LITTLE SURVEY ON I/OM AND ITS VARIANTS AND THEIR RELATION TO (VARIANTS OF) $\widehat{GT}$

— *OLD & NEW* —

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Abstract. This is a short survey on the subject of my talk at the *Hyper-JARCS Memorial Conference for Professor Ştefan Papadima* held at the University of Tokyo in Dec 2019.

1. Motivation

A main theme in Grothendieck's *Esquisse d'un Programme* [G2], see rather [GGA], was to shed new light on the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Aut}(\overline{\mathbb{Q}})$ of $\mathbb{Q}$, e.g. giving a non-tautological description of $G_{\mathbb{Q}}$, studying its open subgroups and finite quotients (the Inverse Galois Problem), its linear representations (the Langlands Program). The proposed way to do that was to study the action of $G_{\mathbb{Q}}$ (and G_K for more general fields K) on combinatorial and/or geometric objects, e.g. the (algebraic) étale fundamental group.

A quite notable development concerning the main theme above was the introduction and quite intensive study of the Grothendieck-Teichmüller group $\widehat{GT}$, and its relationship with yet another idea stemming from the *Esquisse,* namely the automorphism group $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ of the algebraic fundamental group functor $\overline{\pi}_{\mathcal{V}}$ of specific categories of geometrically integral varieties $\mathcal{V}$ related to moduli (stacks) of curves; see Appendix for notation and basic facts on fundamental groups. Despite major progress on understanding the objects under discussion, the precise relationship between $G_{\mathbb{Q}}$ and $\widehat{GT}$ and/or $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ remains largely mysterious to this day. See e.g. the early surveys [N2], [Sch] for some "classical" facts about $\widehat{GT}$.

Another aspect of the search for topological/combinatorial descriptions of $G_{\mathbb{Q}}$ relates to a question by Ihara from the 1980's, which in the 1990's became a conjecture by Oda–Matsumoto, for short (classical) I/OM. In a nutshell, classical I/OM asks whether/conjectures that $G_{\mathbb{Q}}$ is the automorphism group of the algebraic fundamental group functor of the category $\mathfrak{Var}_{\mathbb{Q}}$ (of geometrically integral $\mathbb{Q}$-varieties and dominant morphisms).

To set up notation, let $\mathcal{V}$ be a category of geometrically integral varieties over the base field k, e.g. $k = \mathbb{Q}$, and for $X \in \mathcal{V}$, let $\overline{X} := X \times_k \overline{k}$ be the base change to an algebraic closure $\overline{k}$ of k (which is fixed throughout). Then in notation, definitions and by the facts outlined in the Appendix, one has: The *etale fundamental group functor* of $\mathcal{V}$ defined by

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Date: December 30, 2021.

1991 *Mathematics Subject Classification.* 11G99, 12F10, 12G99, 14A99

2010 MSC. Primary 11G99, 12F10, 12G99, 14A99.

Key words and phrases. Grothendieck-Teichmüller group, Moduli spaces of pointed curves, absolute Galois groups, topological/étale fundamental group, hyperplane arrangements, projective geometry.

$X \mapsto \pi_1(X) \in \boldsymbol{\mathcal{G}}^{\rm out}_{G_k}$ gives rise to a canonical representation

$$\rho_{\mathcal{V}} : G_k \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}), \quad \sigma \mapsto \big(\rho_X(\sigma)\big)_{X\in\mathcal{V}}.$$

This suggests studying G_k via $\rho_{\mathcal{V}}$ for concrete categories $\mathcal{V}$. Concretely, for $k = \mathbb{Q}$, one should give/study categories $\mathcal{V} \subset \mathfrak{Var}_{\mathbb{Q}}$, e.g. subcategories $\mathcal{V} \subset \mathcal{T}$ of the Teichmüller moduli tower $\mathcal{T} = \{\mathcal{M}_{g,n}\}_{g,n}$, such that the following questions have positive answers:

Q1. $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ has a "concrete" combinatorial/topological description.

Q2. The representation $\rho_{\mathcal{V}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ is an isomorphism.

In particular, a category $\mathcal{V}$ for which both questions Q1, Q2 have acceptable answers would give a non-tautological description of the absolute Galois group $G_{\mathbb{Q}}$. Obviously, the classical I/OM is about Q2 having a positive answer for $\mathcal{V} = \mathfrak{Var}_{\mathbb{Q}}$.

This short survey is about the classical I/OM and its variants (birational, tempered, Λ-abelian-by-central). For reader's sake we first very briefly recall how $\widehat{GT}$ fits into the picture above, to be precise, how $\widehat{GT}$ relates to Q1. On the other hand, this short survey is not by any means (even a sketch of) a survey about $\widehat{GT}$.

Acknowledgements. I would like to thank several experts on the matter, especially Yves André, David Harbater, Pierre Lochak, Hiroaki Nakamura, Leila Schneps, Adam Topaz and the referee for the careful reading of the manuscript as well as remarks and suggestions, which were very helpful for improving both content and the presentation.

2. Question **Q1**: $\widehat{GT}$ versus $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$

We recall below the "classical" definition of the Grothendieck-Teichmüller group $\widehat{GT}$ as originating from Drinfel'd [Dr],[1] and after mentioning a few of its basic properties, we recall how $\widehat{GT}$ relates to $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ for specific categories $\mathcal{V}$ of moduli spaces of curves.

2.1. $\widehat{GT}$ and a few "classical" facts about $\widehat{GT}$.

Let $\widehat{F}_2$ be the profinite completion of the discrete free group F_2 on two generators x, y. Then every element $f = f(x,y) \in \widehat{F}_2$ is a proword in x, y. Hence if $\varphi : \widehat{F}_2 \to \tilde{F}$, $x \mapsto \tilde{x}, y \mapsto \tilde{y}$ is a morphism of profinite groups, the proword $f = f(x,y)$ defines uniquely a proword $\tilde{f} \in \tilde{F}$ as follows

$$f(\tilde{x},\tilde{y}) := \tilde{f} := \varphi(f) = \varphi\big(f(x,y)\big) = f\big(\varphi(x),\varphi(y)\big) \in \tilde{F}.$$

In particular, since $\widehat{F}_2$ is profinite free on the generators x, y, it follows that $\mathrm{Hom}(\widehat{F}_2, \tilde{F})$ is in bijection with the set $\tilde{F} \times \tilde{F}$ via $\varphi \mapsto \big(\varphi(x), \varphi(y)\big)$. On the other hand, if $\tilde{F} = \widehat{F}_2$, it is virtually impossible to write down correspondingly the composition of two endomorphisms $\varphi = \varphi_2 \circ \varphi_1$ of $\widehat{F}_2$, and moreover, given $\varphi(x), \varphi(y) \in \widehat{F}_2$, to decide whether $\varphi \in \mathrm{Aut}(\widehat{F}_2)$.

Let $[\widehat{F}_2, \widehat{F}_2] = \ker(\widehat{F}_2 \twoheadrightarrow \widehat{F}_2^{\rm ab})$ be the (closure of the) commutator group in $\widehat{F}_2$, and $\widehat{\mathbb{Z}}^\times$ be the group of invertible elements in the adic completion $\widehat{\mathbb{Z}}$ of the ring of integers $\mathbb{Z}$. Invoking

[1] We should notice that one can define in a similar way the pro-ℓ variant $\widehat{GT}_\ell$ as well as the prounipontent variant $\mathrm{GT}(k)$ of the Grothendieck-Teichmüller group, where k is an arbitrary field with $\mathrm{char}(k) = 0$. We will not discuss these aspects/variants of the Grothendieck-Teichmüller group here.

the discussion above, consider the set of all the automorphisms $\varphi \in \mathrm{Aut}(\widehat{F}_2)$ of the form:

$$(*)_{\widehat{GT}} \qquad \varphi(x) = x^{\lambda}, \quad \varphi(y) = f y^{\lambda} f^{-1} \quad \text{with} \quad \lambda \in \widehat{\mathbb{Z}}^{\times}, \ f \in [\widehat{F}_2, \widehat{F}_2],$$

where $f = f(x,y) \in [\widehat{F}_2, \widehat{F}_2]$ satisfy the following three equations (also called *relations*):

(I) $f(x,y) = f(y,x)$, i.e., the proword $f = f(x,y)$ is symmetric.

(II) $x^{\mu} f(y,x) y^{\mu} f(z,y) z^{\mu} f(x,z) = 1$, where $\mu = \frac{1}{2}(1-\lambda)$, and $z = (xy)^{-1}$.

(III) $f(x_{12}x_{23}) f(x_{34}, x_{45}) f(x_{15}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{15}) = 1$ inside $\widehat{\Gamma}_{0,5} = \widehat{K}(0,5)$.

Here, for (I), (II), one makes the identification $\widehat{F}_2 = \langle x, y, z \mid xyz = 1\rangle$; hence (I) asks that $f = f(x,y)$ is invariant under the involution of $\widehat{F}_2$ defined by $(x,y) \mapsto (y,x)$, whereas $f(z,y)$ and $f(x,z)$ are defined by the automorphisms of $\widehat{F}_2$ defined by $(x,y) \mapsto (z,y)$, respectively $(x,y) \mapsto (x,z)$. Finally, $f(x_{ij}, x_{i'j'})$ in (III) are defined by maps $\widehat{F}_2 \to \widehat{\Gamma}_{0,5}$, $(x,y) \mapsto (x_{ij}, x_{i'j'})$ for $1 \leqslant i < j \leqslant 5$, $1 \leqslant i' < j' \leqslant 5$, see Appendix for notations and fundamental groups.

It turns out that the set of all $\varphi \in \mathrm{Aut}(\widehat{F}_2)$ satisfying $(*)_{\widehat{GT}}$ and (I), (II), (III) build a subgroup $\widehat{GT} < \mathrm{Aut}(\widehat{F}_2)$, called the *Grothendieck–Teichmüller group,* which is easily seen to be a profinite group. Moreover, identifying $\widehat{F}_2$ with $\pi_1^{\mathrm{et}}(\mathbb{P}^1 \backslash \{0,1,\infty\}, \vec{01})$ via the tangential base point $\vec{01}$, see Deligne [De], Ihara [I3,I4,I5] shows that the image of the resulting canonical embedding of $G_{\mathbb{Q}}$ into $\mathrm{Aut}(\widehat{F}_2)$ is a subgroup of $\widehat{GT}$.[2]

Subsequently $\widehat{GT}$ was intensively and extensively studied by many, e.g. (a rather alphabetical order) Hain–Matsumoto [HM], Harbater–Schneps [HS], Ihara–Matsumoto [IM], Ihara–Nakamura [IN], Lochak–Nakamura–Schneps [LNS1,LNS2], Lochak–Schneps [LS1,LS2], Nakamura [N1,N2,N3,N4], Nakamura–Schneps [NS], and more (respectively very) recent by Enriquez [En], (respectively) Hoshi–Minamide–Mochizuki [HMM], Minamide–Nakamura [MN]. This list does not include the long list of papers on (variants of) $\widehat{GT}$ and related topics by math physicists and representation theorists, among other things relating $\widehat{GT}$ to operads, Lie theory (as already present in the work of Drinfel'd), multi-zeta and polylogs, the Deligne-Ihara Conjecture, etc., e.g. work by F. Brown, B. de Brito, Dolgushev, Fresse, Furusho, Goncharov, Horel, Racinet, Robertson, Shabat, Tamarkin, Willwacher, Wojtkoviak, Zapponi, to mention a few names.

From the list of "classical" facts about $\widehat{GT}$ in the above setting we recall the following (by no means a comprehensive list!); see the Appendix for basic facts on fundamental groups.

- Lochak–Schneps [LS2]: *The complex conjugation $\sigma \in G_{\mathbb{Q}} \leqslant \widehat{GT}$ is self-normalizing in $\widehat{GT}$.* This fact extends/generalizes the well known fact that σ is self-normalizing in $G_{\mathbb{Q}}$.
- Nakamura–Schneps [NS]: *There is an explicitly defined closed subgroup $\mathbb{\Gamma} < \widehat{GT}$ with $G_{\mathbb{Q}} \leqslant \mathbb{\Gamma}$ which acts compatibly with $G_{\mathbb{Q}}$ on the tower of fundamental groups $\{\widehat{\Gamma}_{g,n}\}_{g,n}$.* The group $\mathbb{\Gamma} < \mathrm{Aut}(\widehat{F}_2)$ consists of all φ satisfying (I), (II) above, and two further relations: (III)$'$ which implies (III), and (VI) which was introduced in Nakamura [N4], Part I.
- Ihara [I6] defined the "cyclotomic" $GTK \leqslant \widehat{GT}$, a closed subgroup containing $G_{\mathbb{Q}}$, and inquired whether $GTK < \widehat{GT}$ strictly. But Enriquez [En] showed: $GTK = \widehat{GT}$.

[2] In loc.cit. further variants of both fundamental groups an $\widehat{GT}$ were considered.

On the other hand, despite all the effort and that a lot is known, the precise relationship between $\widehat{GT}$ and $G_{\mathbb{Q}} \leqslant \widehat{GT}$—in particular, whether $G_{\mathbb{Q}} = \widehat{GT}$ and/or whether $G_{\mathbb{Q}} \cong \widehat{GT}$ as profinite groups— remains mysterious to this day. See e.g. the early surveys [N2] and [Sch] for lists of open problems concerning $\widehat{GT}$ and related questions.

2.2. $\widehat{GT}$ **as automorphisms group** $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$**.** The question Q1 relates to $\widehat{GT}$ in a rather concrete way as follows. By the discussion in Appendix, the (profinite completion of the) mapping class group $\widehat{\Gamma}_{0,n} = \overline{\pi}_1(\mathcal{M}_{0,n})$ is the algebraic fundamental group of the affine variety $\mathcal{M}_{0,n} = (\mathbb{P}^1 \backslash \{0, 1, \infty\})^{n-3}$ and gives rise to the representation $\rho_{0,n} : G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{\Gamma}_{0,n})$. This being said, Harbater–Schneps [HS] considered the subgroups $\mathrm{Out}^\sharp_n < \mathrm{Out}(\widehat{\Gamma}_{0,n})$ consisting of all the automorphism which preserve the conjugacy classes of *inertia at infinity.* Letting $\widehat{GT}_0 < \mathrm{Aut}(\widehat{F}_2)$ consist of all φ which satisfy $(*)_{\widehat{GT}}$ and the relations (I), (II) above, one has:

- [HS]: *There are canonical isomorphisms* $\mathrm{Out}^\sharp_4 \cong \widehat{GT}_0$ *and* $\mathrm{Out}^\sharp_n \cong \widehat{GT}$ *for* $n > 4$.

It is further shown in [HS] that the isomorphisms $\mathrm{Out}^\sharp_n \cong \widehat{GT}$ are compatible with both the representations $\rho_{0,n} : G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{\Gamma}_{0,n})$ and the canonical morphisms $\mathcal{M}_{0,n} \to \mathcal{M}_{0,m}$ for $m \leqslant n$ in $\mathcal{T}_0 = \{M_{0,n}\}_{n>3}$. In particular, as a consequence of the above facts, [HS] implies:

- $\widehat{GT} = \mathrm{Aut}^\sharp(\overline{\pi}_{\mathcal{T}_0})$ *is the group of all* $\varphi \in \mathrm{Aut}(\overline{\pi}_{\mathcal{T}_0})$ *which preserve inertia at infinity.*

Finally one should mention that there is a host of results concerning the action of $\widehat{GT}$ on geometric objects, e.g. [HLS, IM, IN, MT, N3, N4, Co] which we do not discuss.

We conclude by mentioning two very recent major results concerning Q1 and $\widehat{GT}$.

First, let $\mathcal{V}_0 := \{M_{0,4}, M_{0,5}\}$ with the canonical morphisms $\mathcal{M}_{0,5} \to \mathcal{M}_{04}$. Then Hoshi–Minamide–Mochizuki [HMM] give a ***complete unconditional solution*** to Q1 by proving:

- *For* $n > 4$*, all* $\varphi \in \mathrm{Aut}(\widehat{\Gamma}_{0,n})$ *permute the conjugacy classes of inertia at infinity, and* $\mathrm{Out}(\widehat{\Gamma}_{0,n}) = \mathfrak{S}_n \times \widehat{GT}$*. Moreover, one has that* $\widehat{GT} = \mathrm{Aut}(\overline{\pi}_{\mathcal{V}_0}) = \mathrm{Aut}(\overline{\pi}_{\mathcal{T}_0})$.

Actually, the results in [HMM] are much more general, and show that given (g, r) with $2g - 2 + r > 0$, the algebraic fundamental group $\overline{\pi}_1(X_n)$ of the configuration space X_n of n geometric points on a curve of type (g, r) encodes (n, g, r) and the inertia at infinity. These results generalize/extend previous ones from Mochizuki–Tamagawa [MT] on configuration spaces of curves X of genus $g \geqslant 2$. Further, [HMM] show that similar pro-ℓ versions of these results hold, correspondingly, for the pro-ℓ completions of the groups involved.

Second, let $\widehat{\mathcal{B}}_n = \widehat{B}_n / \widehat{C}_n$ be the quotient of the profinite Artin braid group on n stings by its center $\widehat{C}_n \cong \widehat{\mathbb{Z}}$. Then Minamide–Nakamura [MN] prove the ***quite remarkable facts***:

- *One has canonical isomorphisms* $\widehat{GT} \cong \mathrm{Out}(\widehat{\mathcal{B}}_n)$ *for all* $n > 3$*, and* $\widehat{GT} = \mathrm{Out}(\widehat{\Gamma}_{1,2})$.

Among other things, these results support Grothendieck's "first two levels" philosophy, namely that the subcategories of the Teichmüller moduli tower $\mathcal{T}$ involving $\mathcal{M}_{0,4}$, $\mathcal{M}_{0,5}$, $\mathcal{M}_{1,1}$, $\mathcal{M}_{1,2}$ should "encode everything."

- See also Hatcher–Lochak–Schneps [HLS], where the subgroup $\mathbf{\Lambda} < \widehat{GT}^1 < \widehat{GT}$ is defined in terms of the "first two levels" and it is shown that $\mathbf{\Lambda}$ acts on the whole fundamental group Teichmüller tower $\{\widehat{\Gamma}_{g,n}\}_{g,n}$, and the connection of $\mathbf{\Lambda}$ with the group $\mathbb{I}\Gamma < \widehat{GT}$ is discussed.

On the other hand, to the best of my knowledge, it is not known whether $\widehat{GT}$ acts on the fundamental group Teichmüller tower $\{\widehat{\Gamma}_{g,n}\}_{g,n}$ respectively whether $\widehat{GT}$ equals $\mathrm{Aut}(\overline{\pi}_{\mathcal{T}})$. It would be quite interesting to see whether (refinements of) [HMM] and [MN] could be used to tackle this question. [It seems that one does not know enough about higher genera braid groups and the relationship between configurations spaces X_n and moduli spaces $\mathcal{M}_{g,n}$, to enable extending the above result to the whole $\mathcal{T}$.]

3. Question **Q2** and I/OM

Recall that the question Q2 is about finding explicit "nice" subcategories $\mathcal{V} \subset \mathfrak{Var}_{\mathbb{Q}}$, e.g. $\mathcal{V} \subset \mathcal{T}$, such that $\rho_{\mathcal{V}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ is an isomorphism.

First, concerning the *injectivity* of $\rho_{\mathcal{V}}$, it was remarked by Drinfel'd that Belyi's Theorem implies that if $\mathbb{P}^1\backslash\{0,1,\infty\} \in \mathcal{V}$, then $\rho_{\mathcal{V}}$ is injective. Further, Voevodsky [Vo] showed that $\rho_{\mathcal{V}}$ is injective if $E\backslash\{pt\} \in \mathcal{V}$, where E is an elliptic curve, and Matsumoto [Ma] shows that $\rho_{\mathcal{V}}$ is injective if $\mathcal{V}$ contains any affine hyperbolic curve. Finally, Hoshi–Mochizuki [HMo] shows that $\rho_{\mathcal{V}}$ is injective if $\mathcal{V}$ contains <u>any</u> hyperbolic curve. Hence one has:

- [Be, Vo, Ma, HMo]: $\rho_{\mathcal{V}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ *is injective if* $\mathcal{V}$ *contains a hyperbolic curve.*

Second, the *surjectivity* of $\rho_{\mathcal{V}}$ appears to be more involved, because of lack of insight in the origin of automorphism $\varphi = (\varphi_X)_{X\in\mathcal{V}} \in \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$. An obvious observation is that the more objects $\mathcal{V}$ has, the more possibilities for elements in $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ are there, whereas each morphism in $\mathcal{V}$ imposes a restriction on the elements in $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$.

3.1. **The classical I/OM.** Ihara asked (in the 1980's) whether $G_{\mathbb{Q}} = \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ in the case $\mathcal{V}$ is as rich as possible, i.e., $\mathcal{V} = \mathfrak{Var}_{\mathbb{Q}}$; and based on "some motivic evidence" Oda–Matsumoto conjectured (in the 1990's) that Ihara's question should have a positive answer, i.e., $G_{\mathbb{Q}} = \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ for $\mathcal{V} = \mathfrak{Var}_{\mathbb{Q}}$. For short we will speak about the (classical) I/OM. The classical I/OM was answered in positive in 1998 by the author of this note, but the proof was never published because of subsequent developments superseding that result (namely the stronger forms of I/OM, e.g. the pro-ℓ-abelian-by-central I/OM for connected rigid categories $\mathcal{V}$, see the discussion below).

In a nutshell, the idea to tackle the (classical) I/OM is to reduce it to its *birational variant* $\mathrm{I/OM_{bir}}$ and use birational anabelian type results to tackle the latter. Namely, for $X \in \mathfrak{Var}_k$, let $\mathcal{V}_X \subset \mathfrak{Var}_k$ be the category which contains the (affine) open dense subsets $U \subset X$, $V \subset \mathbb{P}^1_k$, and as morphisms the canonical inclusions $U'' \subset U'$, $V'' \subset V'$ and the dominant k-morphisms $U \to V$. The "generic fiber" of $\mathcal{V}_X$ is the category $\mathcal{F}_X := \{k(X), k(\mathbb{P}^1)\}$ having as objects $k(X), k(\mathbb{P}^1)$ and as morphisms all the k-embeddings $k(\mathbb{P}^1) \hookrightarrow k(X)$. Further, every $\sigma \in \mathrm{Aut}(\overline{\pi}_{\mathcal{V}_X})$ is a family of the form $\sigma = \big((\sigma_U)_U, (\sigma_V)_V\big)$, compatible with the all projections $\overline{\pi}_1(U'') \twoheadrightarrow \overline{\pi}_1(U')$, $\overline{\pi}_1(V'') \twoheadrightarrow \overline{\pi}_1(V')$ and $\overline{\pi}_1(U) \twoheadrightarrow \overline{\pi}_1(V)$. Hence if $K := k(\overline{X}) = \cup_U k[\overline{U}]$, by "taking limits," every $\sigma \in \mathrm{Aut}(\overline{\pi}_{\mathcal{V}_X})$ defines a unique $\sigma_K \in \mathrm{Out}(G_K)$ which is compatible with the (surjective) projections $\pi_{\imath,V} : G_K \to \overline{\pi}_1(V)$, $V \in \mathcal{V}_X$ defined by fixed k-embeddings $\imath : k(\mathbb{P}_k) \hookrightarrow k(X)$. Finally, let $\mathrm{Out}_{\mathcal{V}_X}(G_K) \leqslant \mathrm{Out}(G_K)$ be the subgroup of all $\Phi \in \mathrm{Out}(G_K)$ satisfying the conditions the σ_K satisfy, i.e., for all k-embeddings $\imath : k(\mathbb{P}_k) \hookrightarrow k(X)$ and $V \in \mathcal{V}_X$, one has: Φ is compatible with the projections $\pi_{\imath,V} : G_K \to \overline{\pi}_1(V)$.

Then one has canonical embeddings:

$$G_k \hookrightarrow \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}) \hookrightarrow \mathrm{Out}_{\mathcal{V}_X}(G_K),$$

hence a possible strategy to tackle I/OM over k is to prove its birational variant $\mathrm{I/OM}_{\mathrm{bir}}$, i.e., to show that $G_k = \mathrm{Out}_{\mathcal{V}_X}(G_K)$—and this is how the initial proof of classical I/OM went. Thus it appears that in fact,

(∗) I/OM *is rather a problem of birational nature which has a rich geometric hypothesis.*

Variants of I/OM and $\widehat{GT}$

The variants of I/OM and $\widehat{GT}$ we have in mind and review/discuss briefly below arise from variants of fundamental groups, e.g. the *tempered fundamental group* π_1^{temp} defined for varieties over p-adic fields, the *pro-$\mathcal{C}$ algebraic fundamental group* $\overline{\pi}_1^{\,c}$ of varieties over arbitrary base fields, and *pro-linear/pro-unipotent completions* of the fundamental group, etc.

3.2. **Tempered $\widehat{GT}_p$ and tempered** $\mathrm{I/OM}_p$**.** The tempered variant $\widehat{GT}_p$ of $\widehat{GT}$ and the tempered variant $\mathrm{I/OM}_p$ of I/OM, are introduced/defined in André [An] and are based on the *tempered fundamental group* $\pi_1^{\mathrm{temp}}(X, \overline{x})$, defined for integral varieties $X \in \mathfrak{Var}_{\mathbb{C}_p}$.[3] Precisely, let $X_\nu \to X$ be a finite Galois étale cover, and $\mathcal{X}_\nu \to X_\nu^{\mathrm{an}}$ be the *p-adic analytic universal cover* of X_ν^{an}. Then $\mathrm{Aut}_X(\mathcal{X}_\nu)$ is an extension of $\mathrm{Aut}_X(X_\nu)$ by the possibly infinite discrete group $\mathrm{Aut}_{X_\nu}(\mathcal{X}_\nu)$. Finally (choosing base points, which we do not write), one defines

$$\pi_1^{\mathrm{temp}}(X) := \varprojlim_{\nu} \mathrm{Aut}_X(\mathcal{X}_\nu),$$

hence $\pi_1^{\mathrm{temp}}(X)$ is a projective limit of discrete possibly infinite groups. By mere definitions, π_1^{temp} is compatible with morphisms $X \to Y$ in $\mathfrak{Var}_{\mathbb{C}_p}$. We notice that after choosing base points, one has canonical morphisms $\pi_1^{\mathrm{temp}}(X) \to \overline{\pi}_1(X)$, but $\pi_1^{\mathrm{temp}}(X)$ encapsulates rather specific information about X^{an} and it is <u>not</u> a pro-$\mathcal{C}$ completion of $\pi_1^{\mathrm{top}}(X)$ in the usual sense. For instance, let E be an elliptic curve over $\mathbb{C}_p$. Then if E has good reduction, $\pi_1^{\mathrm{temp}}(E) = \overline{\pi}_1(E)$, whereas if E is a Tate elliptic curve, then $\pi_1^{\mathrm{temp}}(E) \cong \mathbb{Z} \times \widehat{\mathbb{Z}}$.

For a p-adic field k and $X \in \mathfrak{Var}_k$, let $X_{\mathbb{C}_p} := X \times_k \mathbb{C}_p$. Setting $\pi_1^{\mathrm{temp}}(X) := \pi_1^{\mathrm{temp}}(X_{\mathbb{C}_p})$, there are: (i) the tempered short exact sequence $1 \to \pi_1^{\mathrm{temp}}(X) \to \pi_1^{(\mathrm{temp})}(X) \to G_k \to 1$. (ii) a functorial morphism $\pi_1^{\mathrm{temp}}(X) \to \overline{\pi}_1(X)$ with dense image such that the tempered exact sequence (i) maps funtorially to $1 \to \overline{\pi}_1(X) \to \pi_1(X) \to G_k \to 1$. In particular, one gets:

$$\pi_1^{(\mathrm{temp})} : \mathfrak{Var}_k \to \boldsymbol{\mathcal{G}}^{\mathrm{out}}_{G_k}, \quad X \mapsto \pi_1^{(\mathrm{temp})}(X).$$

Hence for a subcategory $\mathcal{V} \subset \mathfrak{Var}_k$, one gets a representation

$$\rho_{\mathcal{V}} : G_k \to \mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{temp}}), \quad \sigma \mapsto \big(\rho_X(\sigma)\big)_{X \in \mathcal{V}} \text{ with } \rho_X(\sigma) \in \mathrm{Out}\left(\pi_1^{\mathrm{temp}}(X)\right).$$

In this setup, André [An] defines the tempered Grothendieck–Teichmüler group $\widehat{GT}_p$, which is a closed subgroup $\widehat{GT}_p < \mathrm{Aut}\big(\pi_1^{\mathrm{temp}}(\mathbb{P}^1 \backslash \{0, 1, \infty\})\big)$, and proves:

- *$\widehat{GT}_p \leqslant \widehat{GT}$ is closed, $G_{\mathbb{Q}_p} \leqslant \widehat{GT}_p$ canonically, and $G_{\mathbb{Q}_p} = G_{\mathbb{Q}} \cap \widehat{GT}_p$ inside $\widehat{GT}$.*

Finally, using the classical I/OM, André [An] concludes that the *tempered* $\mathrm{I/OM}_p$ holds:

- *The representation $\rho_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \to \mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{temp}})$ is an isomorphism for $\mathcal{V} = \mathfrak{Var}_{\mathbb{Q}_p}$.*

[3] Here $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic absolute value.

3.3. **Pro-$\mathcal{C}$ (birational) variants of** I/OM **and $\widehat{GT}$.** Let $\mathcal{C}$ be a category of topological groups which is closed with respect to fiber products and taking closed subgroups. Given a topological group G, we set $\mathcal{N}_G := \{N \lhd G \mid G/N \in \mathcal{C}\}$, and notice that since $\mathcal{C}$ is closed with respect to fiber products and closed subgroups, $\mathcal{N}_G$ is closed with respect to intersection. In particular, $(G/N)_{N\in\mathcal{N}}$ is canonically a surjective projective system, which is compatible with the system of projections $G \twoheadrightarrow G/N$, $N \in \mathcal{N}_G$. Its projective limit G^c endowed with the canonical morphism $\widehat{\imath}^{\,c} : G \to \widehat{G}^{\,c}$ is the *pro-$\mathcal{C}$ completion* of G.

Notice that the pro-$\mathcal{C}$ completion is functorial, i.e., it is compatible with continuous morphisms of topological groups, and if G is a discrete free group, say on generators $(g_i)_i$, then $\widehat{G}^{\,c}$ is the pro-$\mathcal{C}$ free group on the generators $(g_i)_i$. Further, if $\mathcal{C}$ is a category of finite groups, then $\widehat{G}^{\,c}$ is a profinite group, whose finite quotients lie in $\mathcal{C}$. Hence if $\mathcal{C}$ consists of all finite groups, then $\widehat{G}^{\,c}$ is the *profinite completion* of G. Some notable pro-$\mathcal{C}$-completions are the (*level m*) *pro-solvable*/*pro-nilpotent*/*pro-ℓ* completions.

Of particular interest is the Λ-*abelian-by-central* completion of G. Here $\mathbb{Z}_\ell \twoheadrightarrow \Lambda$ is a quotient of $\mathbb{Z}_\ell$, and $\mathcal{C}$ is the category of level two nilpotent groups of the form $\Lambda^m \rtimes \Lambda^n$, $m, n \geqslant 0$.

Finally, if $\mathcal{C}$ is a category of linear groups over a base field κ, then $\widehat{G}^{\,c}$ is the corresponding *pro-linear* completion of G. In the case $\mathcal{C}$ is the category of all reductive/unipotent/linear groups over κ, one speaks about the prolinear reductive/unipotent κ-completion of G.

This being said, suppose that $\mathcal{C}$ consists of *finite groups.* For an arbitrary base field k, and $\mathcal{V}$ a subcategory of $\mathfrak{Var}_k$, recalling notation, the definitions and facts from Appendix, one has: Since $\overline{\pi}_1(X)$ is a profinite group for $X \in \mathfrak{Var}_k$, so is $\overline{\pi}_1^c(X)$, and $\overline{\pi}_1(X) \twoheadrightarrow \overline{\pi}_1^c(X)$ is surjective. Further, $1 \to \overline{\pi}_1(X) \to \pi_1(X) \to G_k \to 1$ has $1 \to \overline{\pi}_1^{\,c}(X) \to \pi_1^{(c)}(X) \to G_k \to 1$ as a canonical quotient, and one gets the *pro-$\mathcal{C}$ algebraic fundamental group functor*

$$\overline{\pi}_1^c : \mathfrak{Var}_k \to \boldsymbol{\mathcal{G}}_{G_k}^{\mathrm{out}}, \quad X \mapsto \pi_1^{(c)}(X).$$

In particular, one gets a representation

$$\rho_{\mathcal{V}}^{\mathcal{C}} : G_k \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}^{\mathcal{C}}), \quad \sigma \mapsto \big(\rho_X(\sigma)\big)_{X\in\mathcal{V}} \ \text{ with } \ \rho_X(\sigma) \in \mathrm{Out}\left(\overline{\pi}_1^{\,c}(X)\right).$$

In the above context, let $\ell \neq \mathrm{char}(k)$, and $\mathcal{C}$ consist of the Λ-abelian-by-central groups. Set $\Pi_X^{\mathrm{c}} := \overline{\pi}_1^{\,c}(X)$, $\Pi_X =: \overline{\pi}_1^{\,c,\mathrm{ab}}(X)$, and notice that Π_X^{c} is encoded in the cup product $H^1_{\mathrm{et}} \times H^1_{\mathrm{et}} \xrightarrow{\cup} H^2_{\mathrm{et}}$ and the Bockstein operator $H^1_{\mathrm{et}} \to H^2_{\mathrm{et}}$.[4] Further, $pr_X : \Pi_X^{\mathrm{c}} \twoheadrightarrow \Pi_X$ has $\ker(pr_X) = [\Pi_X^{\mathrm{c}}, \Pi_X^{\mathrm{c}}]$, hence pr_X gives rise to a canonical morphism $\mathrm{Aut}(\Pi_X^{\mathrm{c}}) \to \mathrm{Aut}(\Pi_X)$, and since Π_X is a $\mathbb{Z}_\ell$-module, the action of $\mathbb{Z}_\ell^\times$ by multiplication on Π_X commutes with $\mathrm{Aut}(\Pi_X)$. Hence setting $\mathrm{Aut}^{\mathrm{c}}(\Pi_X) := \mathrm{im}\big(\mathrm{Aut}(\Pi_X^{\mathrm{c}}) \to \mathrm{Aut}(\Pi_X)\big)/\mathbb{Z}_\ell{}^\times$, we get:

$$\rho_{\mathcal{V}}^{\mathrm{c}} : G_k \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}}), \quad \sigma \mapsto \big(\rho_X^{\mathrm{c}}(\sigma)\big)_{X\in\mathcal{V}} \ \text{ with } \ \rho_X^{\mathrm{c}}(\sigma) \in \mathrm{Aut}^{\mathrm{c}}(\Pi_X).$$

In this setup, the following much stronger forms of both the classical I/OM and $\mathrm{I/OM_{bir}}$ were proved as follows. First, one replaces $\overline{\pi}_1(X)$ by Π_X^{c}, which is of "motivic nature" and carries less information than $\overline{\pi}_1(X)$. Second, one reduces to, and proves, birational variants for categories $\mathcal{V}_X$ (as explained in subsection 4.1), in which only "few" open subset $V \subset \mathbb{P}_k^1$ and morphisms $U \to V$ are involved (necessary to rigidify $\mathcal{V}_X$). Precisely, setting $K = k(\overline{X})$ for $X \in \mathfrak{Var}_k$, we denote $\Pi_K^{\mathrm{c}} \to \Pi_K$ the projection of the Λ-abelian-by-central Galois group

[4] The cup product alone recovers the "Zassenhaus quotient" of Π_X^{c}, which would do the job as well.

$\Pi_K^{\rm c}$ to the Λ-abelian Galois group Π_K of K. Then for $\mathcal{V}_X$ as introduced in subsection 4.1, every $\sigma \in \mathrm{Aut}^c(\Pi_{\mathcal{V}_X})$ defines a unique $\sigma_K \in \mathrm{Out}^{\rm c}_{\mathcal{V}_X}(\Pi_K)$, thus getting embeddings

$$G_k \hookrightarrow \mathrm{Aut}^c(\Pi_{\mathcal{V}_X}) \hookrightarrow \mathrm{Out}^{\rm c}_{\mathcal{V}_X}(\Pi_K).$$

• In Pop [P] one considers the following context: Let $U_0 := \mathbb{P}^1\backslash\{0,1,\infty\}$ have standard parameter t_0, and x, y be the standard affine coordinates on $\mathbb{A}^2 \supset \mathcal{M}_{0,5}$. Consider the category $\mathcal{V}_0^{\rm bir}$ having as objects U_0, and $U = U_f := \mathcal{M}_{0,5}\backslash V(f)$, for all $f \in \mathbb{Q}[x,y]$ divisible by $f_0 = x(1-x)y(1-y)(y-x)$, and as morphisms $U_g \hookrightarrow U_f$ for $f|g$, and the projections $p_t : U \to U_0$ defined by $t_0 \mapsto t \in \Sigma_0 := \{x, y, y-x\}$. Inspired by $\widehat{GT} = \mathrm{Aut}(\overline{\pi}_{\mathcal{V}_0})$, denote

$$\widehat{GT}_{\rm bir} := \mathrm{Aut}(\overline{\pi}_{\mathcal{V}_0^{\rm bir}}), \quad \widehat{GT}^{\rm c}_{\rm bir} := \mathrm{Aut}^{\rm c}_{\mathcal{V}_0^{\rm bir}}(\Pi_{\mathcal{V}_0^{\rm bir}}) \ \text{ with } \ \Lambda = \mathbb{Z}_\ell,$$

the *birational,* respectively *pro-ℓ abelian-by-central* birational variants of $\widehat{GT}$. Then recalling that $\mathcal{M}_{0,5}$ has $\mathbb{Q}(x,y)$ as function field, hence $K = \overline{\mathbb{Q}}(x,y)$, one has:

$$\rho_{\mathcal{V}_0^{\rm bir}} : G_{\mathbb{Q}} \to \widehat{GT}_{\rm bir} \to \mathrm{Out}_{\mathcal{V}_0^{\rm bir}}(G_K), \ \ \rho^{\rm c}_{\mathcal{V}_0^{\rm bir}} : G_{\mathbb{Q}} \to \widehat{GT}^{\rm c}_{\rm bir} \to \mathrm{Out}^c_{\mathcal{V}_0^{\rm bir}}(\Pi_K) \ \textit{are isoms.}$$

Actually, much more general results are proved in [P] as follows. Let k be any perfect field, and $\mathcal{V} \subset \mathfrak{Var}_k$ be a connected rigid category containing some X with $\dim(X) > 1$, e.g. for $k = \mathbb{Q}$ one can choose the higher dimensional variant $\mathcal{V} = \mathcal{V}_{0,n} := \{\mathcal{M}_{0,4}, \mathcal{M}_{0,n}\}$ of $\mathcal{V}_0$, or for k general, can choose $\mathcal{V} = \mathcal{V}_X$, provided $\mathcal{V}_X$ is rigid, $\dim(X) > 1$, and $U_0 \in \mathcal{V}_X$. Then $\rho^{\rm c}_{\mathcal{V}} : G_k \to \mathrm{Aut}^{\rm c}(\Pi_{\mathcal{V}})$ is an isomorphism. Further, it is shown that $\rho^{\rm c}_{\mathcal{V}}$ being an isomorphism implies the full profinite variant, i.e., $\rho_{\mathcal{V}} : G_k \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ is an isomorphism as well. In particular, if $\mathcal{V} = \mathfrak{Var}_k$, one gets the *pro-$\ell$ abelian-by-central* I/OM over arbitrary base fields k, which in turn implies the *full profinite* I/OM over k.

• In Topaz [T], one proves a similar results for $\Lambda = \mathbb{Z}/\ell$, thus a purely combinatorial hypothesis, but the categories $\mathcal{V}$ are more restrictive: First, $\mathcal{V}$ should contain at least one k-variety X with $\dim(X) \geqslant 5$, and second, the morphisms should include (among other things) the k-morphisms $U \to U_0$ defined by all the rational maps $t_0 \mapsto t \in k(U)$, $U \in \mathcal{V}$. Under these hypotheses, [T] shows that the representation $\rho^{\rm c}_{\mathcal{V}} : G_k \to \mathrm{Aut}^{\rm c}(\Pi_{\mathcal{V}})$ is an isomorphism. In particular, this is so for $\mathcal{V} = \mathfrak{Var}_k$, thus one gets the mod ℓ*-abelian-by-central* I/OM over arbitrary perfect fields k. Hence the mod ℓ*-abelian-by-central* form of classical I/OM holds.

Finally we notice that the birational form(s) of I/OM and $\widehat{GT}$ are proved by solving the so called *Bogomolov program* (BP) in the situations under discussion, see [P], Introduction. The BP is about reconstructing function fields $K = k(\overline{X})$ from $\Pi_K^{\rm c}$ for $\dim(X) > 1$, and it is essentially open. But under the supplementary information encoded in $\mathrm{Out}^{\rm c}_{\mathcal{V}}(\Pi_K)$, one can show that every $\Phi \in \mathrm{Out}^{\rm c}_{\mathcal{V}}(\Pi_K)$ originates from $\mathrm{Aut}(K)$ up to Frobenius twists, etc.

4. Line/Hyperplane Λ-abelian-by-central variants of $\widehat{GT}$

As remarked in subsection 3.1, the approaches to tackle I/OM (and its variants) are based on solving partially the Bogomolov program under the supplementary hypothesis of geometric nature of I/OM. In very recent work, Pop–Topaz [PT] introduced/defined so called (Λ-abelian-by-central) line/hyperplane variants of $\widehat{GT}$, which are not of birational nature, hence closer in nature to the original $\widehat{GT}$. On the other hand, one of the points to be stressed in the case of $\widehat{GT}$—as well as the groups $\mathbf{\Lambda}, \mathbb{I}\Gamma < \widehat{GT}$ from [HLS, NS] defined in connection with $\widehat{GT}$ —is that these groups are defined by *finitely many relations* (or equations) inside $\mathrm{Aut}(\widehat{F}_2)$, with those for $\mathbf{\Lambda}$ originating from the "first two levels" of the Teichmüller moduli tower. Although the line/hyperplane variants of $\widehat{GT}$ are much closer in nature to $\widehat{GT}$ than the birational variants of $\widehat{GT}$, the elements of the line/hyperplane variants of $\widehat{GT}$ have to satisfy *infinitely many "relations"* originating form the infinitely many lines and/or hyperplanes used in the definition of the corresponding line/hyperplane variants of $\widehat{GT}$. For the moment, it is unclear how/whether there are line/hyperplane variants of $\widehat{GT}$ which involve only finitely many line/hyperplane arrangements (and/or some moduli spaces like of such) defining some line/hyperplane variant of $\widehat{GT}$ which equals $G_{\mathbb{Q}}$.

4.1. **Complements of line and hyperplane arrangements.** To begin with, we notice/recall that the complements of line arrangements in $\mathbb{A}^2$, and more general, hyperplane arrangements in $\mathbb{A}^N$ are generalizations of $\mathcal{M}_{0,5}$, respectively of $\mathcal{M}_{0,n}$ for $N = n - 3 \geqslant 2$. Precisely, let x, y be the standard affine coordinates in $\mathbb{A}^2$, respectively $x_1, \dots, x_N$ be the standard coordinates in $\mathbb{A}^N$. Recalling that $\mathcal{M}_{0,n} = (\mathbb{P}^1\backslash\{0,1,\infty\})^N \backslash \Delta \subset \mathbb{A}^N$ with Δ the fat diagonal, $\mathcal{M}_{0,5} \subset \mathbb{A}^2$ is the complement of the line arrangement $\mathcal{L}_0 = V(f_0) \subset \mathbb{A}^2$ which is the zero set of $f_0 = x(1-x)y(1-y)(y-x) \in \mathbb{Q}[x,y]$; and in general, $\mathcal{M}_{0,n} \subset \mathbb{A}^N$ is the complement of the hyperplane arrangement $\mathcal{H}_0$ defined by the $2N + \frac{1}{2}N(N-1)$ hyperplanes $x_i = 0, 1 - x_i = 0, x_j - x_i = 0$ with $1 \leqslant i, j \leqslant N$ and $i < j$.

The study of (complements of) hyperplane arrangements is a classical research topic which is extremely active today, see e.g. the surveys/monographs/books/proceedings [AM, Di, CS, M, OT, S] for literature. A special class of line $\mathcal{L} \subset \mathbb{A}^2$, respectively hyperplane arrangements $\mathcal{H} \subset \mathbb{A}^N$, are the ones containing $\mathcal{L}_0$, respectively $\mathcal{H}_0$. Notice that these are *spectral* in the sense of Deligne, see e.g. [Pa] for details. In particular, setting $U_{\mathcal{L}} := \mathbb{A}^2\backslash\mathcal{L}$, respectively $U_{\mathcal{H}} := \mathbb{A}^{n-3}\backslash\mathcal{H}$, one has: $\mathcal{M}_{0,5} = U_{\mathcal{L}_0}$ and $\mathcal{M}_{0,n} = U_{\mathcal{H}_0}$ for $n \geqslant 5$. Concerning fundamental groups, $\overline{\pi}_1(U_{\mathcal{L}})$ and $\overline{\pi}_1(U_{\mathcal{H}})$ have well known presentations as successive semi-direct products of profinite free groups—generalizing among other things well known facts about the structure of $\widehat{\Gamma}_{0,n} = \widehat{K}(0,n)$, etc., see Paris [Pa].

4.2. **Line/Hyperplane $\widehat{GT}$.** The line/hyperplane variants of $\widehat{GT}$ are based on the category $\boldsymbol{\mathcal{L}}$ and its higher dimensional variant $\boldsymbol{\mathcal{H}}$, the former being the "line arrangements" variant of $\mathcal{V}_{0,5} := \mathcal{V}_0$ and $\mathcal{V}_{0,5}^{\mathrm{bir}} := \mathcal{V}_0^{\mathrm{bir}}$ considered above, whereas the latter is the higher dimensional "hyperplane arrangements" variant of $\mathcal{V}_{0,n} := \{\mathcal{M}_{0,n}, \mathcal{M}_{0,4}\}$ and its birational variant $\mathcal{V}_{0,n}^{\mathrm{bir}}$. In contrast to $\mathcal{V}_0^{\mathrm{bir}}$ and $\mathcal{V}_{0,n}^{\mathrm{bir}}$ and their generalizations $\mathcal{V}_X$, the categories $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{H}}$ are <u>not</u> of birational nature, being rather direct *line/hyperplane generalizations* of $\mathcal{V}_0$ and its higher dimensional variant $\mathcal{V}_{0,n}$, thus much closer in nature to $\mathcal{V}_{0,5}$ and $\mathcal{V}_{0,n}$, which define $\widehat{GT}$. As a cautionary note, we should mention the following: Let $N = n-3$ be the dimension of $\mathcal{M}_{0,n}$. Then setting $\boldsymbol{\mathcal{H}}_N := \boldsymbol{\mathcal{H}}$, one obviously has $\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{H}}_2$. On the other hand, besides this obvious formal fact, we do not see at the moment a way to relate the $\boldsymbol{\mathcal{H}}_N$ to each other for various dimensions $N = n-3$. In particular, we do not know whether/how the answer to Question 4 in section 5 might depend on the dimension $N = n - 3$.

4.2.1. *The category $\boldsymbol{\mathcal{L}}$ and $G_{\mathbb{Q}} = \widehat{GT}^{\mathrm{c}}_{\boldsymbol{\mathcal{L}}}$.* The objects of $\boldsymbol{\mathcal{L}}$ are $U_0 := \mathbb{P}^1\backslash\{0,1,\infty\}$ and the complements $U_{\mathcal{L}}$ of the $\mathbb{Q}$-rational line arrangements $\mathcal{L} \subset \mathbb{A}^2$, and the morphisms are the canonical inclusions $U_{\mathcal{L}''} \subset U_{\mathcal{L}'}$ for $\mathcal{L}' \subset \mathcal{L}''$ together with the projections $p_t : U_{\mathcal{L}} \to U_0$

defined by the three projections $p_t : \mathcal{M}_{0,5} \to U_0$, $t_0 \mapsto t \in \Sigma_0 := \{x, y, y-x\}$. Obviously one has $\boldsymbol{\mathcal{L}} \subset \mathcal{V}_0^{\rm bir}$ strictly, because for $f \in \mathbb{Q}[x,y]$ one has that $U_f \in \boldsymbol{\mathcal{L}}$ iff f splits in linear factors over $\overline{\mathbb{Q}}$. Recalling the Λ-abelian-by-central fundamental group $\Pi_X^{\rm c}$ and $\mathrm{Aut}^c(\Pi_{\boldsymbol{\mathcal{L}}})$, one has the following Λ-*abelian-by-central line* variant of $\widehat{GT}$, see [PT]:

- *The representation* $\rho_{\boldsymbol{\mathcal{L}}}^{\rm c} : G_{\mathbb{Q}} \to \widehat{GT}_{\boldsymbol{\mathcal{L}}}^{\rm c} := \mathrm{Aut}^{\rm c}(\Pi_{\boldsymbol{\mathcal{L}}})$ *is an isomorphism.*

Actually, as in the birational case of $\mathcal{V}_X$, the result proved in [PT] is much more general, and holds over arbitrary base (perfect) fields k, by defining $\boldsymbol{\mathcal{L}} := \boldsymbol{\mathcal{L}}_S$ as follows: Let $S \subset k$ with $0 \in S$ be a system of generators of k over its prime field. Define $\boldsymbol{\mathcal{L}}_S$ to have as objects all the complements $U_{\mathcal{L}} \subset \mathbb{A}_k^2$ of the k-rational line arrangements $\mathcal{L}$ which contain $\mathcal{L}_0$ together with the lines $x = s, y = s$, $s \in S$, and as morphisms the inclusions $U_{\mathcal{L}''} \subset U_{\mathcal{L}'}$ for $\mathcal{L}' \subset \mathcal{L}''$ and the projections $p_t : U_{\mathcal{L}} \to U_0$ defined by $t_0 \mapsto t \in \Sigma_S = \Sigma_0 \cup \{x-s, y-s \,|\, s \in S\}$, provided $\mathcal{L}$ is large enough, so that p_t is defined. Then $\rho_{\boldsymbol{\mathcal{L}}_S}^{\rm c} : G_k \to \mathrm{Aut}^{\rm c}(\Pi_{\boldsymbol{\mathcal{L}}_S})$ is an isomorphism.

4.2.2. *The category* $\boldsymbol{\mathcal{H}}$ *and* $G_{\mathbb{Q}} = \widehat{GT}_{\boldsymbol{\mathcal{H}}}^{\rm c}$. The objects of $\boldsymbol{\mathcal{H}}$ are $U_0 = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and $U_{\mathcal{H}}$ for all $\mathbb{Q}$-rational hyperplane arrangements $\mathcal{H} \subset \mathbb{A}^N$, and the morphisms are the canonical inclusions $U_{\mathcal{H}''} \subset U_{\mathcal{H}'}$ for $\mathcal{H}' \subset \mathcal{H}''$ together with all the projections $p_t : U_{\mathcal{H}} \to U_0$ defined by $p_t : \mathcal{M}_{0,n} \to U_0$, $t_0 \mapsto t \in \Sigma_{\boldsymbol{\mathcal{H}}} := \{x_i, y_j - x_i\}_{i,j}$ with $1 \leqslant i, j \leqslant N$ and $i < j$. As in the case of $\boldsymbol{\mathcal{L}}$, one has $\boldsymbol{\mathcal{H}} \subset \mathcal{V}_{0,n}^{\rm bir}$ strictly, because for $f \in \mathbb{Q}[x_1, \dots, x_N]$, one has: $U_f \in \boldsymbol{\mathcal{H}}$ iff f splits in linear factors over $\overline{\mathbb{Q}}$. Finally, the Λ-*abelian-by-central hyperplane* $\widehat{GT}_{\boldsymbol{\mathcal{H}}}^{\rm c}$ satisfies:

- *The representation* $\rho_{\boldsymbol{\mathcal{H}}}^{\rm c} : G_{\mathbb{Q}} \to \widehat{GT}_{\boldsymbol{\mathcal{H}}}^{\rm c} := \mathrm{Aut}^{\rm c}(\Pi_{\boldsymbol{\mathcal{H}}})$ *is an isomorphism.*

Actually, as in case of $\boldsymbol{\mathcal{L}}$, the result proved in [PT] is much more general, and holds over arbitrary base (perfect) fields k, by defining $\boldsymbol{\mathcal{H}} := \boldsymbol{\mathcal{H}}_S$ as follows: Let $S \subset k$ be a system of generators containing $0 \in k$ over its prime field. Define $\boldsymbol{\mathcal{H}}_S$ to have as objects all the complements $U_{\mathcal{H}} \subset \mathbb{A}_k^N$ of the k-rational line arrangements $\mathcal{H}$ which contain $\mathcal{H}_0$ together with the hyperplanes $s - x_i = 0$, $1 \leqslant i \leqslant N$, $s \in S$, and as morphisms the canonical inclusions $U_{\mathcal{H}''} \subset U_{\mathcal{H}'}$ for $\mathcal{H}' \subset \mathcal{H}''$ together with the projections $p_t : U_{\mathcal{H}} \to U_0$ define by $t_0 \mapsto t \in \Sigma_S = \Sigma_{\boldsymbol{\mathcal{H}}} \cup \{s - x_i \,|\, 1 \leqslant i \leqslant N, s \in S\}$, provided $\mathcal{L}$ is large enough, so that p_t is defined. Then $\rho_{\boldsymbol{\mathcal{H}}_S}^{\rm c} : G_k \to \mathrm{Aut}^{\rm c}(\Pi_{\boldsymbol{\mathcal{H}}_S})$ is an isomorphism.

Finally, concerning the proofs, recall that the methods developed to tackle the classical I/OM are based on solving Bogomolov Program (BP) for $K = k(\overline{X})$ using the extra information encoded in $\mathrm{Out}_{\mathcal{V}_X}^{\rm c}(\Pi_K)$. Obviously that information (and the category$\mathcal{V}_X$) are of birational nature. On the other hand, both categories $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{H}}$ are obviously not of birational nature, thus so are the corresponding automorphism groups $\mathrm{Aut}^{\rm c}(\Pi_{\bullet})$. Therefore some new methods are needed to tackle the problem. In a nutshell, given $\Pi_{\bullet}^{\rm c}$, one recovers the lines and the colineations in $\mathbb{A}^2$ in the case of $\boldsymbol{\mathcal{L}}$, respectively the planes and plane incidence in $\mathbb{A}^N$ in the case of $\boldsymbol{\mathcal{H}}$. One concludes the proofs in a way similar to the birational case, by invoking the Fundamental Theorem of Projective Geometries, see Artin [Ar].

5. A few Open Questions

There are many open questions concerning $\widehat{GT}$, the most important ones being whether $G_{\mathbb{Q}} = \widehat{GT}$, respectively whether $G_{\mathbb{Q}} \cong \widehat{GT}$. Below I mention a few open questions directly relating to the themes discussed in this short survey.

1) Does $\widehat{GT}$ embed into $\mathrm{Aut}(\overline{\pi}_{\mathcal{T}})$, and if so, is $\widehat{GT}$ equal to $\mathrm{Aut}(\overline{\pi}_{\mathcal{T}})$?
2) For $\mathbb{I}\Gamma$ as defined in [NS], is there some subcategory $\mathcal{V} \subset \mathfrak{Var}_{\mathbb{Q}}$ such that $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}}) = \mathbb{I}\Gamma$?
2) Does the mod ℓ-abelian-by-central I/OM hold for $\mathcal{V}_0^{\mathrm{bir}}$? If not, what are the "canonical minimal" categories for which the mod ℓ-abelian-by-central I/OM holds?
3) Are there Λ-abelian-by-central line/hyperplane variants of $\widehat{GT}$ which involve a bounded number of lines and/or hyperplanes only, and if so, how do those related to Galois groups?
4) Does the full-profinite line $\widehat{GT}_{\boldsymbol{\mathcal{L}}}$ and/or hyperplane $\widehat{GT}_{\boldsymbol{\mathcal{H}}}$ equal $G_{\mathbb{Q}}$?
5) Do prolinear/prounipontent variants I/OM hold, and if so, what is their significance for Galois theory and/or studying multi-zeta and/or polylogs?

6. Appendix: Notation/Basics

6.1. **The categories $\boldsymbol{\mathcal{G}}_\Gamma$ and $\boldsymbol{\mathcal{G}}_\Gamma^{\mathrm{out}}$.** For a fixed group Γ, let $\mathbf{Group}_\Gamma$ be the category of groups above Γ, and for $\varphi_G : G \to \Gamma$ in $\mathbf{Group}_\Gamma$, let $\overline{G} := \ker(\varphi_G)$ be the "geometric part" of G. Further let $\mathbf{Group}_\Gamma^{\mathrm{out}}$ be the category having the same objects as $\mathbf{Group}_\Gamma$, and as morphisms the *outer Γ-morphisms,* i.e., $\mathrm{Hom}_{\mathbf{Group}_\Gamma^{\mathrm{out}}}(G,H) := \mathrm{Hom}_{\mathbf{Group}_\Gamma}(G,H)/\!\sim$, where $f \sim g$, provided $f \circ \mathrm{Isom}(\overline{H}) = g \circ \mathrm{Isom}(\overline{H})$. Notice that every group G can be viewed as an object in $\mathbf{Group}_\Gamma$ via the trivial morphism $G \to \Gamma$. In particular, $\mathbf{Group}_1 = \mathbf{Groups}$ can be embedded in $\mathbf{Group}_\Gamma$, and correspondingly, $\mathbf{Group}^{\mathrm{out}} := \mathbf{Group}_1^{\mathrm{out}}$ is the category of groups with *outer morphisms.* Further, if $\Gamma_G := \varphi_G(G)$, the exact sequence $1 \to \overline{G} \to G \to \Gamma_G \to 1$ gives rise to a "representation" $\rho_G : \Gamma_G \to \mathrm{Out}(\overline{G})$, which is functorial in G in the sense that given $\phi : G \to H$, the induced map $\overline{\phi} : \overline{G} \to \overline{H}$ satisfies $\rho_H(\sigma) = \overline{\phi} \circ \rho_G(\sigma)\ \forall\, \sigma \in \Gamma_G$.

Next suppose that the groups under discussion (including Γ) are topological groups, e.g. profinite groups. Since inner conjugation in topological groups is a topological automorphism, the categories $\mathbf{Group}_\Gamma^{\mathrm{out}}$ and $\mathbf{Group}^{\mathrm{out}}$ are defined for the category of topological groups, e.g. profinite groups. And if $\overline{G} \in \mathbf{Group}^{\mathrm{out}}$ has finite corank (i.e., for every $N > 0$ there are only finitely many open subgroups $G' < \overline{G}$ of index N), e.g. $\overline{G}$ is topologically finitely generated, then $\mathrm{Out}(\overline{G})$ is profinite and topologically finitely generated. In particular, by mere definitions it follows that the representation $\rho_G : \Gamma_G \to \mathrm{Out}(\overline{G})$ is continuous.

Finally let $\boldsymbol{\mathcal{G}}_\Gamma$ be the full subcategory of $\mathbf{Group}_\Gamma$ consisting of *surjective objects,* i.e., $\varphi_G : G \to \Gamma$ is onto, and the corresponding full subcategory $\boldsymbol{\mathcal{G}}_\Gamma^{\mathrm{out}}$ of $\mathbf{Group}_\Gamma^{\mathrm{out}}$. In particular, for every $G \to \Gamma$ in $\boldsymbol{\mathcal{G}}_\Gamma^{\mathrm{out}}$, one has canonical representations $\rho_G : \Gamma \to \mathrm{Out}(\overline{G})$, and these representations are compatible with morphisms $\boldsymbol{\mathcal{G}}_\Gamma^{\mathrm{out}}$. Thus we get a "representation"

$$\rho_{\boldsymbol{\mathcal{G}}_\Gamma} : \Gamma \to \mathrm{Aut}(\boldsymbol{\mathcal{G}}_\Gamma^{\mathrm{out}}).$$

6.2. **(Algebraic) étale fundamental group.** For a base field k, e.g. $k = \mathbb{Q}$, let $\overline{k}|k$ denote some fixed algebraic closure of k, and $k^{\mathrm{s}}|k$ be the separable closure of k in $\overline{k}$. In particular, $G_k = \mathrm{Aut}_k(k^{\mathrm{s}}) = \mathrm{Aut}_k(\overline{k})$ denotes the absolute Galois group of k. Let $\mathfrak{Var}_k$ be the category of geometrically integral k-varieties, and for $X \in \mathfrak{Var}_k$, let $\overline{X} := X \times_k \overline{k}$ be the base change of X under $\overline{k}|k$. In particular, every morphism $f : X \to Y$ in $\mathfrak{Var}_k$ gives rise to its base change $\overline{f} : \overline{X} \to \overline{Y}$. By the theory of étale fundamental groups, the following hold:

- First, for every geometric point $\overline{x} \in X(\overline{k})$ as above, one has the canonical exact sequence

$$1 \to \pi_1(\overline{X}, \overline{x}) \to \pi_1(X, \overline{x}) \to G_k \to 1,$$

in particular, $\pi_1(X, \overline{x}) \in \boldsymbol{\mathcal{G}}_{G_k}$. Moreover, if $\overline{x}' \in X(\overline{k})$ is another geometric point of X, and $\imath : \overline{x} \to \overline{x}'$ is the path from $\overline{x}$ to $\overline{x}'$, then $\imath$ identifies $\pi_1(\overline{X}, \overline{x})$ with $\pi_1(\overline{X}, \overline{x}')$ up to inner conjugation inside $\pi_1(\overline{X}, \overline{x})$. In particular, viewing/considering $\pi_1(\overline{X}, \overline{x})$ and $\pi_1(\overline{X}, \overline{x}')$ as objects in $\boldsymbol{\mathcal{G}}^{\text{out}}_{G_k}$, one has that $\pi_1(\overline{X}, \overline{x}), \pi_1(\overline{X}, \overline{x}') \in \mathbf{Group}^{\text{out}}_{G_k}$ are canonically identified. We will view $\pi_1(\overline{X}, \overline{x})$ as an object of $\mathbf{Group}^{\text{out}}_{G_k}$, and setting $\overline{\pi}_1(X) := \pi_1(\overline{X}, \overline{x})$, we call it the *geometric fundamental group* of X. Finally, by the discussion in subsection 2.1 above, the exact sequence above gives rise to the representation $\rho_X := \rho_{\pi_1(X,\overline{x})}$ below, which turns out to be always a continuous morphism of profinite groups

$$\rho_X : G_k \to \operatorname{Out}\big(\overline{\pi}_1(X)\big) = \operatorname{Aut}_{\boldsymbol{\mathcal{G}}^{\text{out}}}(\overline{\pi}_1(X)).$$

- Second, let $f : X \to Y$ be a morphism in $\mathfrak{Var}_k$, $\overline{f} : \overline{X} \to \overline{Y}$ be the induced morphism, and $\overline{y} = f(\overline{x})$. Then f gives rise functorially to the commutative diagram below:

$$\begin{array}{ccccccc} 1 \to & \overline{\pi}_1(X, \overline{x}) & \to & \pi_1(X, \overline{x}) & \to & G_k & \to 1 \\ & \downarrow \scriptstyle{\overline{\pi}_1(\overline{f})} & & \downarrow \scriptstyle{\pi_1(f)} & & \| & \\ 1 \to & \overline{\pi}_1(Y, \overline{y}) & \to & \pi_1(Y, \overline{y}) & \to & G_k & \to 1 \end{array}$$

In particular, the representations $\rho_X : G_k \to \operatorname{Out}\big(\overline{\pi}_1(X)\big)$, $X \in \mathfrak{Var}_k$ are compatible with morphisms $f : X \to Y$, i.e., $\rho_Y = \overline{\pi}_1(\overline{f}) \circ \rho_X$. Hence by the discussion in subsection 2.1 above and mere definitions, one gets a representation:

$$\rho_k : G_k \to \operatorname{Aut}(\overline{\pi}_{\mathfrak{Var}_k}), \quad \sigma \mapsto \big(\rho_X(\sigma)\big)_X.$$

- Third, for $\overline{y} = f(\overline{x})$, let $\overline{X}_{\overline{y}} \subset X$ be the geometric fiber of $f : X \to Y$ above $\overline{y}$, and suppose that the $\overline{X}_{\overline{y}}$ is integral. Then one has an exact sequence:

$$\overline{\pi}_1(\overline{X}_{\overline{y}}) \to \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y}) \to 1,$$

which in many situations of interest fits into a short exact sequence, see the discussion below.

Next let $k \subset \overline{k} \subset \mathbb{C}$, and $\mathfrak{X} := X(\mathbb{C})$ be the corresponding complex analytic space. Then $\mathfrak{X}$ is a nice topological space, and $\overline{\pi}_1(X)$ equals the profinite completion of the topological fundamental group $\pi_1^{\text{top}}(\mathfrak{X}, *) \in \mathbf{Group}^{\text{out}}$. In particular, if $\pi_1^{\text{top}}(\mathfrak{X}, *)$ has a well known/understood structure as a discrete group, its profinite completion $\overline{\pi}_1(X)$ is known as well. Examples of this instance which are significant in our context here are: The fundamental groups of smooth curves; the fundamental group of configuration spaces, and of the moduli spaces of pointed curves; the fundamental groups the complements of line arrangements in $\mathbb{P}^2$ and more general, of complements of hyperplane arrangements in $\mathbb{P}^N$. For reader's sake we briefly review the well known facts and introduce the relevant notation.

6.2.1. *Curves of type* (g, r). A curve of *type* (g, r) over k is a smooth curve $X \in \mathfrak{Var}_k$ which has a smooth completion $\widehat{X}$ of genus genus g such that $\widehat{X} \backslash X$ consists of $r \geqslant 0$ geometric points. We will usually (tacitly) assume that $2g - 2 + r > 0$. Then $\mathfrak{X} = X(\mathbb{C}) \subset \widehat{X}\mathbb{C}) = \widehat{\mathfrak{X}}$ are Riemann surfaces, and $\pi_1^{\text{top}}(\mathfrak{X}, *) \in \boldsymbol{\mathcal{G}}^{\text{out}}_1$ is

$$\Pi_{g,r} := \pi_1^{\text{top}}(\mathfrak{X}) = \big\langle \alpha_1, \beta_1, \ldots \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_r \bigm| \textstyle\prod_i [\alpha_i, \beta_i] \prod_j \gamma_j = 1 \big\rangle.$$

It is well known that $\Pi_{g,n}$ is *residually finite,* i.e., it embeds into its profinite completion, and therefore, $\widehat{\Pi}_{g,n} = \overline{\pi}_1(X) = \widehat{\pi}_1^{\text{top}}(\mathfrak{X})$ depends on g, r only. Further, if $r > 0$, then $\overline{\pi}_1(X)$

is the free profinite group on $2g+r-1$ generators. In particular, if $g=0$, i.e., $\widehat{X}=\mathbb{P}^1_k$, then $\pi_1^{\rm top}(\mathfrak{X})=\langle\gamma_1,\dots,\gamma_r\,|\,\gamma_1\cdots\gamma_r=1\rangle$ is the free discrete group on $r-1$ generators $\gamma_1,\dots,\gamma_{r-1}$. Hence $\overline{\pi}_1(X)=\widehat{F}_{r-1}$ is the profinite free group on $\gamma_1,\dots,\gamma_{r-1}$.

6.2.2. *Configuration spaces and moduli spaces of curves.* For $X\subset\widehat{X}$ as above, the configuration space of systems of n distinct geometric points of X is parametrized by $X_n:=X^n\backslash\Delta$, where Δ is the fat diagonal, hence $X_n(\overline{k})=X^n(\overline{k})\backslash\Delta(\overline{k})$. Further, for $0<m<n$ and $I=\{i_1,\dots,i_m\}$ with $i_\nu<i_{\nu+1}$, the $I^{\rm th}$ projection $p_I:X_n\to X_m$ is surjective, and p_I is defined on $X(\overline{k})$ by $\overline{\mathbf{x}}:=(x_1,\dots,x_n)\mapsto(x_{i_1},\dots,x_{i_m})=:\overline{\mathbf{x}}_m$. In particular, the geometric fiber of $p_I:X_n\to X_m$ at $\overline{\mathbf{x}}_m$ is $\overline{k}$-isomorphic to $\overline{Y}_{n-m}$, where $\overline{Y}=\overline{X}\backslash\{x_{i_1},\dots,x_{i_m}\}$, thus $\overline{Y}\subset\widehat{X}$ is a $(g,r+m)$ curve. Further, if $\{x_{i_1},\dots,x_{i_m}\}\subset\widehat{X}$ is a closed subset defined over k, then $\overline{Y}$ is defined over k. Concerning fundamental groups, we notice that the corresponding topological groups $\Pi_{g,r;\,n}:=\pi_1^{\rm top}(\mathfrak{X}_n,*)$ are in principle known and finitely generated, and their structure depends on (g,r) and n only. Hence the profinite completion $\widehat{\Pi}_{g,r;\,n}$, which is the geometric fundamental group $\pi_1(\overline{X}_n,\overline{\mathbf{x}}_n)=\widehat{\Pi}_{g,r;\,n}$, is topologically finitely generated and has a structure which is in principle known. Further, the projection $p_I:X_n\to X_m$ defines a surjective projection of étale fundamental groups $\pi_1(X_n,\overline{\mathbf{x}}_n)\to\pi_1(X_m,\overline{\mathbf{x}}_m)$. Moreover, if $2g-2+r>0$, then $\pi_2^{\rm top}(\mathfrak{X}_m)=1$, hence the short exact fiber homotopy exact sequence gives rise by completion to an exact sequence of geometric fundamental groups

$$1\to\overline{\pi}_1(\overline{Y})\to\pi_1(\overline{X}_n,\overline{\mathbf{x}}_n)\to\pi_1(\overline{X}_m,\overline{\mathbf{x}}_m)\to 1.$$

Finally, if $\overline{Y}=\overline{X}\backslash\{x_{i_1},\dots,x_{i_m}\}$ is defined over k, the above sequence is the geometric part of

$$1\to\overline{\pi}_1(\overline{Y})\to\pi_1(X_n,\overline{\mathbf{x}}_n)\to\pi_1(X_m,\overline{\mathbf{x}}_m)\to 1,$$

hence by the general discussion above, one has canonical "representations"

$$\rho_{\overline{X},n}:\pi_1(\overline{X}_m,\overline{\mathbf{x}}_m)\to\operatorname{Out}\big(\overline{\pi}_1(\overline{Y})\big),\quad \rho_{X,n}:\pi_1(X_m,\overline{\mathbf{x}}_m)\to\operatorname{Out}\big(\overline{\pi}_1(\overline{Y})\big).$$

Parallel to the configuration spaces X_n, one considers the *moduli stacks* $\mathcal{M}_{g,n}$ of n-pointed genus g projective smooth curves $\widehat{X}$. The moduli stacks $\mathcal{M}_{g,n}$ under discussion are smooth and defined over $\mathbb{Q}$. Although $\mathcal{M}_{g,n}$ are not schemes in general, by Oda [O], one can define the "fundamental group" $\pi_1(\mathcal{M}_{g.n},\overline{\mathbf{x}})$, and its "algebraic part" turns out to be the profinite completion $\widehat{\Gamma}_{g,n}$ of the mapping class group $\widehat{\Gamma}_{g,n}$. Further, one has a canonical exact sequence of the form $1\to\widehat{\Gamma}_{g,n}\to\pi_1(\mathcal{M}_{g,n},\overline{\mathbf{x}})\to G_{\mathbb{Q}}\to 1$. The full Teichmüller moduli tower $\mathcal{T}$ is the category with objects $\mathcal{M}_{g,n}$ and all "natural $\mathbb{Q}$-morphisms" between its objects. The (algebraic) fundamental group Teichmüller tower is the set of (profinite) mapping class groups $\{\widehat{\Gamma}_{g,n}\}_{g,n}$ endowed with "canonical morphisms originating from geometry," see the discussion in Hatcher–Lochak–Schneps [HLS] for more about this. For instance, given any $0\leqslant m\leqslant n$ and $g\geqslant 0$ such that $2g-2+m>1$, for every $I\subset\{1,\dots,n\}$ with $|I|=m$, one has a canonical morphisms of stacks $\mathcal{M}_{g,n}\to\mathcal{M}_{g,m}$ by "forgetting" the marked points indexed by $i\notin I$. By Knudsen [Kn], the projection $\mathcal{M}_{g,n+1}\to\mathcal{M}_{g,n}$ renders $\mathcal{M}_{g,n+1}$ canonically isomorphic to the universal n-pointed genus g curve. In particular, if $\overline{\mathbf{x}}\in\mathcal{M}_{g,n}(\overline{k})$, then the fiber $\overline{X}_{g,n}:=\mathcal{M}_{g,n;\overline{\mathbf{x}}}$ of $\mathcal{M}_{g,n+1}\to\mathcal{M}_{g,n}$ above $\overline{\mathbf{x}}$ is a $\overline{k}$-curve of type (g,n), thus giving rise to an exact sequence:

$$1\to\widehat{\Pi}_{g,n}\to\pi_1(\mathcal{M}_{g,n+1},\overline{\mathbf{x}}_{n+1})\to\pi_1(\mathcal{M}_{g,n},\overline{\mathbf{x}})\to 1.$$

Hence by the general discussion above, one gets a "representation"

$$\rho_{g,n} : \pi_1(\mathcal{M}_{g,n}, \overline{\mathbf{x}}) \to \mathrm{Out}(\widehat{\Pi}_{g,n}).$$

A quite notable special case of this is the case $g = 0$, i.e., $X = \widehat{X} = \mathbb{P}^1$. Then $X_n = \mathcal{M}_{0,n}$ is the *moduli space of curves of genus* $g = 0$ *with* n *marked points.* Since $\mathrm{Aut}_k(\mathbb{P}^1)$ acts simply transitively on ordered systems of three points, it follows that $X_n = \{\eta_0\} = \mathcal{M}_{0,n}$ for $n \leqslant 3$, $X_4 = \mathbb{P}^1 \backslash \{0, 1, \infty\} = \mathcal{M}_{0,4}$, and in general, $X_n = (\mathbb{P}^1 \backslash \{0, 1, \infty\})^{n-3} \backslash \Delta = \mathcal{M}_{0,n}$ for $n > 3$. Finally, $\overline{\pi}_1(\mathcal{M}_{0,n}) = \widehat{\Gamma}_{0,n}$, which is also denoted by $\overline{\pi}_1(\mathcal{M}_{0,n}) = \widehat{K}(0, n)$ by many authors, is the profinite completion of the *pure mapping class group* $\Gamma_{0,n} = K(0, n)$. The latter has canonical generators x_{ij}, $1 \leqslant i < j \leqslant n$, satisfying well known relations. Hence for $X_{0,n} \subset \mathbb{P}^1$, $n > 3$, one has $\overline{\pi}_1(\overline{X}_{0,n}) = \widehat{\Pi}_{0,n} \cong \widehat{F}_{n-1}$, and one gets canonical exact sequences:

$$1 \to \widehat{\Pi}_{0,n} \to \widehat{\Gamma}_{0,n+1} \to \widehat{\Gamma}_{0,n} \to 1, \quad 1 \to \widehat{\Gamma}_{0,n} \to \pi_1(\mathcal{M}_{0,n}, \overline{\mathbf{x}}) \to G_{\mathbb{Q}} \to 1.$$

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