ s.t. } \phi(u) = t_\phi^{p^e},\ t_\phi \in \{t, 1-t, 1/t, 1/(1-t), t/(t-1), (t-1)/t\},$$

and this completes the proof of assertion ii) of Theorem 2.9.

6. Proof of Theorem 2.6

A) *Proof of assertion* 1)

Recall that $k := \overline{k}$, $K_0 = k_0(X)$, and $K = k(\overline{X})$, hence $\mathrm{Gal}_{k_0} = \mathrm{Aut}^{\mathrm{i}}(k|k_0) = \mathrm{Aut}^{\mathrm{i}}(K|K_0)$. Therefore, the injectivity of the canonical maps $\mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(K^{\mathrm{i}}) \to \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)$ follows by the fact that the canonical maps $\mathrm{Gal}_{k_0} = \mathrm{Aut}^{\mathrm{i}}(K|K_0) \to \mathrm{Aut}(\Pi_K)$ are obviously injective.

Concerning the surjectivity of $\mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)$, let $\Phi \in \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)$ be given. Then by mere definition, Φ satisfies condition i) from Definition/Remark 2.4, 1), hence Φ is Θ-compatible with respect to the bijection $\theta = \mathrm{id}_\Theta$ of Θ. Therefore by Theorem 2.9, there exists a $\phi \in \mathrm{Aut}_\Theta(K^{\mathrm{i}})$ which defines Φ as indicated in loc. cit. Recalling that $K'|K$ is the maximal abelian pro-ℓ extension of K^{i}, let $\phi' : K' \to K'$ be the prolongation of ϕ to K'.

Claim. *ϕ is $\mathcal{V}_X$-compatible in the sense of Definition/Remark 2.2, 2).*

Indeed, since ϕ is Θ-compatible with respect to $\mathrm{id}_\Theta : \Theta \to \Theta$, it follows that ϕ satisfies condition i) from Definition/Remark 2.2, 2). To show that ϕ satisfies condition ii), from loc. cit., let $\varphi : \overline{X} \dashrightarrow \overline{X}$ be the birational map corresponding to ϕ. Since $\Phi \in \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K)$ is $\mathcal{V}_X$-compatible, one has by mere definitions: First, for all $U_i \in \mathcal{B}$ on which φ is defined, one has $\Phi\big(\ker(p_{U_i})\big) = \ker(p_{U_i})$, where $p_{U_i} : \Pi_K \to \Pi_{U_i}$ is the canonical projection. In particular, if $K_{U_i}|K$ is the fixed field of $\ker(p_{U_i})$ in K', it follows that ϕ' maps K_{U_i} isomorphically onto itself. On the other hand, setting $V_i := \varphi(\overline{U}_i)$, it follows by Galois functionality that the fix field K_{V_i} of $\ker(p_{V_i})$ in K' is nothing but $K_{V_i} = \phi'(K_{\overline{U}_i})$. Hence we conclude that $\ker(p_{V_i}) = \ker(p_{\overline{U}_i})$. Hence by Definition/Remark 2.1 one has: First, $V_{i,\max} = \overline{U}_{i,\max}$. Second, since $V_i = \varphi(\overline{U}_i)$, it follows by mere definitions that $V_{i,\max} = \varphi(\overline{U}_i)_{\max}$. Third, if φ is defined on $\overline{U}_{i,\max}$, one has $\varphi(\overline{U}_{i,\max}) = \varphi(\overline{U}_i)_{\max}$. Hence finally, if φ is defined on $\overline{U}_{i,\max}$, one must have $\varphi(\overline{U}_{i,\max}) = \overline{U}_{i,\max}$, thus proving condition ii) from Definition/Remark 2.2, 2).

Finally, since ϕ is $\mathcal{V}_X$-compatible, it follows by Remark/Definition 2.2, 3), that ϕ lies in the image of $\mathrm{Gal}_{k_0} \to \mathrm{Aut}(K^{\mathrm{i}})$. This completes the proof of assertion 1) of Theorem 2.6.

Language: If $\Phi_\sigma := \rho^{\mathrm{c}}_{\mathcal{V}_X}(\sigma)$, we will simply say that Φ_σ is defined by σ.

B) *Proof of assertion* 2)

For every $X \in \mathcal{V}$, let $K_0 := k_0(X) \hookrightarrow k(\overline{X}) =: K$ be the corresponding embedding of function fields, and consider the subextension $K_X|K$ of $K'|K$ defining the canonical projection $\Pi_K \to \Pi_X = \mathrm{Gal}\big(K_X|K\big)$. Let $\Phi_{\mathcal{V}} := (\Phi_{X'})_{X'} \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}})$ be given, thus $\Phi_{X'} \in \mathrm{Aut}_{\boldsymbol{\mathcal{G}}^{\mathrm{out}}}(\Pi_{X'})$ for every $X' \in \mathcal{V}$.

Lemma 6.1. *For every $X \in \mathcal{V}$ there exists $\sigma_X \in \mathrm{Gal}_{k_0}$ such that $\Phi_X = \rho^{\mathrm{c}}_X(\sigma_X)$. Moreover, if $Y \prec X$, and Π_Y is torsion free, then $\Phi_Y = \rho^{\mathrm{c}}_Y(\sigma_X)$.*

Proof. First, suppose that $X \in \mathcal{V}$ is such that $\dim(X) > 1$, and $\mathcal{V}$ contains some category $\mathcal{V}_X$ which satisfies Hypothesis (H). Then the restriction of $\Phi_{\mathcal{V}}$ to $\Pi_{\mathcal{V}_X}$ is an automorphism

$$\Phi_{\mathcal{V}_X} := (\Phi_U)_{U \in \mathcal{V}_X} \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X}) \hookrightarrow \mathrm{Aut}^{\mathrm{c}}_{\mathcal{V}_X}(\Pi_K).$$

Hence by the now proven assertion 1) of Theorem 2.6, there exists a unique $\sigma \in \mathrm{Gal}_{k_0}$ with

$$\Phi_{\mathcal{V}_X} = \rho^{\mathrm{c}}_{\mathcal{V}_X}(\sigma) = \big(\rho^{\mathrm{c}}_U(\sigma)\big)_{U \in \mathcal{V}_X}.$$

Now let $X \in \mathcal{V}$ be arbitrary. Since $\mathcal{V}$ satisfies Hypothesis (H), by Definition 2.5, there exists $\tilde{X} \in \mathcal{V}$ such that the following hold: First, $\mathcal{V}$ contains a subcategory $\mathcal{V}_{\tilde{X}}$ which satisfies Hypothesis (H), and second, there exists $U \in \mathcal{V}_{\tilde{X}}$ such that $X \prec U$ and $\Pi_U \to \Pi_X$ is surjective. In particular, by the discussion above, there is a unique $\sigma \in \mathrm{Gal}_{k_0}$ such that

$$\Phi_{\mathcal{V}_{\tilde{X}}} = \rho^{\mathrm{c}}_{\mathcal{V}_{\tilde{X}}}(\sigma) = \big(\rho^{\mathrm{c}}_U(\sigma)\big)_{U \in \mathcal{V}_{\tilde{X}}}.$$

Hence since $\Phi_{\mathcal{V}} = (\Phi_{X'})_{X'}$ is compatible with $\mathcal{V}$-morphisms, one gets: Let $U \to X$ be the dominating morphisms defining $X \prec U$, thus giving rise to the surjective projection $p_{UX} : \Pi_U \to \Pi_X$. Then $\Phi_U = \rho^{\mathrm{c}}_U(\sigma)$ together with compatibility with p_{UX} give:

$$\Phi_X \circ p_{UX} = p_{UX} \circ \Phi_U = p_{UX} \circ \big(\rho^{\mathrm{c}}_U(\sigma)\big) = \big(\rho^{\mathrm{c}}_X(\sigma)\big) \circ p_{UX}.$$

Hence $\Phi_X(g) = \big(\rho^{\mathrm{c}}_X(\sigma)\big)(g)$ for all $g \in \mathrm{im}(p_{UX}) = \Pi_X$, thus concluding that $\Phi_X = \rho^{\mathrm{c}}_X(\sigma)$.

Next let $Y \prec X$, thus by definition, there exists a dominant morphism $X \to Y$ which is a $\mathcal{V}$-morphism. Then the canonical projection $p_{XY} : \Pi_X \to \Pi_Y$ defined by $X \to Y$ has open image. Hence reasoning as above, it follows that $\Phi_Y \circ p_{XY}$ and $\big(\rho^{\mathrm{c}}_Y(\sigma)\big) \circ p_{XY}$ coincide on $\mathrm{im}(p_{XY})$, which is an open subgroup of Π_Y. Since Π_Y has no torsion, we conclude that actually $\Phi_Y = \rho^{\mathrm{c}}_Y(\sigma)$. □

We now complete the proof of assertion 2) as follows. Let $X \in \mathcal{V}$ be such that $\dim(X) > 1$ and $\mathcal{V}$ contains a subcategory $\mathcal{V}_X$ satisfying Hypothesis (H), and $\sigma \in \mathrm{Gal}_{k_0}$ be the unique element such that $\Phi_{\mathcal{V}_X} = \rho^{\mathrm{c}}_{\mathcal{V}_X}(\sigma)$. We claim that $\Phi_{X'} = \rho^{\mathrm{c}}_{X'}(\sigma)$ for all $X' \in \mathcal{V}$. *By contradiction,* suppose that there exists $Y \in \mathcal{V}$ such that $\Phi_Y \neq \rho^{\mathrm{c}}_Y(\sigma)$. Since $\mathcal{V}$ satisfies Hypothesis (H), by Definition 2.5, it follows that there exists $\tilde{Y} \in \mathcal{V}$ with $\dim(\tilde{Y}) > 1$ such that $\mathcal{V}$ contains a subcategory $\mathcal{V}_{\tilde{Y}}$ satisfying Hypothesis (H), and there is some $V \in \mathcal{V}_{\tilde{Y}}$ with $\Pi_V \to \Pi_Y$ surjective. Hence if $\tau \in \mathrm{Gal}_{k_0}$ is the unique element with $\Phi_{\mathcal{V}_{\tilde{Y}}} = \rho^{\mathrm{c}}_{\mathcal{V}_{\tilde{Y}}}(\tau)$, then reasoning as in the proof of Lemma 6.1, it follows that $\Phi_V = \rho^{\mathrm{c}}_V(\tau)$ for all $V \in \mathcal{V}_{\tilde{Y}}$, thus $\Phi_Y = \rho^{\mathrm{c}}_Y(\tau)$ as well. Hence $\tau \neq \sigma$, implies that there exists a finite quotient

$$\psi : \mathrm{Gal}_{k_0} \to G \quad \text{such that} \quad \psi(\sigma) \neq \psi(\tau).$$

Since by assertion 1) of the Theorem, the representations $\rho^{c}_{\mathcal{V}_{\tilde{X}}}$ and $\rho^{c}_{\mathcal{V}_{\tilde{Y}}}$ are injective, by mere definitions one has: For all small enough $U \in \mathcal{V}_{\tilde{X}}$, $V \in \mathcal{V}_{\tilde{Y}}$, one has:

$$\ker(\rho^{c}_{U}), \ker(\rho^{c}_{V}) \subset \ker(\psi), \quad \text{thus} \ \ \Phi_U = \rho^{c}_{U}(\sigma) \neq \rho^{c}_{U}(\tau), \ \Phi_V = \rho^{c}_{V}(\tau) \neq \rho^{c}_{V}(\sigma),$$

and further, for U, V small enough, Π_U and Π_V have no torsion.

Since $\mathcal{V}$ is connected, there exist $m > 0$ and $X_i \in \mathcal{V}$, $0 \leqslant i < 2m$, such that $X_0 = U$, $X_{2m} = V$ and $X_{2i}, X_{2i+2} \prec X_{2i+1}$. We will get a contradiction by induction on m.

$\underline{m = 1}$: We have $U = X_0, V = X_2$, thus $U, V \prec X_1$. Then if $\Phi_{X_1} = \rho^{c}_{X_1}(\sigma_1)$, it follows by Lemma 6.1 applied to $U, V \prec X_1$ that $\Phi_U = \rho^{c}_{U}(\sigma_1)$, $\Phi_V = \rho^{c}_{V}(\sigma_1)$; thus finally getting $\rho^{c}_{U}(\sigma) = \Phi_U = \rho^{c}_{U}(\sigma_1)$ and $\rho^{c}_{V}(\tau) = \Phi_V = \rho^{c}_{V}(\sigma_1)$. We thus get a contradiction, because $\rho^{c}_{U}(\sigma) = \rho^{c}_{U}(\sigma_1)$, $\rho^{c}_{V}(\tau) = \rho^{c}_{V}(\sigma_1)$ together with $\ker(\rho^{c}_{U}), \ker(\rho^{c}_{V}) \subset \ker(\psi)$ imply:

$$\psi(\sigma) = \psi(\sigma_1) = \psi(\tau).$$

$\underline{m \Rightarrow (m+1)}$: By the induction hypothesis, it follows that $\Phi_{X_{2m}} = \rho^{c}_{X_{2m}}(\sigma)$. Second, since $X_{2m}, X_{2m+2} \prec X_{2m+1}$, it follows by the case $m = 1$ that $\Phi_{X_{2m+2}} = \rho^{c}_{X_{2m+2}}(\sigma)$. Thus since $X_{2m+2} = Y$, conclude that $\Phi_Y = \rho^{c}_{Y}(\sigma)$, as claimed.

The proof of Theorem 2.6 is complete.

7. Proof of Theorem 2.7

We first notice that assertion 2) follows from assertion 1) in the same way as assertion 2) of Theorem 2.6 was deduced from assertion 1) of Theorem 2.6, that is, in more or less formal way. Therefore, we will not repeat this standard arguments, but rather concentrate on giving a proof of assertion 1) of Theorem 2.7. Moreover, we will prove this assertion –hence Theorem 2.7 as a whole– in a ***more general situation***, see subsection B) below.

A) *Absolute/tame Galois theory of generalized (quasi) prime divisors*

To begin with, let $\tilde{K}|K$ be a Galois extension of K which is ℓ-closed, i.e., satisfying the equivalent conditions: i) $\tilde{K}$ has no cyclic ℓ-extensions; ii) Every $a \in \tilde{K}$ is an ℓ^{th} power in $\tilde{K}$.

We denote by $\tilde{G}_K := \text{Gal}\big(\tilde{K}|K\big)$ the Galois group of $\tilde{K}|K$, and for subextensions $L|K$ of $\tilde{K}|K$, we set $\tilde{L} := \tilde{K}$, and $\tilde{G}_L := \text{Gal}\big(\tilde{L}|L\big)$. For valuations v of K, and their prolongations $\tilde{v}$ to $\tilde{K}$, we set $w := \tilde{v}|_L$, and notice that w prolongs v to L, and $\tilde{w} := \tilde{v}$ is a prolongation of w to $\tilde{L} := \tilde{K}$. Let $T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ be their inertia/decomposition groups in $\tilde{G}_K$. By general decomposition theory we have: The prolongations $\tilde{v}$ of a fixed v, thus their inertia/decomposition groups $T_{\tilde{v}} \subseteq Z_{\tilde{v}}$, are $\tilde{G}_K$-conjugated. Further, $T_{\tilde{w}} = T_{\tilde{v}} \cap \tilde{G}_L$ and $Z_{\tilde{w}} = Z_{\tilde{v}} \cap \tilde{G}_L$. Finally, if $L|K$ is Galois, then denoting by $T_{w|v} \subseteq Z_{w|v}$ the inertia/decomposition groups of w in $\text{Gal}\big(L|K\big)$, it follows that $T_{w|v} \subseteq Z_{w|v}$ are the images of $T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ under $\tilde{G}_K \to \text{Gal}\big(L|K\big)$.

Remark 7.1. Let $L|K$ be a *finite* subextension of $\tilde{K}|K$. Then since $\tilde{K}$ is ℓ-closed, $\tilde{K}$ contains (isomorphic copies of) the maximal pro-ℓ abelian, respectively abelian-by-central, extensions $L'|L \hookrightarrow L''|L \hookrightarrow \tilde{K}|L$ of L; in particular, $K'|K \hookrightarrow K''|K \hookrightarrow \tilde{K}|K$ holds as well. Recalling the canonical projection $\Pi^{c}_{L} \to \Pi_L$, in the above notation, one has:

a) $\Pi^{c}_{L} \to \Pi^{c}_{K}$, $\Pi_L \to \Pi_K$ have open images, and if $L|K$ is Galois, so are $L'|K$, $L''|K$.

b) For $\tilde{v}|v$ and $w := \tilde{v}|_L$, etc., as above, the images of $T_w \subset Z_w \subset \Pi_L$ under $\Pi_L \to \Pi_K$ are open subgroups of $T_v \subset Z_v \subset \Pi_K$, respectively.

c) Moreover, v is (quasi) prime divisor iff w is a (quasi) prime divisor, and if so, $T^1_v \subset Z^1_v$ is the only (quasi) divisorial subgroup of Π_K containing the image of $T^1_w \subset Z^1_w$.

d) **Conclusion.** *The divisorial subgroups $T_w \subset Z_w$ of Π_L are precisely the quasi divisorial subgroups of Π_L which are mapped into divisorial subgroups of Π_K.*

Lemma 7.2. *In the above notations, for every finite Galois subextension $L|K$ of $\tilde{K}|K$, consider the action of $\mathrm{Gal}(L|K)$ on the subsets of Π_L defined by the conjugation. Then for every generalized (quasi) divisorial subgroup $T_{\mathfrak{w}} \subseteq Z_{\mathfrak{w}}$ of Π_L one has: $Z_{\mathfrak{w}|\mathfrak{v}} \subseteq \mathrm{Gal}(L|K)$ is precisely the stabilizer of $T_{\mathfrak{w}} \subseteq Z_{\mathfrak{w}}$ in $\mathrm{Gal}(L|K)$.*

Proof. For $g \in \mathrm{Gal}(L|K)$ arbitrary, consider the prolongation $\mathfrak{w}_g := \mathfrak{w} \circ g$ of $\mathfrak{v}$ to L. Then $\mathfrak{w}_g$ is a generalized (quasi) prime divisor of $L|k$. And since every generalized (quasi) prime divisor of $L|k$ is uniquely determined by its decomposition group in Π_L, we have: $Z_{\mathfrak{w}} = Z_{\mathfrak{w}_g}$ iff $\mathfrak{w} = \mathfrak{w}_g$ iff $g \in Z_{\mathfrak{w}|\mathfrak{v}}$. Now suppose that $g \in Z_{\mathfrak{w}|\mathfrak{v}}$. Then by the functoriality of decomposition theory as briefly explained above, there exists a preimage $g' \in \mathrm{Gal}(L'|K)$ of g which lies in the decomposition group $Z_{\mathfrak{w}'|\mathfrak{v}}$ of some prolongation $\mathfrak{w}'$ of $\mathfrak{w}$ to L'. But then $\mathfrak{w}' \circ g' = \mathfrak{w}'$, hence $g'^{-1} Z_{\mathfrak{w}'|\mathfrak{v}}\, g' = Z_{\mathfrak{w}'|\mathfrak{v}}$, thus also $g'^{-1} Z_{\mathfrak{w}'|\mathfrak{w}}\, g' = Z_{\mathfrak{w}'|\mathfrak{w}}$, because $Z_{\mathfrak{w}'|\mathfrak{w}} \subseteq Z_{\mathfrak{w}'|\mathfrak{v}}$ is a normal subgroup. Thus conclude that g stabilizes $Z_{\mathfrak{w}} := Z_{\mathfrak{w}'|\mathfrak{w}}$. In the same way, if $g \notin Z_{\mathfrak{w}|\mathfrak{v}}$, then $\mathfrak{w}_g \neq \mathfrak{w}$, thus $Z_{\mathfrak{w}_g} \neq Z_{\mathfrak{w}}$. Then reasoning as above, if g' is some preimage of g in $\mathrm{Gal}(L'|K)$ and $\mathfrak{w}'_g := \mathfrak{w}' \circ g'$, then $\mathfrak{w}'_g$ is a prolongation of $\mathfrak{w}_g$ to L', thus $Z_{\mathfrak{w}'|\mathfrak{w}} =: Z_{\mathfrak{w}} \neq Z_{\mathfrak{w}_g} := Z_{\mathfrak{w}'_g|\mathfrak{w}_g}$. On the other hand, $Z_{\mathfrak{w}'_g|\mathfrak{w}} = g'^{-1} Z_{\mathfrak{w}'|\mathfrak{w}}\, g'$, thus we conclude that $Z_{\mathfrak{w}} \neq Z^g_{\mathfrak{w}}$, as claimed. □

Remarks 7.3. As a corollary of the Lemma 7.2 above we have a description of the inertia/decomposition groups of generalized (quasi) prime divisors $\mathfrak{v}$ in $\tilde{G}_K$ as follows: Let $L_i|K$ be the inductive family of all the finite Galois subextensions of $\tilde{K}|K$. Then $\tilde{K} = \cup_i L_i$, and $\tilde{G}_K$ is the projective limit of the projective surjective system of finite groups $G_i := \mathrm{Gal}(L_i|K)$. And if $\tilde{v}|v$ are as above, and $v_i := \tilde{v}|_{L_i}$ for every i, then $T_{\tilde{v}} \subset Z_{\tilde{v}}$ is the projective system of all the $T_{v_i|v} \subseteq Z_{v_i|v}$. Therefore we have:

1) Giving a compatible system $(\mathfrak{v}_i)_i$ of generalized (quasi) prime divisors of $(L_i)_i$ above $\mathfrak{v}$, i.e., such that $\mathfrak{v}_i = \mathfrak{v}_j|_{L_i}$ for all $L_i \subset L_j$, and $\mathfrak{v} = \mathfrak{v}_i|_K$, is equivalent to giving a compatible system $(Z_{\mathfrak{v}_i})_i$ of generalized (quasi) divisorial subgroups in $(\Pi_{L_i})_i$, i.e., such that the canonical projection $p_{L_j L_i} : \Pi_{L_j} \to \Pi_{L_i}$ maps $Z_{\mathfrak{v}_j}$ into $Z_{\mathfrak{v}_i}$ for all $L_i \subset L_j$, and $p_{L_i} : \Pi_{L_i} \to \Pi_K$ maps $Z_{\mathfrak{v}_i}$ into $Z_{\mathfrak{v}}$.

(∗) Giving the compatible system $(\mathfrak{v}_i)_i$ with $\mathfrak{v} = \mathfrak{v}_i|_K$ is equivalent to giving a prolongation of $\tilde{\mathfrak{v}}$ of $\mathfrak{v}$ to $\tilde{K}$ which is defined by $\tilde{\mathfrak{v}}|_{L_i} := \mathfrak{v}_i$.

2) For $\tilde{\mathfrak{v}} \leftrightarrow (\mathfrak{v}_i)_i$ as above, the decomposition groups $Z_{\mathfrak{v}_i|\mathfrak{v}} \subseteq G_i$ are precisely the stabilizers of $Z_{\mathfrak{v}_i}$ in G_i. And $(Z_{\mathfrak{v}_i|\mathfrak{v}})_i$ is a surjective projective subsystem of $(G_i)_i$, which has $Z_{\tilde{\mathfrak{v}}} \subset \tilde{G}_K$ as a projective limit. Thus one can recover $Z_{\tilde{\mathfrak{v}}} \subset \tilde{G}_K$ from the system of group extensions $1 \to \Pi_{L_i} \to \mathrm{Gal}(L'_i|K) \to \mathrm{Gal}(L_i|K) = G_i \to 1$ together/endowed with the decomposition groups $Z_{\mathfrak{v}_i} \subset \Pi_{L_i}$ for all $L_i|K$.

3) We finally notice that the canonical projection $\tilde{p}_{L_i} : \tilde{G}_{L_i} \to \Pi_{L_i}$ maps $T_{\tilde{\mathfrak{v}}_i} \subseteq Z_{\tilde{\mathfrak{v}}_i}$ onto $T_{\mathfrak{v}_i} \subseteq Z_{\mathfrak{v}_i}$ for all L_i. In particular, $\tilde{p}_K : \tilde{G}_K \to \Pi_K$ maps $T_{\tilde{\mathfrak{v}}} \subseteq Z_{\tilde{\mathfrak{v}}}$ onto $T_{\mathfrak{v}} \subseteq Z_{\mathfrak{v}}$.

Next let $\tilde{\Phi} : \tilde{G}_K \to \tilde{G}_K$ be an automorphism of $\tilde{G}_K$. Since $(\tilde{G}_{L_i})_i$ is the system of all the open normal subgroups of $\tilde{G}_K$, it follows that $\tilde{\Phi}$ maps each $\tilde{G}_{L_i}$ isomorphically onto some $\tilde{G}_{M_i}$, where $(M_i)_i$ is some degree and inclusion preserving permutation of the $(L_i)_i$.

4) Since the kernels of the canonical projections $\tilde{p}^{\mathrm{c}}_K : \tilde{G}_K \to \Pi^{\mathrm{c}}_K$ and $\tilde{p}_K : \tilde{G}_K \to \Pi_K$ are characteristic in $\tilde{G}_K$, it follows that $\tilde{\Phi}$ gives rise to isomorphisms $\Phi^{\mathrm{c}}_K : \Pi^{\mathrm{c}}_K \to \Pi^{\mathrm{c}}_K$ and $\Phi_K : \Pi_K \to \Pi_K$ such that Φ_K is the abelianization of Φ^{c}_K.
5) Moreover, if $L|K$ is one of the $L_i|K$, and $M|K$ is the corresponding $M_i|K$, the same is true correspondingly for each of the canonical projections $\tilde{p}^{\mathrm{c}}_L : \tilde{G}_L \to \Pi^{\mathrm{c}}_L$, $\tilde{p}_L : \tilde{G}_L \to \Pi_L$, respectively $\tilde{p}^{\mathrm{c}}_M : \tilde{G}_M \to \Pi^{\mathrm{c}}_M$, $\tilde{p}_M : \tilde{G}_M \to \Pi_M$. Moreover, $\tilde{\Phi} : \tilde{G}_K \to \tilde{G}_K$ gives rise to isomorphisms $\Phi^{\mathrm{c}}_L : \Pi^{\mathrm{c}}_L \to \Pi^{\mathrm{c}}_M$, $\Phi_L : \Pi_L \to \Pi_M$ which satisfy: $\Phi^{\mathrm{c}}_L \circ \tilde{p}^{\mathrm{c}}_L = \tilde{p}^{\mathrm{c}}_M \circ \tilde{\Phi}$, respectively $\Phi_L \circ \tilde{p}_L = \tilde{p}_M \circ \tilde{\Phi}$.
6) If $p^{\mathrm{c}}_L : \Pi^{\mathrm{c}}_L \to \Pi^{\mathrm{c}}_K$ and $p_L : \Pi_L \to \Pi_K$ are the canonical projections, then $\tilde{p}^{\mathrm{c}}_K = p^{\mathrm{c}}_L \circ \tilde{p}^{\mathrm{c}}_L$ and $\tilde{p}_K = p_L \circ \tilde{p}_L$, and correspondingly for $M|K$. Finally Φ^{c}_L and Φ_L are compatible with $\tilde{p}^{\mathrm{c}}_L$ and $\tilde{p}^{\mathrm{c}}_M$, respectively, $\tilde{p}_L$ and $\tilde{p}_M$, i.e., one has commutative diagrams:

$$(*)\qquad \begin{array}{ccc} \Pi^{\mathrm{c}}_L & \xrightarrow{\Phi^{\mathrm{c}}_L} & \Pi^{\mathrm{c}}_M \\ \downarrow p^{\mathrm{c}}_L & & \downarrow p^{\mathrm{c}}_M \\ \Pi^{\mathrm{c}}_K & \xrightarrow{\Phi^{\mathrm{c}}_K} & \Pi^{\mathrm{c}}_K \end{array} \qquad\qquad \begin{array}{ccc} \Pi_L & \xrightarrow{\Phi_L} & \Pi_M \\ \downarrow p_L & & \downarrow p_M \\ \Pi_K & \xrightarrow{\Phi_K} & \Pi_K \end{array}$$

7) Since Φ_{L_i} is the abelianization of $\Phi^{\mathrm{c}}_{L_i}$, by the characterization of the quasi r-divisorial subgroups, see [P4], especially Proposition 3.5, one has: Let $T_{\mathfrak{v}_i} \subseteq Z_{\mathfrak{v}_i}$ be a quasi r-divisorial subgroup for $L_i|k$. Then $\Phi_{L_i}(T_{\mathfrak{v}_i}) \subseteq \Phi_{L_i}(Z_{\mathfrak{v}_i})$ is a quasi r-divisorial subgroup of Π_{M_i}, say equal to $T_{\mathfrak{w}_i} \subseteq Z_{\mathfrak{w}_i}$ for some quasi prime r-divisor $\mathfrak{w}_i$ of $M_i|k$. Further, setting $\mathfrak{v} := \mathfrak{v}_i|_K$, $\mathfrak{w} := \mathfrak{w}_i|_K$, one has that $p_{L_i}(T_{\mathfrak{v}_i}) \subseteq p_{L_i}(Z_{\mathfrak{v}_i})$ are open subgroups in $T_{\mathfrak{v}} \subseteq Z_{\mathfrak{v}}$, respectively, and correspondingly for $\mathfrak{w}_i|\mathfrak{w}$. Finally, $\Phi_K : \Pi_K \to \Pi_K$ maps $T_{\mathfrak{v}} \subseteq Z_{\mathfrak{v}}$ isomorphically onto $T_{\mathfrak{w}} \subseteq Z_{\mathfrak{w}}$.
8) Performing the above steps for every finite Galois subextension $L_i|K$ of $\tilde{K}|K$ and the corresponding $M_i|K$, one has: The isomorphism $\Phi_{L_i|K} : \mathrm{Gal}\big(L_i|K\big) \to \mathrm{Gal}\big(M_i|K\big)$ induced by $\tilde{\Phi}$ maps the stabilizer of $Z_{\mathfrak{v}_i}$ in $\mathrm{Gal}\big(L_i|K\big)$ isomorphically onto the stabilizer of $Z_{\mathfrak{w}_i}$ in $\mathrm{Gal}\big(M_i|K\big)$. Thus taking limits we get: If the system $(\mathfrak{v}_i|\mathfrak{v})_i$ is compatible, say defining $\tilde{\mathfrak{v}}|\mathfrak{v}$ on $\tilde{K}|K$, then the system $(\mathfrak{w}_i|\mathfrak{w})_i$ is compatible as well, and defines $\tilde{\mathfrak{w}}|\mathfrak{w}$ on $\tilde{K}|K$, and $\tilde{\Phi}$ maps $T_{\tilde{\mathfrak{v}}} \subseteq Z_{\tilde{\mathfrak{v}}}$ isomorphically onto $T_{\tilde{\mathfrak{w}}} \subseteq Z_{\tilde{\mathfrak{w}}}$.

B) *Proof of assertion* 1) *of Theorem* 2.7

We will prove actually a ***stronger result***, in which we replace the maximal tame subextension $K^t|K$ of $\overline{K}|K$, as used in Theorem 2.7, by ***any subextension*** $\tilde{K}|K$ *of* $K^t|K$ *such that* $\tilde{K}$ *is* ℓ*-closed.* Equivalently, $\tilde{K}|K$ is a Galois extension such that $\tilde{K}$ is ℓ-closed, and for all prime divisors v of $K|k$ satisfies the following equivalent conditions:

i) $\tilde{K}|K$ is tamely ramified above v.
ii) The inertia group $T_{\tilde{v}}$ of $\tilde{v}|v$ is a procyclic group of order prime to $\mathrm{char}(k_0)$.

The proof uses in an essential way Theorem 2.6. Reasoning as in the proof of assertion 1) of Theorem 2.6, we instantly see that assertion 1) of Theorem 2.7 is equivalent to the fact that for every $\tilde{\Phi} \in \mathrm{Aut}_{\varTheta}(\tilde{G}_K)$ there exists an automorphism $\tilde{\phi}$ of $\tilde{K}$ which maps $k_0(X)^{\mathrm{i}}$ onto itself and satisfies $\tilde{\Phi}(g) = \tilde{\phi}^{-1} g\, \tilde{\phi}$ for all $g \in \tilde{G}_K$. Note that if $\tilde{\phi}$ exists, then $\tilde{\phi}$ is unique up to Frobenius twists, because k_0 is perfect, thus $k_0(X) \hookrightarrow k(\overline{X}) = K$ is a Galois extension.

- Let $\tilde{\Phi} \in \mathrm{Aut}_\Theta(\tilde{G}_K)$ be given.

First recall that since the kernels of the homomorphisms $\tilde{G}_K \to \Pi^{\mathrm{c}}_K \to \Pi_K$ are characteristic, one has canonical homomorphisms $\mathrm{Aut}(\tilde{G}_K) \to \mathrm{Aut}(\Pi^{\mathrm{c}}_K) \to \mathrm{Aut}(\Pi_K)$, and furthermore, the image of $\mathrm{Aut}(\tilde{G}_K) \to \mathrm{Aut}(\Pi_K)$ equals the image of $\mathrm{Out}(\tilde{G}_K) \to \mathrm{Aut}(\Pi_K)$, and these images are contained in $\mathrm{Aut}^{\mathrm{c}}(\Pi_K)$. And obviously, directly from the definition, it follows that since $\tilde{\Phi} \in \mathrm{Aut}_\Theta(\tilde{G}_K)$, its image $\Phi_K \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ lies actually in $\mathrm{Aut}^{\mathrm{c}}_\Theta(\Pi_K)$. Hence by Theorem 2.6, 1), it follows that there exists (a unique) $\sigma \in \mathrm{Gal}_{k_0}$ which defines Φ_K, i.e., $\Phi_K \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ is the image of σ under the canonical representation $\mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$. Equivalently, recalling that $K_0 := k_0(X)$, one has by mere definitions: There exist $\varepsilon \in \mathbb{Z}_\ell^\times$ and a prolongation ϕ'_σ of σ to K' such that

$$\varepsilon \cdot \Phi_K(g) = \phi'_\sigma \circ g \circ {\phi'_\sigma}^{-1} \quad \text{for all} \;\; g \in \Pi_K.$$

Thus setting $\Phi_\sigma := \varepsilon \cdot \Phi_K$, and letting $\tilde{\Phi}_\sigma$ be any prolongation of Φ_σ to $\tilde{K}$, it follows that $\tilde{\Phi}_\sigma \in \mathrm{Aut}_\Theta(\tilde{G}_K)$, and $\tilde{\Phi}_\sigma$ equals $\rho^{\mathrm{t}}_{\mathcal{V}_X}(\sigma)$ up to inner $\tilde{G}_K$-conjugation. Hence $\tilde{\Phi}^{-1} \circ \tilde{\Phi}_\sigma$ satisfies:

a) $\tilde{\Phi}^{-1} \circ \tilde{\Phi}_\sigma \in \mathrm{Aut}_\Theta(\tilde{G}_K)$, because $\tilde{\Phi}, \tilde{\Phi}_\sigma \in \mathrm{Aut}_\Theta(\tilde{G}_K)$.

b) The image $\Phi_K^{-1} \circ \Phi_\sigma$ of $\tilde{\Phi}^{-1} \circ \tilde{\Phi}_\sigma$ in $\mathrm{Aut}(\Pi_K)$ is the multiplication by $\varepsilon \in \mathbb{Z}_\ell^\times$ on Π_K.

Hence after replacing $\tilde{\Phi}$ by $\tilde{\Phi}^{-1} \circ \tilde{\Phi}_\sigma$, the assertion 1) of Theorem 2.7 follows from:

Key Lemma 7.4. *Let $\tilde{\Phi} \in \mathrm{Aut}_\Theta(\tilde{G}_K)$ be such that its image $\Phi_K \in \mathrm{Aut}(\Pi_K)$ is $\Phi_K = \varepsilon \cdot \mathrm{id}$ for some $\varepsilon \in \mathbb{Z}_\ell^\times$. Then $\tilde{\Phi}$ is the conjugation by some $\tilde{\phi} \in \tilde{G}_K$ on $\tilde{G}_K$, hence $\varepsilon = 1$.*

In order to prove the Key Lemma above, we first notice that by Remarks 7.3, 3), above, in the notations from there, it follows that for every (quasi) prime divisor $\mathfrak{v}$ of $K|k$ and some prolongation $\tilde{\mathfrak{v}}$ to $\tilde{K}$ one has: $p_K(T_{\tilde{\mathfrak{v}}}) = T_{\mathfrak{v}}$ and $p_K(Z_{\tilde{\mathfrak{v}}}) = Z_{\mathfrak{v}}$. Thus letting $\mathfrak{w}$ and $\tilde{\mathfrak{w}}$ be the unique (quasi) prime divisor $K|k$, respectively its prolongation to $\tilde{K}$, with $\tilde{\Phi}(T_{\tilde{\mathfrak{v}}}) = T_{\tilde{\mathfrak{w}}}$ and $\tilde{\Phi}(Z_{\tilde{\mathfrak{v}}}) = Z_{\tilde{\mathfrak{w}}}$ one has: $p_K(Z_{\tilde{\mathfrak{w}}}) = \Phi_K(Z_{\mathfrak{v}}) = \varepsilon \cdot Z_{\mathfrak{v}} = Z_{\mathfrak{v}}$ and $p_K(T_{\tilde{\mathfrak{w}}}) = \Phi_K(T_{\mathfrak{v}}) = \varepsilon \cdot T_{\mathfrak{v}} = T_{\mathfrak{v}}$. We therefore conclude that $\mathfrak{v} = \mathfrak{w}$, thus $\tilde{\mathfrak{w}}$ is itself a prolongation of $\mathfrak{w} = \mathfrak{v}$ to $\tilde{K}$. In other words, for every (quasi) prime divisor $\mathfrak{v}$ of $K|k$, the automorphism $\tilde{\Phi}$ maps the conjugacy class of inertia/decomposition groups $\{\sigma T_{\tilde{\mathfrak{v}}}\, \sigma^{-1} \subset \sigma Z_{\tilde{\mathfrak{v}}}\, \sigma^{-1} \mid \sigma \in \tilde{G}_K\}$ onto itself.

We next recall some facts about valuation-tame fundamental groups. Let $X \to k$ be a proper normal model of $K|k$. For every Zariski open subset $U \subset X$, let $\mathcal{D}_U$ be the set of prime divisors of $K|k$ which have a non-trivial center on U, and $T^t_{\mathcal{D}_U} \subseteq G^t_K$ be the closed subgroup of G^t_K generated by $T_{\tilde{v}}$ for all $\tilde{v}|v$, $v \in \mathcal{D}_U$. We set $\pi^t_{\mathcal{D}_U} := G^t_K / T^t_{\mathcal{D}_U}$ and call $\pi^t_{\mathcal{D}_U}$ the valuation-tame fundamental group of U. Obviously, if $U'' \hookrightarrow U'$, then there is a canonical surjective projection $\pi^t_{\mathcal{D}_{U''}} \to \pi^t_{\mathcal{D}_{U'}}$, and if $(U_i)_{i \in I}$ is a basis of Zariski open neighborhoods of the generic point η_X, then $\pi^t_{\mathcal{D}_{U_i}}$, $i \in I$, is a projective surjective system of profinite groups having G^t_K as projective limit. See e.g., [K–S] for other forms of *tame fundamental groups.*

Claim 1. *$\pi^t_{\mathcal{D}_U}$ is topologically finitely generated.*

Indeed, by the alteration theory, there exists a generically finite cover $Z \to \overline{X}$ such that letting $V \subset Z$ be the preimage of $\overline{U}$ under $Z \to \overline{X}$, one has:

a) Z is a projective smooth k-variety.

b) $Z\backslash V$ is a normal crossings divisor in Z.

Let $K = k(\overline{X}) \hookrightarrow k(Z) =: N$ be the generic fiber of $Z \to \overline{X}$, and $\mathcal{D}_{N|k} \to \mathcal{D}_{K|k}$, $w \mapsto v$, be the (surjective) restriction map for the prime divisors. Then the canonical projection $\mathrm{Gal}^{\mathrm{t}}_N \to \mathrm{Gal}^{\mathrm{t}}_K$ has an open image, and for $w \mapsto v$ one has: w has a center on V if and only if v has a center on U. Therefore, $\mathrm{Gal}^{\mathrm{t}}_N \to \mathrm{Gal}^{\mathrm{t}}_K$ maps $T^t_{\mathcal{D}_V}$ into $T^t_{\mathcal{D}_U}$, thus gives rise to a projection $\pi^t_{\mathcal{D}_V} \to \pi^t_{\mathcal{D}_U}$ which has open image. On the other hand, since Z is smooth and $Z\backslash V$ is a normal crossings divisor, one has that $\pi^t_1(V) = \pi^t_{\mathcal{D}_V}$, and $\pi^t_1(V)$ is topologically finitely generated. (More precisely, by Grothendieck's theory of tame fundamental groups, one has that $\ker\big(\pi^t_1(V) \to \pi^t_1(Z)\big)$ is generated by properly chosen tame inertia elements above the irreducible components of $Z\backslash V$.) Since $\pi^t_{\mathcal{D}_V} \to \pi^t_{\mathcal{D}_U}$ has open image, and the former group is finitely generated, we conclude that $\pi^t_{\mathcal{D}_U}$ is finitely generated as well, as claimed.

Let $\tilde{T}_{\mathcal{D}_U} \subset \tilde{G}_K$ be the image of $T^t_{\mathcal{D}_U}$ under the surjective canonical projection $\mathrm{Gal}^{\mathrm{t}}_K \to \tilde{G}_K$. Then $\tilde{T}_{\mathcal{D}_U}$ is generated by all the inertia groups $T_{\tilde{v}}$ with $\tilde{v}|v$, $v \in \mathcal{D}_U$, and $\tilde{\pi}_{\mathcal{D}_U} := \tilde{G}_K/\tilde{T}_{\mathcal{D}_U}$ is topologically finitely generated. Further, $\tilde{G}_K$ is the projective limit of the surjective projective system $\tilde{\pi}_{\mathcal{D}_{U_i}}$, $i \in I$.

Claim 2. *$\tilde{T}_{\mathcal{D}_U} \subset \tilde{G}_K$ is $\tilde{\Phi}$ invariant, i.e., $\tilde{\Phi}(\tilde{T}_{\mathcal{D}_U}) = \tilde{T}_{\mathcal{D}_U}$.*

Indeed, by the Lemma 7.2 above, the equlity $\tilde{\Phi}(T_{\tilde{v}}) = T_{\tilde{w}}$ implies that $\tilde{v}|_K = \tilde{w}|_K$. Hence $\tilde{\Phi}$ defines a permutation of the system of generators $T_{\tilde{v}}$, $\tilde{v}|v$, $v \in \mathcal{D}_U$, of $\tilde{T}_{\mathcal{D}_U}$, etc.

An immediate consequence of Claim 1 and Claim 2, above is the following: First, $\tilde{\Phi}$ gives rise to an isomorphism $\tilde{\Phi}_U : \tilde{\pi}_{\mathcal{D}_U} \to \tilde{\pi}_{\mathcal{D}_U}$ which is compatible with projections $\pi^t_{\mathcal{D}_{U''}} \to \pi^t_{\mathcal{D}_{U'}}$ for $U'' \hookrightarrow U'$, and $\tilde{\Phi}$ is the projective limit of the system of isomorphisms $(\tilde{\Phi}_{U_i})_i$. Second, since $\tilde{\pi}_{\mathcal{D}_U}$ is topologically finitely generated, for every positive bound $c > 0$, there exist only finitely many open normal subgroups $\Delta \subset \tilde{\pi}_{\mathcal{D}_U}$ with $|\tilde{\pi}_{\mathcal{D}_U}/\Delta| \leqslant c$. Hence the intersection $\Delta_c := \cap_\Delta \Delta$ of all such Δ is an open characteristic subgroup of $\tilde{\pi}_{\mathcal{D}_U}$. In particular, if $K_{U,c}|K$ is the finite Galois subextension of $\tilde{K}|K$ with $\mathrm{Gal}\big(K_{U,c}|K\big) = \tilde{\pi}_{\mathcal{D}_U}/\Delta_c$, then $\mathrm{Gal}\big(\tilde{K}|K_{U,c}\big)$ is invariant under $\tilde{\Phi}$.

Using this we get: Let $L|K$ be a *finite* Galois subextension of $\tilde{K}|K$, and set $c := [L : K]$. Since $\tilde{G}_K$ is the projective limit of the projective surjective system $\tilde{\pi}_{\mathcal{D}_{U_i}}$, $i \in I$, there exists some U_i such that the canonical projection $\tilde{G}_K \to \mathrm{Gal}\big(L|K\big)$ factors through $\tilde{G}_K \to \tilde{\pi}_{\mathcal{D}_{U_i}}$. Thus in the above notations, $\mathrm{Gal}\big(\tilde{K}|K_{U_i,c}\big)$ is invariant under $\tilde{\Phi}$.

We conclude that there exists an inductive system $(G_\mu)_\mu$ of open normal subgroups of $\tilde{G}_K$ having $\cap_\mu G_\mu = 1$ such that $\tilde{\Phi}(G_\mu) = G_\mu$ for all μ. For every G_μ, let $K_\mu|K$ be the finite Galois subextension of $\tilde{K}|K$ with $G_{K_\mu} = G_\mu$, hence $\tilde{K} = \cup_\mu K_\mu$. Further, $\tilde{\Phi}$ gives rise to a compatible system of automorphisms $\Phi_\mu : \mathrm{Gal}\big(K_\mu|K\big) \to \mathrm{Gal}\big(K_\mu|K\big)$, and $\tilde{\Phi}$ is the projective limit of the system $(\Phi_\mu)_\mu$.

To simplify notations, let $L := K_\mu$ be fixed, $\overline{\Phi} := \Phi_\mu$, and $G := \mathrm{Gal}\big(L|K\big)$. Using the usual notation, let $L'|L \hookrightarrow L''|L$ be the maximal abelian, respectively abelian-by-central pro-ℓ extensions of L. Then $L'|K \hookrightarrow L''|K$ are Galois extensions, and $L|K$ being invariant under $\tilde{\Phi}$, implies that $L'|K \hookrightarrow L''|K$ are Galois extensions which are invariant under $\tilde{\Phi}$ as well. Hence $\tilde{\Phi}$ gives rise by restriction to an automorphism $\Phi_L \in \mathrm{Aut}^{\mathrm{c}}(L)$, which fits into the canonical diagram below, having exact rows and isomorphism as columns:

$$\begin{array}{ccccccccc} 1 \to & \Pi_L & \to & \mathrm{Gal}\big(L'|K\big) & \to & G & \to 1 \\ & \downarrow \Phi_L & & \downarrow \Phi_{LK} & & \downarrow \overline{\Phi} & \\ 1 \to & \Pi_L & \to & \mathrm{Gal}\big(L'|K\big) & \to & G & \to 1 \end{array}$$

Lemma 7.5. *Φ_L maps the set of generalized divisorial groups of Π_L isomorphically onto itself, thus defines to an isomorphisms $\Phi_L : \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$.*

Proof. Indeed, by Remark 7.1, c), d), it follows that $\Phi_L : \Pi_L \to \Pi_L$ maps the set $T_w \subset Z_w$ of divisorial groups in Π_L isomophically onto itself. Therefore Φ_L maps the set of divisorial inertia elements $\cup_w T_w$ homomorphically onto itself. Thus recalling that the set of all the k-inertia $\mathfrak{Inr}_k(L) \subset \Pi_L$ is the topological closure of $\cup_w T_w$, it follows that Φ_L maps $\mathfrak{Inr}_k(L)$ homomorphically onto itself. Thus by Proposition 3.11, it follows that the total decomposition graph $\mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$ can be reconstructed from $\Pi_L^{\mathrm{c}} \to \Pi_L$ and Π_L endowed with $\mathfrak{Inr}_k(L)$, and further, $\Phi_L : \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$ is an isomorphism. □

We next show that Φ_L is compatible with the rational quotients of $\mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$. Let $Y \to k$ be some projective normal model of $L|k$ on which $G = \mathrm{Gal}\big(L|K\big)$ acts. Without loss of generality, we can assume that Y is complete regular like (in the sense of the discussion in Remark 3.2, 4), and that the quotient $X := G\backslash Y$ of Y by G is a complete regular like model for $K|k$. Further, let $\tilde{u} \in L$ be such that its G-conjugates $(\tilde{u}_g)_{g\in G}$ are K-linearly independent. Let $t \in K$ be non-constant such that the pole divisor $(\tilde{u})_\infty$ of $\tilde{u}$ is contained in the pole divisor $(t)_\infty$ of t, and t is not in the k-subspace generated by $(\tilde{u}_g)_g$. Then for almost all $c \in k$, all the G-conjugates $u_g = g(u)$ of $u := \tilde{u}/t + c \in L$ satisfy:

a) u_g are general elements of L, i.e., $\kappa_{u_g} := k(u_g)$ are relatively algebraically closed in L.
b) $(u_g)_g$ are K-linearly independent.
c) The pole divisor of u_g is $(u_g)_\infty = (t)_0$ thus it lies in the image of $\mathrm{Div}(X) \to \mathrm{Div}(Y)$.

Next recall that denoting by $\mathbb{P}^1_{u_g}$ the projective u_g-line over k, the embedding $k(u_g) \hookrightarrow L$ is defined by a k-rational map $\varphi_g : Y \dashrightarrow \mathbb{P}^1_{u_g}$, and let $U_g \subseteq Y$ be the domain of φ_g. Notice that if $g' = hg$ in G, then $u_{g'} = h(u_g)$, and $\varphi_{g'} = \varphi_g^h$. Thus in particular, $U_{g'} = U_g^h$. Therefore, setting $V := \cap_g U_g$, it follows that G acts on $V \subset Y$, and all the rational maps φ_g are defined on V. Finally, since $Y\backslash V$ is G-invariant, after performing a properly chosen sequence of G-invariant blowups and normalizing again, one can suppose that $V = Y$, i.e., the rational maps φ_g are actually morphisms $\varphi_g : Y \to \mathbb{P}^1_{u_g}$. Now since the geometric generic fiber of each φ_g is integral –which is equivalent to the fact that $k(u_g)$ is relatively algebraically closed in L, it follows that the fibers of the k-morphisms $\varphi_g : Y \to \mathbb{P}^1_{u_g}$ are geometrically integral on an open subset $U \subset \mathbb{P}^1_{u_g}$. This means that for all $a \in U(k)$ and all $g \in G$, the fibers $X_{g,a} \subset Y$ of φ_g at $u_g = a$ are geometrically integral Weil prime divisors of Y. In other words, the Y-divisor of the function $u_g - a$ is of the form $(u_g - a) = v_{g,a} - (t)_0$ with $(t)_0$ the zero-divisor of t on Y, thus the image of the zero-divisor of t on X under $\mathrm{Div}(X) \to \mathrm{Div}(Y)$, and $v_{g,a} = (u_g - a)_0$ the zero diver of $u_g - a$. Notice that since $h(u_g - a) = h(u_g) - a = u_{hg} - a$, one has that $v_{g,a}^h = v_{hg,a}$. In other words, the free action of G on $\{u_g\}_g$ gives rise to a free action of G on $\{v_{g,a}\}_g$ via $v_{g,a}^h = v_{hg,a}$.

Recall the ℓ-adic completion morphisms $\jmath_K : K^\times \to \widehat{K}$ and $\jmath_L : L^\times \to \widehat{L}$, and that the canonical divisorial $\widehat{U}_L$-lattice $\mathcal{L}_L \subset \widehat{L}$ is the unique divisorial $\widehat{U}_L$-lattice in $\widehat{L}$ which contains $\jmath_L(L^\times)$. Further, via $p_L : \Pi_L \to \Pi_K$ one gets a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \to & \widehat{U}_K & \to & \widehat{K} & \xrightarrow{\text{div}} & \widehat{\text{Div}}(X)_{(\ell)} & \to & \widehat{\mathfrak{Cl}}(X) \\ & & \downarrow \imath & & \downarrow \imath & & \downarrow \text{div}_{p_L} & & \downarrow \text{can}_{p_L} \\ 1 & \to & \widehat{U}_L & \to & \widehat{L} & \xrightarrow{\text{div}} & \widehat{\text{Div}}(Y)_{(\ell)} & \to & \widehat{\mathfrak{Cl}}(Y) \end{array}$$

and Φ_L give rise to a commutative diagram with exact rows of the form:

$$\begin{array}{ccccccccc} 1 & \to & \widehat{U}_L & \to & \widehat{L} & \xrightarrow{\text{div}} & \widehat{\text{Div}}(Y) & \to & \widehat{\mathfrak{Cl}}(Y) \\ & & \downarrow \hat{\psi} & & \downarrow \hat{\psi} & & \downarrow \text{div}_{\Phi_L} & & \downarrow \text{can}_{\Phi_L} \\ 1 & \to & \widehat{U}_L & \to & \widehat{L} & \xrightarrow{\text{div}} & \widehat{\text{Div}}(Y) & \to & \widehat{\mathfrak{Cl}}(Y) \end{array}$$

where $\hat{\psi}$ is the Kummer homomorphism of Φ_L (that is, the ℓ-adic dual of Φ_L), div_{Φ_L} is the canonical abstract divisor map defined by Φ_L, and can_{Φ_L} is the canonical isomorphism making the diagram commutative. We proceed as follows:

Since $\text{div}\big(\jmath_L(u_g - a)\big) = v_{g,a} - (t)_0$, and $(t)_0$ is G-invariant, and taking into account the identity $\text{div} \circ \hat{\psi} = \text{div}_{\Phi_L} \circ \text{div}$, it follows that the divisor of $\hat{\psi}\big(\jmath_L(u_g - a)\big)$ is of the form

$$\begin{aligned} \text{div}\big(\hat{\psi} \circ \jmath_L(u_g - a)\big) &= \text{div}_{\Phi_L}\big(\text{div}(\jmath_K(u_g - 1))\big) \\ &= \text{div}_{\Phi_L}\big(v_{g,a} - (t)_0\big) = \text{div}_{\Phi_L}(v_{g,a}) - \text{div}_{\Phi_L}\big((t)_0\big). \end{aligned}$$

On the other hand, one has that $\text{div}_{\Phi_L}(v_{g,a}) = \eta \cdot w$ for some $\eta \in \mathbb{Z}_\ell$, where w is the image of $v_{g,a}$ under Φ_L, and $\text{div}_{\Phi_L}\big((t)_0\big) = \varepsilon \cdot (t)_0$, because $t \in K$, etc. Since by the discussion above, w is G-conjugate to $v_{g,a}$, one has $w = v_{h,a}$ for some $h \in G$. We thus conclude that $\text{div}\big(\hat{\psi} \circ \jmath_L(u_g - a)\big) = \eta \cdot v_{h,a} - \varepsilon \cdot (t)_0$ for some $h \in G$ and $\eta \in \mathbb{Z}_\ell$. Now since $v_{h,a} - (t)_0 = (u_h - a)$ is a principal divisor, its image in $\widehat{\mathfrak{Cl}}(X)$ is trivial, hence the image of

$$\eta \cdot v_{h,a} - \varepsilon \cdot (t)_0 = (\eta - \varepsilon) \cdot v_{h,a} + \varepsilon \cdot \big(v_{h,a} - (t)_0\big) = (\eta - \varepsilon) \cdot v_{h,a} + \varepsilon \cdot (u_h - a)$$

in $\widehat{\mathfrak{Cl}}(Y)$ equals $(\eta - \varepsilon) \cdot [v_{h,a}]$, where $[v_{h,a}]$ is the image of $v_{g,a}$ in $\widehat{\mathfrak{Cl}}(Y)$. On the other hand, $v_{g,a} - (t)_0 = (u_g - a)$ is principal and equals the divisor of $\jmath_L(u_g - a)$, hence it has a trivial image in $\widehat{\mathfrak{Cl}}(Y)$. Thus by the commutativity of the diagram above, it follows that $(\eta - \varepsilon) \cdot [v_{h,a}] = 0$. Since $[v_{h,a}] \neq 0$, we conclude that $\eta = \varepsilon$. Therefore, we get:

$$\hat{\psi}\big(\jmath_L(u_g - a)\big) = \varepsilon \cdot \big(\boldsymbol{u}\, \jmath_L(u_h - a)\big)$$

for some $\boldsymbol{u} \in \widehat{U}_L$. Notice that for a fixed $g \in G$, the elements $h \in G$ as well as $\boldsymbol{u}$ could *anteriori* depend on $a \in U(k) \subset \mathbb{P}^1_{u_g}(k)$. The more precise assertion we prove is:

Lemma 7.6. *There is a bijection $\theta_G : G \to G$, $g \mapsto h$, such that for all $a \in U(k)$, one has:*

a) $\hat{\psi}\big(\jmath_L(u_g - a)\big) = \varepsilon \cdot \jmath_L(u_h - a)$

b) $\hat{\psi}\big(\jmath_L(\mathcal{K}^\times_{u_g})\big) = \varepsilon \cdot \jmath_L(\mathcal{K}^\times_{u_h})$

Proof. Since $U(k)$ is infinite, there exists $h \in \text{Gal}\big(L|K\big)$ such that the set

$$\Sigma_h := \{a \in U(k) \mid \hat{\psi}\big(\jmath_L(u_g - a)\big) = \varepsilon \cdot \big(\boldsymbol{u}\, \jmath_L(u_h - a)\big) \text{ for some } \boldsymbol{u} \in \widehat{U}_L\}$$

is infinite. For a fixed $b \in k$, let $\mathsf{x} := \jmath_L(u_g - b)$ and $\boldsymbol{x} := \hat\psi(\mathsf{x})$. To simplify notations, for $a \in U(k)$ and $g, h \in G$, let $\kappa_{g,a}$ be the residue field of $v_{g,a}$, and $\jmath_{g,a} : \widehat{U}_{v_{g,a}} \to \widehat{\kappa}_{g,a}$ be the reduction homomorphism as introduced at the beginning of section 3. Further define $\kappa_{h,a}$ and $\jmath_{h,a} : \widehat{U}_{v_{h,a}} \to \widehat{\kappa}_{h,a}$ correspondingly. Then for all $a \in U(k)$ with $a \neq b$, one has: x is a $v_{g,a}$-unit with $v_{g,a}$-residue equal to $a - b \in k^\times$, thus $\jmath_{g,a}(\mathsf{x}) = 1$. Hence if $a \in \Sigma$, and $v_{h,a}$ is the prime divisor of $L|k$ corresponding to $v_{g,a}$ under Φ_L, one gets commutative diagrams:

$$\begin{array}{ccc} \widehat{U}_{v_{g,a}} & \xrightarrow{\hat\psi} & \widehat{U}_{v_{h,a}} \\ \downarrow \jmath_{g,a} & & \downarrow \jmath_{h,a} \\ \widehat{\kappa}_{g,a} & \xrightarrow{\hat\phi_{g,a}} & \widehat{\kappa}_{h,a} \end{array}$$

where $\hat\phi_{g,a}$ is defined by the residual isomorphism $\Phi_{g,a} : \Pi_{K v_{g,a}} \to \Pi_{K v_{h,a}}$. Hence $\jmath_{g,a}(\mathsf{x}) = 1$ implies $\jmath_{h,a}\big(\hat\psi(\mathsf{x})\big) = \hat\phi_{g,a}\big(\jmath_{g,a}(\mathsf{x})\big) = 1$ for all $a \in \Sigma$. Next recall that by the discussion at the beginning of Section 3, especially the proof of Proposition 3.1, for every $\boldsymbol{y} \in \widehat{L}_{\text{fin}} \backslash \widehat{\kappa}_{u_h}$ and almost all $a \in k$, one has: If $v_{h,a}$ is the (unique) zero of $u_h - a$, then $\jmath_{h,a}(\boldsymbol{y}) \neq 1$. In particular, since $\mathsf{x} = \jmath_L(u_g - a) \in \mathcal{L}_L \subset \widehat{L}_{\text{fin}}$ and $\boldsymbol{x} := \hat\psi(\mathsf{x}) \in \widehat{L}_{\text{fin}}$ satisfy $\jmath_{h,a}(\boldsymbol{x}) = \jmath_{h,a}\big(\hat\psi(\mathsf{x})\big) = 1$ for all $a \in \Sigma$, it follows that $\boldsymbol{x} \in \varepsilon \cdot \mathcal{L}_L \cap \widehat{\kappa}_{u_h} = \varepsilon \cdot \jmath_L(\kappa^\times_{u_h})_{(\ell)}$. Since the set of all the $\mathsf{x} = \jmath_L(u_g - b)$ with $b \in k$ generate $\jmath_L(\kappa^\times_{u_g})$, we conclude that $\hat\psi\big(\jmath_L(\kappa^\times_{u_g})\big) \subseteq \varepsilon \cdot \jmath_L(\kappa^\times_{u_h})$. By symmetry, the opposite inclusion holds too, thus finally $\hat\psi\big(\jmath_L(\kappa^\times_{u_g})\big) = \varepsilon \cdot \jmath_L(\kappa^\times_{u_h})$. Moreover, for all $a \in U(k)$ one has $\hat\psi\big(\jmath_L(u_g - a)\big) = \varepsilon \cdot \jmath_L(u_h - a)$. □

Coming back to the proof of Key Lemma 7.4, we notice that setting

$$\Theta_L := \Theta \cup \{u_g \mid g \in G\}$$

one has: First, by the hypothesis of Theorem 2.7, one has $K = k(\Theta)$; and by the definition of $(u_g)_g$ one has $L = K[(u_g)_g]$; hence $L = k(\Theta_L)$. Second using Lemma 7.6, it follows that the map $\theta_L : \Theta_L \to \Theta_L$, defined by $t \mapsto t$ for $t \in \Theta$, and $u_g \mapsto u_h$ for $g \in G$, is a bijection which makes $\Phi_L \in \mathrm{Aut}^{\mathrm{c}}(\Pi_L)$ into a weakly Θ_L compatible automorphism. Thus $\Phi_L \in \mathrm{Aut}^{\mathrm{c}}(\Pi_L)$ is weakly Θ_L-compatible as defined in Definition/Remark 2.8, hence Theorem 2.9 is applicable. Therefore, there exists an automorphism $\phi_L \in \mathrm{Aut}_{\Theta_L}(L^{\mathrm{i}})$ and a unique $\varepsilon_L \in \mathbb{Z}_\ell^\times$ such that $\hat\psi = \varepsilon_L \cdot \hat\phi_L$ is the Kummer morphism of $\hat\psi$. Thus recalling that $\Phi_K : \Pi_K \to \Pi_K$ is $\Phi_K = \varepsilon^{-1} \cdot \mathrm{id}$ with ε from the Key Lemma 7.4, the functoriality gives rise to commutative diagrams:

$$(\dagger) \qquad \begin{array}{ccc} \Pi_L & \xrightarrow{\Phi_L} & \Pi_L \\ \downarrow p_L & & \downarrow p_L \\ \Pi_K & \xrightarrow{\Phi_K} & \Pi_K \end{array} \qquad\qquad \begin{array}{ccc} \widehat{L} & \xleftarrow{\hat\psi} & \widehat{L} \\ \uparrow \imath & & \uparrow \imath \\ \widehat{K} & \xleftarrow{\cdot\,\varepsilon} & \widehat{K} \end{array}$$

in which $\cdot\,\varepsilon$ is the multiplication by ε. But from the commutativity of the right diagram from (†), it follows that $\hat\psi = \varepsilon_L \cdot \hat\phi_L$ and $\varepsilon \cdot \mathrm{id}$ coincide on $\widehat{K}$. In particular, $\hat\phi_L$ must map $\widehat{K}$ isomorphically onto itself, and therefore ϕ_L maps K^{i} isomorphically onto itself. We thus conclude that $\hat\phi_L$ equals $\varepsilon \cdot \varepsilon_L^{-1}$ on $\widehat{K}$. Since $\hat\phi_L|_{\widehat{K}}$ is the completion of the field isomorphism $\phi_L|_K$ of K, one must have $\varepsilon \cdot \varepsilon_L^{-1} = 1$, hence $\varepsilon_L = \varepsilon$ is indpendent of L.

Recalling the notations and discussion before Lemma 7.5, we conclude that for every $K_\mu \subseteq \tilde{K}$ as there, there is a (unique) field K-automorphism $\phi_\mu : K_\mu^{\mathrm{i}} \to K_\mu^{\mathrm{i}}$ such that $\varepsilon \cdot \hat\phi_\mu$ is the Kummer isomorphism of the group automorphism $\Phi_{K_\mu} : \Pi_{K_\mu} \to \Pi_{K_\mu}$. Then reasoning as above, it follows immediately that for $K_\mu \subset K_\nu$, one must have $\phi_\mu = \phi_\nu|_{K_\mu}$. Thus finally the compatible system of K-automorphisms $(\phi_\mu)_\mu$ gives rise to a K-automorphism $\tilde{\phi}$ of $\tilde{K}$ defined by $\tilde{\phi}|_{K_\mu} := \phi_\mu$ such that the Kummer isomorphisms of each Φ_μ is precisely $\varepsilon \cdot \phi_\mu$.

Finally, in order to conclude the proof of the Key Lemma 7.4, let $\tilde{\Phi}_\phi$ be the automorphism of $\tilde{G}_K$ defined by the $\tilde{\phi}$-conjugation. Then $\tilde{\Phi} \circ \tilde{\Phi}_\phi^{-1}$ is an automorphism of $\tilde{G}_K$ which induces on every Π_{K_μ} the multiplication by ε^{-1} for the fixed given $\varepsilon \in \mathbb{Z}_\ell^\times$ independent of K_μ. We claim that $\varepsilon = 1$. Indeed, let $\tilde{K} := K(\ell)$ be the maximal pro-ℓ subextension of $\tilde{K}|K$. Then by a standard argument it follows that the automorphism of $\mathrm{Gal}_K(\ell) := \mathrm{Gal}\big(\tilde{K}\,|\,K\big)$ defined by $\tilde{\Phi} \circ \tilde{\Phi}_\phi^{-1}$ maps every $g \in \tilde{G}_K(\ell)$ to its power $g^{\varepsilon^{-1}}$. From this easily follows that $\varepsilon = 1$. This concludes the proof of the Key Lemma 7.4, thus of assertion 1) of Theorem 2.7.

References:

[An] André, Y., *On a geometric description of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q})$ *and a p-adic avatar of of* $\widehat{GT}$*,* Duke Math. J. **119** (2003), 1–39.

[Be] Belyi, G. V., *On Galois extensions of a maximal cyclotomic field,* Mathematics USSR Izvestija, Vol. **14** (1980), no. 2, 247–256. (Original: Izvestiya Akademii Nauk SSSR, vol. **14** (1979), no. 2, 269–276.)

[Bo] Bogomolov, F. A., *On two conjectures in birational algebraic geometry,* in: Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, eds A. Fujiki et all, Springer Verlag Tokyo 1991.

[De] Deligne, P., *Le groupe fondamental de la droite projective moins trois points,* in: Galois groups over **Q**, eds. Ihara, Ribet, Serre; Math. Sci. Res. Inst. Publ. **16**, Springer 1989; pp. 79–297.

[Dr] Drinfeld, V. G., *On quasi-triangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*,* Leningrad Math. J. **2**, no. 4 (1991), 829–860. Original: Algebra i Analiz **2**, no. 4 (1990), 149–181.

[F1] Furusho, H., *Geometric and arithmetic subgroups of the Grothendieck-Teichmüller group,* Math. Res. Letters **10** (2003), 97–108

[F2] Furusho, H., *Multiple zeta values and Grothendieck-Teichmüller groups,* AMS Contemp. Math. **416** (2006), 49–82.

[GGA] Geometric Galois Actions I, LMS LNS Vol **242**, eds L. Schneps – P. Lochak, Cambridge Univ. Press 1998.

[G1] Grothendieck, A., *Letter to Faltings, June 1983*, See [GGA].

[G2] Grothendieck, A., *Esquisse d'un programme, 1984.* See [GGA].

[H–Ma] Hain, R. and Matsumoto, M., *Tannakian fundamental groups associated to Galois groups,* in: Galois groups and Fundamental groups, ed. Schneps, MSRI Pub. Series **41**, 2003, pp.183–216.

[H–Sch] Harbater, D. and Schneps, L., *Fundamental groups of moduli and the Grothendieck-Teichmüller group,* Trans. AMS, Vol. **352** (2000), 3117-3148.

[HLS] Hatcher, A., Lochak, P. and Schneps, L., *On the Teichmüller tower of mapping class groups,* J. reine angew. Math. **521** (2000), 1–24.

[H–Mz] Hoshi, Y. and Mochizuki, Sh., *On the combinatorial anabelian geometry of nodally nondegenerate outer representations,* Hiroshima Math. J. **41**, Number 3 (2011), 275–342.

[I1] Ihara, Y., *On Galois represent. arising from towers of covers of* $\mathbb{P}^1\backslash\{0,1,\infty\}$, Invent. Math. **86** (1986), 427–459.

[I2] Ihara, Y., *Braids, Galois groups, and some arithmetic functions,* Proceedings of the ICM'90, Vol. I, II, Math. Soc. Japan, Tokyo, 1991, pp.99–120.

[I3] Ihara, Y., *On the embedding of* $\mathrm{Gal}(\mathbb{Q})$ *into* $\widehat{GT}$*, The Grothendieck theory of dessins d'enfants,* (Luminy, 1993); LMS LNS, **200**, Cambridge Univ. Press, Cambridge, 1994; pp.289–321.

[I–M] Ihara, Y. and Matsumoto, M., *On Galois actions on profinite completions of braid groups,* in: Recent developments in the inverse Galois problem (Seattle, WA, 1993), eds. Abhyankar et al, Contemp. Math. **186** AMS Providence, RI, 1995; pp. 173–200.

[LNS1] Lochak, P., Nakamura, H. and Schneps, L., *On a new version of the Grothendieck-Teichmüller group,* CRAS **325** (1997), 11–16.

[LNS2] Lochak, P., Nakamura, H. and Schneps, L., *Eigenloci of 5 point configurations on the Riemann sphere and the Grothendieck-Teichmueller group,* Math. J. Okayama Univ. **46** (2004), 39–75.

[L–Sch] Lochak, P., and Schneps, L., *A cohomological interpretation of the Grothendieck-Teichmüller group. Appendix by C. Scheiderer,* Invent. Math. **127** (1997), 571–600.

[Ma] Matsumoto, M., *Galois representations on profinite braid groups on curves,* J. reine angew. Math. **474** (1996), 169–219.

[M-T] Matsumoto, M. and Tamagawa, A., *Mapping class-group action versus Galois action on profinite fundamental groups,* Am. J. Math. **122** (2000), 1017–1026.

[Na] Nakamura, H., *Limits of Galois representations in fundamental groups along maximal degeneration of marked curves II,* Proc. Symp. Pure Math. **70** (2002), 43–78.

[N–Sch] Nakamura, H. and Schneps, L., *On a subgroup of the Grothendieck–Teichmüller group acting on the profinite Teichmüller modular group,* Invent. Math. **141** (2000), 503–560.

[P1] Pop, F., *Pro-l abelian-by-central Galois theory of Zariski prime divisors,* Israel J. Math. **180** (2010), 43–68.

[P2] Pop, F., *Inertia elements versus Frobenius elements,* Math. Annalen **438** (2010), 1005–1017.

[P3] Pop, F., *Recovering fields from their decomposition graphs,* in: Number theory, Analysis and Geometry –In memory of Serge Lang, Springer special Volume 2010; eds: Goldfeld, Jorgenson, Jones, Ramakrishnan, Ribet, Tate; pp. 519–594.

[P4] Pop, F., *On the birational anabelian program initiated by Bogomolov I,* Invent. Math. **187** (2012), 511–533.

[P5] Pop, F., *On I/OM.* See: `https://www.math.upenn.edu/~pop/Research/Papers.html`.

[Sch] Schneps, L., *Dessins d'enfants on the Riemann sphere,* in: The Grothendieck theory of dessins denfants (Luminy, 1993), LMS LNS **200**, Cambridge Univ. Press, Cambridge, 1994; pp. 47–77.

[To1] Topaz, A., *Commuting-Liftable Subgroups of Galois Groups II,* J. reine angew. Math. **730** (2017), 65–133.

[To2] Topaz, A., *On the Nature of Base Fields,* An appendix to: *On The Minimized Decomposition Theory of Valuations,* by F. Pop, Bull. Math. Soc. Sci. Math.Roumanie, Tome **58 (106)**, No. 3 (2015), 331–357.

[To3] Topaz, A., *The Galois action on geometric lattices and the mod-ℓ I/OM.* See `http://arXiv:1510.08836v2 [math.AG]`

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