ON A CONJECTURE OF COLLIOT-THÉLÈNE

FLORIAN POP

Abstract. In this note we extend results by Denef and Loughran, Skorobogatov, Smeets concerning a conjecture of Colliot-Thélène. The question is about giving necessary and sufficient birational conditions for morphisms of varieties to be surjective on local points for almost all localizations of the base field.

1. Introduction/Motivation

The aim of this note is to shed new light on a conjecture by Colliot-Thélène, cf. [CT], concerning the image of local rational points under dominant morphisms of varieties over global fields (and beyond). The precise context is as follows:

- Let k be a global field, $\mathbb{P}(k)$ be the places of k, and k_v be the completion of k at $v \in \mathbb{P}(k)$.
- Let $f : X \to Y$ be a morphism of k-varieties, $X(k_v)$, $Y(k_v)$ the k_v-rational points.

For every $v \in \mathbb{P}(k)$, the k-morphism f gives rise to a canonical map $f^{k_v} : X(k_v) \to Y(k_v)$. There are obvious examples showing that, in general, f^{k_v} is not surjective, e.g. $f : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ of degree two. Therefore, for $f : X \to Y$ as above, it is natural to consider the basic property:

(Srj) $\qquad f^{k_v} : X(k_v) \to Y(k_v)$ *is surjective for almost all* $v \in \mathbb{P}(k)$.

and to ask the following fundamental:

Question: *Give necessary and sufficient conditions for $f : X \to Y$ to have property* (Srj).

This problem was considered in a systematic way by Colliot-Thélène [CT], under the following restrictive but to some extent natural hypothesis:

$(*)_{\mathsf{CT}}$ $\qquad$ *k is a number field, X and Y are projective smooth integral k-varieties, and $f : X \to Y$ is a dominant morphism with geometrically integral generic fiber.*

In particular, if $L := k(Y)$ is the function field of Y, the generic fiber X_L of f can be viewed as an L-variety. In this notation, for morphisms $f : X \to Y$ satisfying $(*)_{\mathsf{CT}}$, Colliot-Thélène considered the hypothesis (CT) and made the conjecture (CCT) below:

(CT) $\qquad$ *For each discrete valuation k-ring $R \subset L$, and its residue field κ_R, there is a regular flat R-model $\mathfrak{X}_R$ of X_L whose special fiber $\mathfrak{X}_{\kappa_R}$ has an irreducible component $\mathfrak{X}_\alpha$ which is κ_R-geometrically integral.*

Date: Variant of April 15, 2021.

Key words and phrases. Rational points of varieties, localizations of global fields, function fields, valuations and prime divisors, Galois theory, Ax–Kochen–Ershov Principle, ultraproducts, pseudo finite fields.

2010 MSC. Primary: 11Gxx, 11Uxx 14Dxx, 14Gxx.

NOT funded by the NSF: The ANT research directors (Douglas, Hodges, Libgober and Pollington), in their higher understanding of this matter, overruled the expert reviewers and panelists recommendation and decided not to fund this research.

Conjecture of Colliot-Thélène (CCT)**.** *Let $f : X \to Y$ be a dominant morphism of proper smooth geometrically integral varieties over a number field k, and suppose that hypotheses $(*)_{\mathsf{CT}}$ and* (CT) *are satisfied. Then $f : X \to Y$ has the property* (Srj).

In a recent paper, Denef [Df2] proved a stronger form of the conjecture (CCT), by replacing the hypothesis (CT) by the weaker hypothesis (D) below. In order to explain Denef's result, recall the following terminology: Let $f : X \to Y$ be a morphism satisfying hypothesis $(*)_{\mathsf{CT}}$. A *smooth modification* of f is any morphism $f' : X' \to Y'$ satisfying hypothesis $(*)_{\mathsf{CT}}$ such that there exist modifications (i.e., birational morphisms) $p : X' \to X$, $q : Y' \to Y$ satisfying $q \circ f' = f \circ p$. Given a smooth modification $f' : X' \to Y'$ of f, for every Weil prime divisor $E' \subset Y'$, and the Weil prime divisors D' of X' above E', consider: First, the multiplicity $e(D'|E')$ of D' in $f'^*(E') \in \mathrm{Div}(X')$; second, the restriction $f'_{D'} : D' \to E'$ of f' to $D' \subset X'$, which is a morphism of integral k-varieties. For $f : X \to Y$ satisfying $(*)_{\mathsf{CT}}$, it turns out that the hypothesis (CT) above implies that following obviously weaker hypothesis:

(D) *For all smooth modifications f' and every $E' \in \mathrm{Div}(Y')$ prime, there is D' above E' with $e(D'|E) = 1$ and $f'_{D'} : D' \to E'$ having geometrically integral generic fiber.*

Theorem (Denef [Df2], Main Theorem 1.2)**.**
Let $f : X \to Y$ satisfy the hypotheses $(*)_{\mathsf{CT}}$ and (D)*. Then f has the property* (Srj).

Finally we recall the very recent results by Loughran–Skorobogatov–Smeets [LSS] which, for morphisms $f : X \to Y$ satisfying the hypothesis $(*)_{\mathsf{CT}}$ above, give *necessary and sufficient conditions* such that $f : X \to Y$ has property (Srj). Namely, following [LSS], in the notation introduced above, let $f' : X' \to Y'$ be a smooth modification of $f : X \to Y$. For a Weil prime divisor E' of Y' and a Weil prime divisor D' of X' above E', let $k(D') \,|\, k(E')$ be the function field extension defined by the dominant map $f'_{D'} : D' \to E'$. One says that E' is *pseudo-split* under $f' : X' \to Y'$, if for every element of the absolute Galois group $\sigma \in G_{k(E')}$, there is some Weil prime divisor D' of X' above E' satisfying:

$$e(D'|E') = 1 \ \text{ and } \ k(D') \otimes_{k(E')} \overline{k(E')} \text{ has a factor stabilized by } \sigma.$$

Following Loughran–Skorobogatov–Smeets [LSS], consider the hypothesis:

(LSS) *For all smooth modifications f' of f, all Weil prime divisors $E' \subset Y'$ are pseudo-split.*

Note that if D', E' satisfy hypothesis (D), then $k(D') \,|\, k(E')$ is a regular field extension, hence $k(D') \otimes_{k(E')} \overline{k(E')}$ is a field stabilized by all $\sigma \in G_{\kappa_{E'}}$ (and E' is called *split*). Hence hypothesis (D) implies (LSS), leading to the following sharpening of Denef's result above:

Theorem (Loughran–Skorobogatov–Smeets [LSS], Theorem 1.4)**.**
Let $f : X \to Y$ satisfy $(*)_{\mathsf{CT}}$. Then f satisfies hypothesis (LSS) *iff f has property* (Srj).

About this paper. In this note we provide a different approach to the basic problem (CCT) considered above, and using completely different techniques, we give wide generalizations of the results from [Df2], [LSS], see e.g. Theorems 1.1 and Theorem 1.2 below. The context and form in which these results hold and will be proved is as follows.

• In stead of number fields, we will consider base fields k satisfying the hypothesis $(\mathsf{H})_k$ below, and consider the corresponding generalization $(\mathsf{Srj})_k$ of the property (Srj).

$(\mathsf{H})_k$ *k is* (i) *finitely generated, or* (ii) *finitely generated over a pseudo-finite field k_0.*[1]

[1] Recall that pseudo-finite field is a perfect PAC field with pro-cyclic free absolute Galois group.

Let $\mathbb{P}(k)$ denote the set of *discrete valuations* v of k having residue field kv finite in case (i), respectively finite over k_0 in case (ii). Recall that a model of k is any separated integral scheme S of finite type with function field $\kappa(S) = k$ in case (i), respectively an integral k_0-variety S with function field $k = k_0(S)$ in case (ii). For every model S of k we denote:

$$\mathbb{P}_S(k) := \{v \in \mathbb{P}(k) \,|\, v \text{ has a center } x_v \in S\}.$$

In particular, x_v must be a closed point of S, and conversely, for every closed point $s \in S$ there are valuations $v_x \in \mathbb{P}_S(k)$ having center x on S. Further we notice: First, since any models S_1 and S_2 are birationally equivalent, there is a model S which has open embeddings $S \hookrightarrow S_1$ and $S \hookrightarrow S_2$, hence $\mathbb{P}_S(k) \subset \mathbb{P}_{S_1}(k), \mathbb{P}_{S_2}(k)$. Second, $S_{\mathrm{reg}} \subset S$ is Zariski open dense, and for $x \in S_{\mathrm{reg}}$ there are $v \in \mathbb{P}(k)$ with $x_v = x$ and $kv = \kappa(x)$. In particular one has:

(†) $\quad \mathcal{P}_k := \{\, \mathbb{P}_S(k) \,|\, S \text{ is regular model of } k\,\}$ is a prefilter on $\mathbb{P}(k)$ of k.

Recall that if k is a global field, then k has a unique proper regular model S_0, and $\mathbb{P}(k)$ is in bijection with the closed points of S_0 via $\mathcal{O}_s = \mathcal{O}_v$ with $s \in S_0$ closed, $v \in \mathbb{P}_S(k)$. Therefore, $\mathbb{P}_S(k) \subset \mathbb{P}(k)$ is always cofinite if k is a global field. This being said, the natural generalization the property (Srj) is:

$(\mathsf{Srj})_k$ $\quad$ *k has a model S such that $f^{k_v} : X(k_v) \to Y(k_v)$ is surjective for all $v \in \mathbb{P}_S(k)$.*

We next give the (fully) birational form of the pseudo-splitness hypothesis (LSS) from [LSS], and define/introduce the pseudo-splitness of morphisms of *arbitrary* k-varieties.

• *Pseudo-splitness of prime divisors in function field extensions over k.* Let $F|k$ be a function field over an arbitrary base field k. For valuations $w \in \mathrm{Val}(F)$, we denote by wF the value group of w, by $\mathcal{O}_w, \mathfrak{m}_w$ the valuation ring/ideal of w, and by Fw the residue field of w. A *prime divisor* of $F|k$ is any w which satisfies the following equivalent conditions:

(i) There is a projective normal model Z of $F|k$ and $x \in Z$, $\mathrm{codim}_Z(x) = 1$, with $\mathcal{O}_w = \mathcal{O}_x$.

(ii) w is a k-valuation of F, i.e., w is trivial on k, and $\mathrm{td}(Fw|k) = \mathrm{td}(F|k) - 1$.

Let $\mathcal{D}(F|k)$ denote the *set of prime divisors of $F|k$ together with the trivial valuation.*

For extensions of function fields $E|F$ over k, the restriction map $\mathcal{D}(E|k) \to \mathcal{D}(F|k)$, $v \mapsto w := v|_F$ is well defined and surjective. In particular, if $v \in \mathcal{D}(E|k)$ and $w = v|_F$, then there is a canonical k-embedding of the residue function fields $Fw := \kappa(w) \hookrightarrow \kappa(v) =: Ev$, and $e(v|w) := (vE : wF)$ is finite if either v is trivial or w is non-trivial.

We say that $w \in \mathcal{D}(F|k)$ is *pseudo-split* in $\mathcal{D}(E|k)$, if for every $\sigma \in G_{Fw}$, there is some $v \in \mathcal{D}(E|k)$ satisfying: (i) $w = v|_F$; (ii) $e(v|w) = 1$ (in particular w is trivial iff v is so); (iii) $Ev \otimes_{Fw} \overline{Fw}$ has a factor which is a field stabilized by σ.

We say that $\mathcal{D}(F|k)$ is *pseudo-split* in $\mathcal{D}(E|k)$, if all $w \in \mathcal{D}(F|k)$ are pseudo-split in $\mathcal{D}(E|k)$.

This notion of pseudo-splitness relates to the hypothesis (LSS) as follows: Let $f : X \to Y$ be a dominant morphism of proper smooth varieties over a field k with $\mathrm{char}(k) = 0$, and setting $K = k(X)$, $L = k(Y)$, let $K \,|\, L$ be the corresponding k-extension of function fields. By Hironaka's Desingularization Theorem, the system of projective smooth models $(X_\alpha)_\alpha$ and $(Y_\alpha)_\alpha$ are cofinal (w.r.t. the domination relation) in the system of all the proper models of $K|k$, respectively $L|k$. Hence if $f_\alpha \colon X_\alpha \to Y_\alpha$, $\alpha \in I$ is the (projective) system of all the smooth modifications of f satisfying the hypothesis $(*)_{\mathsf{CT}}$, by mere definitions one has:

Fact. *The hypothesis* (LSS) *implies that $\mathcal{D}(L|k)$ is pseudo-split in $\mathcal{D}(K|k)$.*

• *Pseudo-splitness of morphisms of arbitrary k varieties.* Let $f : X \to Y$ be a morphism of *arbitrary varieties* over an arbitrary base field k, and for every $y \in Y$, let X_y be the reduced fiber of f at $y \in Y$. For $y \in Y$ and $x \in X_y$, we denote $L_y := \kappa(y)$, $K_x := \kappa(x)$, hence f defines canonically an extension of function fields $K_x \,|\, L_y$ over k. In particular, one has the canonical restriction map $\mathcal{D}(K_x|k) \to \mathcal{D}(L_y|k)$, $v_x \mapsto w_y := (v_x)|_{L_y}$. To simplify notation, we set $l_y := L_y w_y$ and $k_x := K_x v_x$, hence $k_x|\, l_y$ is canonically a function field extension over k.

We say that $w_y \in \mathcal{D}(L_y|k)$ is *pseudo-split* under f, if for every $\sigma \in G_{l_y}$ there are $x \in X_y$ and $v_x \in \mathcal{D}(K_x|k)$ satisfying: $w_y = (v_x)|_{L_y}$, $e(v_x|w_y) = 1$ if w_y is non-trivial, and $k_x \otimes_{l_y} \overline{l}_y$ has a factor which is a field stabilized by σ. Further, we say that $y \in Y$ is *pseudo-split under* f if all $w_y \in \mathcal{D}(L_y|k)$ are pseudo-split under f, and that f is *pseudo-split* if all $y \in Y$ are pseudo-split under f. Finally consider the following hypothesis:

$(\mathsf{p.s.})_k$ $\qquad$ *$f : X \to Y$ is a pseudo-split morphism of k-varieties.*

This being said, the results extending/generalizing and shedding new light on the afore mentioned [Df2], Main Theorem 1.2, and [LSS], Theorem 1.4, are as follows:

Theorem 1.1. *Let k have* $\mathrm{char}(k) = 0$ *and satisfy* $(\mathsf{H})_k$*, and $f : X \to Y$ be a morphism of arbitrary k-varieties. Then f has property* $(\mathsf{Srj})_k$ *iff f satisfies hypothesis* $(\mathsf{p.s.})_k$.

Theorem 1.2. *Let k have* $\mathrm{char}(k) = 0$ *and satisfy* $(\mathsf{H})_k$*. Let $f : X \to Y$ be a dominant morphism of proper smooth k-varieties, and $K = k(X)$, $L = k(Y)$. Then f has property* $(\mathsf{Srj})_k$ *iff $\mathcal{D}(L|k)$ is pseudo-split in $\mathcal{D}(K|k)$. Hence* $(\mathsf{Srj})_k$ *is a birational property for f.*

The main point in our approach is to use Ax–Kochen–Ershov Principle (AKE) type results (together with some general model-theoretical facts about rational points and ultraproducts of local fields), as originating from [Ax, A-K1, A-K2], see e.g. [P-R] for details on AKE. Moreover, one should notice that in the realm of "conjectural math," a weak form of AKE in char $= p > 0$, see hypothesis $(\mathsf{qAKE})_{\Sigma_k}$ after Fact 2.7, would imply that $(\mathsf{p.s.})_k$ implies $(\mathsf{Srj})_k$ in char $= p > 0$ as well. Finally, one should mention that [Df2], subsection 6.3, gives a sketch of a quite short proof of (CCT)—as initially stated by Colliot-Thélène—using the AKE Principle, but not of the stronger final results from in [Df2]. Actually, the main results of both [Df2] and [LSS] are based on quite deep desingularization facts, e.g. [ADK, A-K], and build on previous results and ideas by the authors, cf. [Df1, L-S, Sk], aimed at—among other things—giving arithmetic geometry proofs of AKE. It would be interesting to see whether the methods of this note could be used to extend results of Gvirtz [Gv].

Here is an example—pointed out to me by Daniel Loughran, showing the relation between Theorem 1.1 above, and the previous results.

Example 1.3. Let $Y = \mathrm{Spec}\, k[t]$, $X = V(T_0^2 + T_1^2 - t^2T_2^2) \subset Y \times_k \mathrm{Proj}\, k[T_0, T_1, T_2]$. One checks directly that for $k = \mathbb{Q}$ the canonical projection $f : X \to Y$ has the property (Srj), and f is smooth and split above all points $y \in Y$ satisfying $y \neq (1:0)$. Further, for the k-rational point $(1:0) \in Y$ one has: The fiber X_y above $(1:0) \in Y$ is smooth, except the point $x = (0:0:1) \in X_y$, which is a non-rationally double point of X. In particular the "smooth" results do not apply. On the other hand, f satisfies hypothesis $(\mathsf{p.s.})_k$: Namely, all $y \neq (1:0)$ are split under f, thus pseudo-split under f; and for $y = (1:0)$ one has $X_y \ni x = (0:0:1) \mapsto (1:0) = y \in Y$, $K_x = k = L_y$, and $\mathcal{D}(K_x|k) = \{v_k^0\} = \mathcal{D}(L_y|k)$ with v_k^0 the trivial valuation of k. Hence y is pseudo-split under f in the sense defined above.

Acknowledgements. I would like to thank several people at the IHP special programs in the spring 2018 and summer 2019, especially Zoe Chatzidakis, J.-L. Colliot-Thélène, E. Hrushovski, J. Koenigsmann, F. Loeser, Junguk Lee, D. Loughran, B. Poonen, A. Skorobogatov, Arne Smeets, S. Starchenko, Tamás Szamuely, O. Wittenberg, for discussions, comments and suggestions concerning the subject and content of this note.

2. Ultraproducts and rational Points/generalized Pseudospitness

2.1. Ultraproducts and approximation results for points.

We begin by recalling a few facts, which are/might be well known to experts; see e.g. [B-S], [Ch], [F-J], Ch.7, for details on ultraproducts and other model theoretical facts.

Fact 2.1. *Let $(k_i\,|\,k)_{i\in I}$ be a family of field extensions, $\mathcal{P}_I$ be a fixed prefilter on I, and for every ultrafilter $\mathcal{U}$ on I with $\mathcal{P}_I \subset \mathcal{U}$, let ${}^*k_{\mathcal{U}} := \prod_{i\in I} k_i/\mathcal{U}$ be the corresponding ultraproduct. Then for every morphism $f : X \to Y$ of k-varieties, the following are equivalent:*

i) *There is $I_0 \in \mathcal{P}_I$ such that the map $f^{k_i} : X(k_i) \to Y(k_i)$ is surjective for all $i \in I_0$.*

ii) *The map $f^{*k_{\mathcal{U}}} : X({}^*k_{\mathcal{U}}) \to Y({}^*k_{\mathcal{U}})$ is surjective for all ultrafilters $\mathcal{U} \supset \mathcal{P}_I$.*

In particular, if I is infinite, then $f^{k_i} : X(k_i) \to Y(k_i)$ is surjective for almost all $i \in I$ if and only if $f^{*k_{\mathcal{U}}} : X({}^*k_{\mathcal{U}}) \to Y({}^*k_{\mathcal{U}})$ is surjective for all non-principal ultrafilters $\mathcal{U}$ in I.

Proof. To i) $\Rightarrow$ ii): To simplify notation, we can suppose that $I = I_0$, or equivalently, $f^{k_i} : X(k_i) \to Y(k_i)$ is surjective for every $i \in I$. Let $\mathcal{U}$ be an ultrafilter on I with $\mathcal{P}_I \subset \mathcal{U}$, and ${}^*y_{\mathcal{U}} \in Y({}^*k_{\mathcal{U}})$ be defined by $\kappa(y) \hookrightarrow {}^*k_{\mathcal{U}}$ for some $y \in Y$. Let $V \subset Y$ be an affine open neighborhood of y, say $k[V] = k[\boldsymbol{u}] =: S$ with $\boldsymbol{u} := (u_1, \dots, u_n)$ a system of generators of the k-algebra S. Then by mere definitions, there is a system $\boldsymbol{a}_{\mathcal{U}} = (a_1, \dots, a_n)$ of n elements of ${}^*k_{\mathcal{U}}$ such that ${}^*y_{\mathcal{U}}$ is defined by the morphism of k-algebras

$$ {}^*\psi_{\mathcal{U}} : S \to S/y \hookrightarrow {}^*k_{\mathcal{U}}, \quad \boldsymbol{u} \mapsto \boldsymbol{a}_{\mathcal{U}}. $$

Hence, $\mathcal{U}$-locally, there are $\boldsymbol{a}_i = (a_{i1}, \dots, a_{in}) \in k_i^n$ and morphisms of k-algebras

$$ \psi_i : S \to S/y \to k_i, \quad \boldsymbol{u} \mapsto \boldsymbol{a}_i, $$

defining ${}^*\psi_{\mathcal{U}}$, i.e., $\boldsymbol{a}_{\mathcal{U}} = (\boldsymbol{a}_i)_i/\mathcal{U}$, and let $y_i \in Y(k_i)$ be the k_i-rational point defined by ψ_i.

Finally, let $(U_\alpha)_\alpha$, $U_\alpha = \operatorname{Spec} R_\alpha$, be a finite open affine covering of $f^{-1}(V) \subset X$. Then $X(k_i) = \cup_\alpha U_\alpha(k_i)$ for all k_i, and $y_i \in \cup_\alpha f\big(U_\alpha(k_i)\big)$ for every $i \in I$. Since $(U_\alpha)_\alpha$ is finite, there exists some $U := U_{\alpha_0}$ such that $\mathcal{U}$-locally one has: $y_i \in f\big(U(k_i)\big)$. Equivalently, $\mathcal{U}$-locally, there exists $x_i \in U(k_i)$ such that $f^{k_i}(x_i) = y_i$. Let $R := k[U]$ be the k-algebra of finite type with $U = \operatorname{Spec} R$. Then $f|_U : U \to V$ is defined by a unique morphism $f^{\#}_{UV} : S \to R$ of k-algebras, and there is a unique k-morphism

$$ \phi_i : R \to R/x_i \hookrightarrow k_i $$

defining $x_i \in U(k_i)$. Further, the fact that $f^{k_i}(x_i) = y_i$ is equivalent to $\phi_i \circ f^{\#}_{UV} = \psi_i$. Hence if ${}^*\phi_{\mathcal{U}} : R \to {}^*k_{\mathcal{U}}$ is the k-morphism having $\mathcal{U}$-local representatives $\phi_i : R \to k_i$, then one has

$$ {}^*\phi_{\mathcal{U}} \circ f^{\#}_{UV} = {}^*\psi_{\mathcal{U}}\,. $$

Hence if ${}^*x_{\mathcal{U}} \in X({}^*k_{\mathcal{U}})$ is the ${}^*k_{\mathcal{U}}$-rational point of X defined by ${}^*\phi_{\mathcal{U}}$, then $f^{*k_{\mathcal{U}}}({}^*x_{\mathcal{U}}) = {}^*y_{\mathcal{U}}$.

To ii) $\Rightarrow$ i): By contradiction, suppose that for every $J \in \mathcal{P}_I$ there exists $j \in J$ such that $f^{k_j} : X(k_j) \to Y(k_j)$ is not surjective. Then setting $I' := \{i \in I \,|\, f^{k_i} \text{ is not surjective}\}$, one has: $\mathcal{P}'_I := \{J \cap I' \mid J \in \mathcal{P}_I\}$ is a prefilter on I', and since $\mathcal{P}_I \prec \mathcal{P}'_I$, every ultrafilter $\mathcal{U}'$ on I' containing $\mathcal{P}'_I$ is the restriction $\mathcal{U}' = \mathcal{U}|_{I'}$ of an ultrafilter $\mathcal{U}$ on I containing $\mathcal{P}_I$. Hence *mutatis mutandis,* w.l.o.g., we can suppose that there is an ultrafilter $\mathcal{U}$ continuing $\mathcal{P}_I$ and a set $J \in \mathcal{U}$ such that f^{k_i} is not surjective for all $i \in J$. Let $(V_\beta)_\beta$ be a finite open affine covering of Y. Then reasoning as above, there exists some $V := V_{\beta_0}$ such that $\mathcal{U}$-locally one has: $V(k_i) \not\subset f^{k_i}\big(X(k_i)\big)$. Equivalently, $\mathcal{U}$-locally, there exists $y_i \in V(k_i)$ such that $y_i \notin f^{k_i}\big(X(k_i)\big)$. That being said, let $\psi_i : S := k[V] \to k_i$ be the morphism of k-algebras defining $y_i \in V(k_i)$, and ${}^*\psi_{\mathcal{U}} : S \to {}^*k_{\mathcal{U}}$ be the k-morphism defined by $(\psi_i)_i$. Then ${}^*\psi_{\mathcal{U}} : S \to {}^*k_{\mathcal{U}}$ defines a ${}^*k_{\mathcal{U}}$-rational point ${}^*y_{\mathcal{U}} \in V({}^*k_{\mathcal{U}}) \subset Y({}^*k_{\mathcal{U}})$. Hence by the hypothesis, there is ${}^*x_{\mathcal{U}} \in X({}^*k_{\mathcal{U}})$ such that $f^{{}^*k_{\mathcal{U}}}({}^*x_{\mathcal{U}}) = {}^*y_{\mathcal{U}}$. Let $y \in V$ and $x \in X$ be such that ${}^*y_{\mathcal{U}}$ and ${}^*x_{\mathcal{U}}$ are defined by k-embeddings $\kappa(y) \hookrightarrow {}^*k_{\mathcal{U}}$, respectively $\kappa(x) \hookrightarrow {}^*k_{\mathcal{U}}$. Then choosing $U \subset X$ affine open with $x \in U$ and $f(U) \subset V$, and setting $R := k[U]$, the following hold:

a) $f|_U : U \to V$ is defined by a unique morphism of k-algebras $f^{\#}_{UV} : S \to R$.

b) ${}^*x_{\mathcal{U}}$ is defined by a unique morphism of k-algebras ${}^*\phi_{\mathcal{U}} : R \to R/x \to {}^*k_{\mathcal{U}}$.

c) One has that ${}^*\psi_{\mathcal{U}} = {}^*\phi_{\mathcal{U}} \circ f^{\#}_{UV}$.

Therefore, letting $\phi_i : R \to k_i$ be $\mathcal{U}$-local representatives for ${}^*\phi_{\mathcal{U}}$, $\mathcal{U}$-locally one has:

$$\psi_i = \phi_i \circ f^{\#}_{UV}\,.$$

Hence if x_i is the k_i-rational point of X defined by $\phi_i : R \to k_i$, it follows that $f^{k_i}(x_i) = y_i$. Therefore, $\mathcal{U}$-locally, one must have that $y_i \in f\big(X(k_i)\big)$, contradiction!

Finally, for the last assertion of Fact 2.1, we notice: First, the set $\mathcal{P}_I$ of all the cofinite subsets of I is a prefilter on I, and $I' \in \mathcal{P}_I$ iff $I \backslash I'$ is finite. Second, an ultrafilter $\mathcal{U}$ on I is non-principal iff $\mathcal{P}_I \subset \mathcal{U}$. Conclude by applying the equivalence i) $\Leftrightarrow$ ii) to this situation. $\square$

Definition 2.2. A field k-extension $k' \to l'$ is called *quasi-elementary*, if there are field k-extensions $k' \to l' \to k'' \to l''$ with $k''|\,k'$ and $l''|\,l'$ elementary k-embeddings.

Fact 2.3. *Let $f : X \to Y$ be a morphism of varieties over an arbitrary base field k, and let $\mathcal{C}_f$ be the class of all the field extensions $k'|k$ with $f^{k'} : X(k') \to Y(k')$ surjective. One has:*

1) *$\mathcal{C}_f$ is an elementary class, i.e., $\mathcal{C}_f$ is closed w.r.t. ultraproducts and sub-ultrapowers.*

2) *Let $k' \hookrightarrow l'$ be a quasi-elementary k-field extension. Then $k' \in \mathcal{C}_f$ iff $l' \in \mathcal{C}_f$.*

Proof. Assertion 1) follows from Fact 2.1 by mere definition. To 2): We begin by noticing that $X(\tilde{k}) \subset X(\tilde{l})$ for all k-field extensions $\tilde{k} \subset \tilde{l}$. First, consider the case $l' \in \mathcal{C}_f$. Then one has $Y(k') \subset Y(l') = f^{l'}\big(X(l')\big) \subset f^{k''}\big(X(k'')\big)$, hence $Y(k') \subset f^{k'}\big(X(k')\big)$, because k' is existentially closed in k''. Hence finally $Y(k') = f^{k'}\big(X(k')\big)$. Second, let $k' \in \mathcal{C}_f$. Embeddings $k' \hookrightarrow l' \hookrightarrow k'' \hookrightarrow l''$ as in Definition 2.2 imply: First, $k'' \in \mathcal{C}_f$, by assertion 1) above, and second, l' is existentially closed in l''. Hence reasoning as in the first case, one gets $l' \in \mathcal{C}_f$. $\square$

2.2. **Ultraproducts of localizations of arithmetically significant fields.**

We introduce notation and recall well known facts and generalize the context in which the conclusion of Theorems 1.1, 1.2 hold, finally allowing to announce Theorems 3.1, 4.1 below. We first collect basic facts in a general setting and subsequently discuss the more special situation of fields satisfying Hypothesis $(\mathrm{H})_k$ as introduced in the Introduction.

2.2.1. *Basics and Notation.* For *arbitrary fields* k we consider the following.

Notations/Remarks 2.4. First, let $\Sigma_k \subset \mathrm{Val}(k)$ be sets of *discrete valuations* v with residue field kv perfect if $\mathrm{char}(k) = p > 0$ satisfying the hypothesis:

($\mathcal{P}$) $\Sigma_A := \{v \in \Sigma_k \,|\, A \subset \mathcal{O}_v^\times\} \neq \emptyset\ \forall\, A \subset\ k^\times$ finite, i.e., $\mathcal{P}_{\Sigma_k} := \{\Sigma_A\}_A$ *is a prefilter on* Σ_k.

For $v \in \Sigma_k$, let k_v be the completion of k at $v \in \Sigma_k$, and $\mathcal{U}$ always be ultrafilters on Σ_k with $\mathcal{P}_{\Sigma_k} \subset \mathcal{U}$. Given $\mathcal{U}$, consider the ultraproducts:

$$^*k_{\mathcal{U}} := \prod_v k_v/\mathcal{U}, \quad ^*\mathcal{O}_{\mathcal{U}} := \prod_v \mathcal{O}_v/\mathcal{U}, \quad ^*\mathfrak{m}_{\mathcal{U}} := \prod_v \mathfrak{m}_v/\mathcal{U}, \quad ^*\kappa_{\mathcal{U}} := \prod_v kv/\mathcal{U}.$$

Then $^*\mathcal{O}_{\mathcal{U}}$ is the valuation ring of $^*k_{\mathcal{U}}$, say $^*\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{^*v_{\mathcal{U}}}$ of the valuation $^*v_{\mathcal{U}}$, with valuation ideal $\mathfrak{m}_{^*v_{\mathcal{U}}} = {}^*\mathfrak{m}_{\mathcal{U}}$, residue field $^*k_{\mathcal{U}}{}^*v_{\mathcal{U}} = {}^*\kappa_{\mathcal{U}}$, and value group $^*v_{\mathcal{U}}{}^*k_{\mathcal{U}} = \prod_v vk/\mathcal{U} = \mathbb{Z}^{\Sigma_k}/\mathcal{U} = {}^*\mathbb{Z}_{\mathcal{U}}$.

1) One has the (canonical) diagonal field embedding $^*\imath_{\mathcal{U}} : k \hookrightarrow {}^*k_{\mathcal{U}}$, and $^*v_{\mathcal{U}}$ is trivial on k (by the fact that $\mathcal{P}_{\Sigma_k} \subset \mathcal{U}$).
2) If $\omega_v \subset \mathcal{O}_v$ is a set of representatives of kv, then $^*\omega_{\mathcal{U}} := \prod_v \omega_v \subset {}^*\mathcal{O}_{\mathcal{U}}$ is a system of representatives of $^*k_{\mathcal{U}}{}^*v_{\mathcal{U}}$ and further, if ω_v are multiplicative, so is $^*\omega_{\mathcal{U}}$.
3) The value group $^*v_{\mathcal{U}}{}^*k_{\mathcal{U}} = {}^*\mathbb{Z}_{\mathcal{U}}$ is a $\mathbb{Z}$-group. Further, if $\pi_v \in k_v$ is a uniformizing parameter for $v \in \Sigma_k$, then $\pi_{\mathcal{U}} = (\pi_v)_v/\mathcal{U}$ is an element of minimal value in $^*v_{\mathcal{U}}{}^*k_{\mathcal{U}}$.
4) The field $^*k_{\mathcal{U}}$ is Henselian with respect to $^*v_{\mathcal{U}}$, and one has:
   a) Let $\mathrm{char}(k) = 0$. Then $^*v_{\mathcal{U}}$ is trivial on $\mathbb{Q} \subset \kappa_{\mathcal{U}}$, and if $\mathcal{T} \subset {}^*\mathcal{O}_{\mathcal{U}}$ is any lifting of a transcendence basis of $\kappa_{\mathcal{U}} \,|\, \mathbb{Q}$, by Hensel Lemma one has: The relative algebraic closure $\kappa_{\mathcal{U}} \subset {}^*\mathcal{O}_{\mathcal{U}}$ of $\mathbb{Q}(\mathcal{T})$ in $^*k_{\mathcal{U}}$ is a field of representatives for $^*\kappa_{\mathcal{U}}$.
   b) Let $\mathrm{char}(k) = p > 0$. Then by hypothesis, kv is perfect for all $v \in \Sigma_k$, thus the Teichmüller system of representatives $\mathbb{F}_p \subset k_v$ for kv is a field and $k_v = \mathbb{F}_v((\pi'_v))$ for any $\pi'_v \in k$ with $v(\pi'_v) = 1$. Hence $\kappa_{\mathcal{U}} = \mathbb{F}_{\mathcal{U}} := \prod_v \mathbb{F}_v/\mathcal{U} \subset {}^*\mathcal{O}_{\mathcal{U}}$ is a perfect field and a system of representatives for $^*\kappa_{\mathcal{U}}$, the "Teichmüller system" of representatives.

$*$ **Note** that in both cases a), b) above, the fields of representatives $\kappa_{\mathcal{U}} \subset {}^*\mathcal{O}_{\mathcal{U}}$ for $\kappa_{\mathcal{U}}$ defined there are relatively algebraically closed in $^*k_{\mathcal{U}}$.

5) Finally, for $\kappa_{\mathcal{U}} \subset k_{\mathcal{U}}$ as above, let $k_{\mathcal{U}} := \kappa_{\mathcal{U}}(\pi_{\mathcal{U}})^h \subset {}^*k_{\mathcal{U}}$ be the Henselization of $\kappa_{\mathcal{U}}(\pi_{\mathcal{U}})$ with respect to the $\pi_{\mathcal{U}}$-adic valuation, and set $v_{\mathcal{U}} := ({}^*v_{\mathcal{U}})|_{k_{\mathcal{U}}}$.

$*$ **Note** that $k_{\mathcal{U}} \subset {}^*k_{\mathcal{U}}$ is nothing but the relative algebraic closure of $\kappa_{\mathcal{U}}(\pi_{\mathcal{U}})$ in $^*k_{\mathcal{U}}$.

2.2.2. *Hypothesis* $(\mathrm{H})_k$ *revisited.*

Let k be as in Hypothesis $(\mathrm{H})_k$ from the Introduction, i.e., k satisfies one of the hypotheses:

(i) k is a finitely generated field. (ii) k is the function field $k|k_0$ with k_0 pseudo-finite.

Recall the basic definitions/facts from Introduction: First, $\mathbb{P}(k) \subset \mathrm{Val}(k)$ is the set of all discrete valuations v of k having finite residue field kv in case (i), respectively finite over k_0 in case (ii). Second, for models S of k, we denote by $\mathbb{P}_S(k) \subset \mathbb{P}(k)$ the set of valuations $v \in \mathbb{P}(k)$ which have a center x_v on S. In particular, the center $x_v \in S$ of $v \in \mathbb{P}_S(k)$ is a closed point of S, and conversely, every closed point $x \in S$ is the center of some $v \in \mathbb{P}_S(k)$.

Finally, let $\mathbb{P}^0_S(k) \subset \mathbb{P}_S(k)$ be the set of all $v \in \mathbb{P}_S(k)$ such that $kv = \kappa(x)$. Notice that if $x \in S_{\mathrm{reg}}$ is closed, then $\exists\, v_x \in \mathbb{P}_S(k)$ having center x on S and $kv_x = \kappa(x)$, hence $v_x \in \mathbb{P}^0_S(k)$.

Next, for arbitrary non-empty subsets $\Sigma_k \subset \mathbb{P}(k)$ we denote:

$$S_{\Sigma_k} := \{x \in S \,|\, \exists\, v \in \Sigma_k \ \text{ such that } x \text{ is the center of } v \text{ on } S\}.$$

Fact 2.5 (**Hypothesis (H)$_k$ revisited**). *Let k satisfy Hypothesis* (H)$_k$*, S denote models of k and $\Sigma_k \subset \mathbb{P}(k)$ be non-empty. Then the following hold:*

($*$) *Σ_k satisfies $(\mathcal{P})$ iff S_{Σ_k} is Zariski dense in S iff $U_{\Sigma_k} \neq \emptyset$ $\forall\, U \subset S$ open non-empty.*

1) *Since $S_{\rm reg} \subset S$ is Zariski dense, the same holds correspondingly for subsets $\Sigma_k^0 \subset \mathbb{P}_S^0(k)$.*

2) *In case* (ii)*, suppose that $k_0 = \overline{k}_0 \cap k$, i.e., S is geometrically integral over k_0. Then $S_{\rm reg}(k_0)$ is Zariski dense, hence one can choose Σ_k such that $kv = k_0$ for all $v \in \Sigma_k$.*

In the following Fact 2.6 and Fact 2.7, one works under the hypothesis:

- k and $\Sigma_k \subset \mathbb{P}(k)$ satisfy condition $(\mathcal{P})$ as in Fact 2.5, and $\mathcal{U} \supset \mathcal{P}$ is a ultrafilter on Σ_k.
- $\kappa_{\mathcal{U}} \subset {}^*\mathcal{O}_{\mathcal{U}}$ is the field of representatives for ${}^*\kappa_{\mathcal{U}} = {}^*k_{\mathcal{U}}{}^*v_{\mathcal{U}}$ from Notations/Remarks 2.4, 4).
- $k_{\mathcal{U}} = \kappa_{\mathcal{U}}(\pi_{\mathcal{U}})^h \hookrightarrow {}^*k_{\mathcal{U}}$ is the k-embedding of valued fields from Notations/Remarks 2.4, 5).

Fact 2.6 (**Hypothesis (H)$_k$/Residue fields**). *By* [Ch] *and* [F-J], Ch. 11*, one has:*

1) *In case* (i)*, $\kappa_{\mathcal{U}}$ is an $\aleph_1$-saturated pseudo-finite field.*

2) *In case* (ii)*, $\kappa_{\mathcal{U}}$ is a $\aleph_\bullet$-saturated pseudo-finite field, where $\aleph_\bullet = \max(\aleph_1, \aleph_{|k|^+})$.*

Fact 2.7 (**Hypothesis (H)$_k$/AKE**). *The k-embedding of valued fields $k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}$ satisfies:*

(i) *${}^*v_{\mathcal{U}}$ is trivial on $\kappa_{\mathcal{U}}$ and one has canonical k-identifications $\kappa_{\mathcal{U}} = k_{\mathcal{U}}v_{\mathcal{U}} = {}^*k_{\mathcal{U}}{}^*v_{\mathcal{U}}$.*

(ii) *$v_{\mathcal{U}}k_{\mathcal{U}} = \mathbb{Z} \hookrightarrow {}^*\mathbb{Z}_{\mathcal{U}} = {}^*v_{\mathcal{U}}{}^*k_{\mathcal{U}}$ are $\mathbb{Z}$-groups with minimal positive element $v_{\mathcal{U}}(\pi_{\mathcal{U}}) = {}^*v_{\mathcal{U}}(\pi_{\mathcal{U}})$.*

In particular, if $\mathrm{char}(k) = 0$*, by the Ax–Kochen–Ershov Principle* (AKE) *one has:*

$$(*)\qquad k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}} \text{ is an elementary } k\text{-embedding of (valued) fields.}$$

Remarks 2.8. Let k satisfy Hypothesis (H)$_k$ and have $\mathrm{char}(k) = p > 0$. Unfortunately, it is unknown whether the conclusion ($*$) of Fact 2.7 holds in this case, that is, whether the k-embedding $k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}$ is an elementary embedding (of abstract and/or valued fields). One could conjecture that the weaker assertion below holds, and that would be enough for extending—at least partially—some of the results of this note to positive characteristic.

$$(\mathsf{qAKE})_{\Sigma_k}\qquad k_{\mathcal{U}} \to {}^*k_{\mathcal{U}} \text{ is a quasi-elementary } k\text{-embedding for every } \mathcal{U}.$$

2.2.3. Σ_k*-pseudo-splitness* (for short Σ_k-p.s.)

Throughout this subsection, the field k satisfies hypothesis $(\mathsf{H})_k$ from Introduction and $\Sigma_k \subset \mathbb{P}(k)$ satisfies condition $(\mathcal{P})$, as considered in Fact 2.5. Further, in the case (ii), i.e., k is the function field over a pseudo-finite field k_0, we fix a generator σ_0 of G_{k_0}, and for finite extensions $l_0|k_0$ we define $\mathrm{Frob}_{l_0} := \sigma_0^n$ with $n = [l_0 : k_0]$. Hence if $l|k$ is finite Galois and $v \in \Sigma_k$ is unramified in $l|k$, then $\mathrm{Frob}(v) \in \mathrm{Gal}(l|k)$ is well defined up to conjugation.

Definition 2.9. For k, Σ_k as above, $\sigma \in G_k$ and the co-procyclic extension $\overline{k}^{\sigma}|\,k$ of k is called Σ_k*-definable*, if for all finite Galois extensions $l\,|\,k$, and all $\Sigma_A \in \mathcal{P}_{\Sigma_k}$, one has:

$$U_{A,\,l|k}(\sigma) := \{v \in U_A \,|\, v \text{ unramified in } l\,|\,k \text{ and } \mathrm{Frob}(v) := \sigma|_l\} \neq \emptyset.$$

Notice that if S is a model of k and $\Sigma_k \subset \mathbb{P}_S(k)$, for all $v \in \Sigma_k$ one has:

- If $S_{\Sigma_k} \subset S$ has Dirichlet density $\delta(S_{\Sigma_k}) = 1$, e.g. if $S_{\Sigma_k} \subset S$ is open dense, by the Chebotarev Density Theorem, see e.g. Serre [Se1], all $\sigma \in G_k$ are Σ_k-definable.
- If $S_{\Sigma_k} \subset S$ is *Frobenian*, cf. Serre [Se2], 3.3, say defined by a finite Galois extension $k_1|k$ and a set of conjugacy classes $\Phi \subset \mathrm{Gal}(k_1|k)$, then $\sigma \in G_k$ is Σ_k-definable iff $\sigma|_{k_1} \in \Phi$.

Fact 2.10. *In the above notation, $\sigma \in G_k$ is Σ_k-definable iff $\overline{k}^{\sigma} = {}^*k_{\mathcal{U}} \cap \overline{k}$ for some $\mathcal{U}$.*

Proof. For the direct implication, notice that $\mathcal{P}_{\Sigma_k}(\sigma) := \{U_{A,\,l|k}\}_{A,\,l|k}$ is a prefilter on Σ_k such that any ultrafilter $\mathcal{U}$ containing $\mathcal{P}_{\Sigma_k}(\sigma)$ contains $\mathcal{P}_{\Sigma_k}$. Let $l \,|\, k$ be a finite Galois extension. Then for $v \in U_{A,\,l|k}(\sigma) \in \mathcal{U}$, setting $l_v := lk_v$ one has: $l_v|k_v$ is unramified and $l^\sigma = l \cap k_v$. Hence $l^\sigma = l \cap {}^*k_{\mathcal{U}}$, and finally $\overline{k}^{\sigma} = \overline{k} \cap {}^*k_{\mathcal{U}}$.

Conversely, let $\mathcal{U}$ be such that $\overline{k}^{\sigma} = {}^*k_{\mathcal{U}} \cap \overline{k}$. To show that σ is Σ_k-definable, we have to show that all the sets $U_{A,\,l|k}(\sigma)$ are non-empty. First, since $\overline{k}^{\sigma} = {}^*k_{\mathcal{U}} \cap \overline{k}$, it follows that for every finite Galois extension $l|k$, one has $l^\sigma = {}^*k_{\mathcal{U}} \cap l$. Hence for every $l|k$ there exists a set $V_l \in \mathcal{U}$ such that for all $v \in V_l$ one has $l^\sigma = k_v \cap l$. Further, let $U_A \subset \Sigma_k$ be given. Since $\mathcal{P}_{\Sigma_k} \subset \mathcal{U}$, hence $U_A \in \mathcal{U}$, w.l.o.g., we can suppose that $V_l \subset U_A$. Second, let $B \subset k^\times$ be a finite set such that all discrete valuations w of k with $w(B) = 0$ are unramified in $l|k$. (Note that such sets B exist: If $\mathcal{S}_l \to \mathcal{S}$ is the normalization of $\mathcal{S}$ in the finite Galois extension $l|k$, then there exists an affine open dense subset $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{S}_l$ is étale above $\mathcal{S}'$. Hence if w has its center in $\mathcal{S}'$, then w is unramified in $l|k$, etc.) Then setting $A_l := A \cup B$, one has: $V_l \cap U_{A_l} \in \mathcal{U}$, and all $v \in V_l \cap U_{A_l}$ are unramified in $l|k$. Hence $U_{A_l,\,l|k} \neq \emptyset$, thus $U_{A,k|l} \supset U_{A_l,\,l|k}$ is non-empty as well, concluding that σ is Σ_k-definable. ∎

Definition 2.11. In the context of Definition 2.9, let $E|F$ be function fields over k, and $F'|F$ be an algebraic extension.

1) $F'|F$ is called *co-procyclic Σ_k-definable,* if $F' = \overline{F}^{\sigma_F}$ for some $\sigma_F \in G_F := \mathrm{Aut}_F(\overline{F})$ such that $\sigma := (\sigma_F)|_{\overline{k}} \in G_k$ is Σ_k-definable.
2) $E|F$ is called F'-pseudo-split, or pseudo-split above F', if the F'-algebra $E \otimes_F F'$ has a factor E' which is a field and $E'|F'$ is a regular field extension.

Proposition 2.12. *In the above notation, let $E \,|\, F$ be function fields over k.*

1) *An algebraic extension $F'|F$ is co-procyclic Σ_k-definable if and only if there is $\mathcal{U}$ and a k-embedding $F \hookrightarrow \kappa_{\mathcal{U}}$ such that $F' = \overline{F} \cap \kappa_{\mathcal{U}}$.*
2) *Let $F' = \overline{F} \cap \kappa_{\mathcal{U}}$ as above be given. Then $E|F$ is split above F' iff $E|F$ is separably generated and $F \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to a field embedding $E \hookrightarrow \kappa_{\mathcal{U}}$.*

Proof. To 1): To the direct implication: Since $\kappa_{\mathcal{U}}$ is a perfect pseudo-finite field, $k \hookrightarrow F \hookrightarrow \kappa_{\mathcal{U}}$ gives rise to embedding of perfect fields $k' = \overline{k} \cap \kappa_{\mathcal{U}} \hookrightarrow F' = \overline{F} \cap \kappa_{\mathcal{U}} \hookrightarrow \kappa_{\mathcal{U}}$ and to surjective projections $\widehat{\mathbb{Z}} \cong G_{\kappa_{\mathcal{U}}} \twoheadrightarrow G_{F'} \twoheadrightarrow G_{k'}$. Hence $F'|F$ is by mere definitions co-procyclic and Σ_k-definable. For the converse implication, let $F'|F$ be co-procyclic and Σ_k-definable. Then $k' := \overline{k} \cap F'$ is obviously co-procyclic and Σ_k-definable. Hence, there is some $\mathcal{U}$ such that $k' = \overline{k} \cap \kappa_{\mathcal{U}}$, and obviously, $F'|k'$ is a regular field extension. We claim that there is a k-embedding $F \hookrightarrow \kappa_{\mathcal{U}}$ such that $F' = \overline{F} \cap \kappa_{\mathcal{U}}$, hence $k' \subset F'$. First, $F'_0 := Fk' \subset F'$ is a regular function field over k', and setting $\tilde{F}_0 = F'_0$, there is an increasing sequence of cyclic field subextensions $(\tilde{F}_i|F'_i)_{i\in\mathbb{N}}$ of $\overline{F}|F'$ such that $F' = \cup_{i\in\mathbb{N}} F'_i$, $\overline{F} = \cup_{i\in\mathbb{N}} \tilde{F}_i$, and $\tilde{F}_i|F'_i$ is the maximal subextension of $\overline{F}|F'$ of degree $\leqslant i$. By algebra general non-sense, the sequence $(\tilde{F}_i|F'_i)_i$ and the conditions it satisfies are expressible by a type $p(\boldsymbol{t})$ over k', where $\boldsymbol{t}$ is a transcendence basis of $F_0|k'$; and since $\kappa_{\mathcal{U}}$ is a perfect PAC pseudo-finite field, the type $p(\boldsymbol{t})$ is finitely satisfiable. Since $\kappa_{\mathcal{U}}$ is $\aleph_1$-saturated in case (i), and $\aleph_{|k|}$-saturated in case (ii), the type $p(\boldsymbol{t})$ is satisfiable in $\kappa_{\mathcal{U}}$, thus $F = F_0$ has a k'-embedding $F \hookrightarrow \kappa_{\mathcal{U}}$ such that $F' = \overline{F} \cap \kappa_{\mathcal{U}}$.

To 2): For the direct implication, let E' be a factor of $E \otimes_F F'$ such that $E'|F'$ is a regular field extension. Since $F'|F$ contains the perfect closure of F, it follows that $E\,|\,F$ must be separably generated (because otherwise all the factors of $E \otimes_F F'$ have non-trivial nilpotent elements). Hence $E = F(Z_F)$ for an integral F-variety Z_F such that $Z_F \times_F F'$ has a geometrically integral irreducible component $Z_{F'}$ of multiplicity one with $E' = F'(Z_{F'})$. Since $\kappa_{\mathcal{U}}$ is $\aleph_1$-saturated in case (i), and $\aleph_{|k|}$-saturated in case (ii), $Z_{F'}(\kappa_{\mathcal{U}})$ contains "generic points" of $X_{F'}$, that is, E' is F'-embeddable into $\kappa_{\mathcal{U}}$.

For the converse implication, since $E\,|\,F$ is separably generated, it follows that $E \otimes_F F'$ is a product of fields. Let $E \hookrightarrow \kappa_{\mathcal{U}}$ be a prolongation of $F \hookrightarrow \kappa_{\mathcal{U}}$. Then

$$F' := \overline{F} \cap \kappa_{\mathcal{U}} \hookrightarrow \overline{E} \cap \kappa_{\mathcal{U}} =: E' \hookrightarrow \kappa_{\mathcal{U}}$$

are co-procyclic extensions, and $E \otimes_F F'$ has a factor $E_{F'}$ which is F'-embeddable in E'. Since F' is perfect, $F' = \overline{F} \cap E' \hookrightarrow E'$ is regular, hence $E_{F'}|F'$ is regular. $\square$

2.3. **Setup for Generalizations of Theorem 1.1 and Theorem 1.2.**

The generalizations of Theorem 1.1 and Theorem 1.2 we aim at are based on generalizing the notions $(\mathsf{Srj})_k$, the pseudo-splitness of prime divisors in function field extension over k and pseudo-splitness $(\mathsf{p.s.})_k$ of morphisms of arbitrary varieties as defined in the Introduction. These generalizations are obtained by considering arbitrary base fields k endowed with sets $\Sigma_k \subset \mathrm{Val}(k)$ of discrete valuations of k satisfying Hypothesis $(\mathcal{P})$ above, as in Notations/Remarks 2.4 above, and defining $(\mathsf{Srj})_{\Sigma_k}$, the Σ_k-generalized-pseudo-splitness of prime divisors in function field extension over k and of morphisms of arbitrary k-varieties. Then Theorem 1.1 and Theorem 1.2 from the Introduction are consequence of Theorems 3.1 and Theorem 4.1 below, which are a kind of general non-sense type results.

2.3.1. Σ_k-*generalized-pseudo-splitness* (for short Σ_k-*g.p.s.*)

The Proposition 2.12 above hints at the following *generalization* of Σ_k-pseudo-splitness. Let k, Σ_k satisfy condition $(\mathcal{P})$ from Notations/Remarks 2.4, but otherwise be arbitrary.

Definition 2.13. In Notations/Remarks 2.4, let $E|F$ be k-field extension.

1) For an ultrafilter $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$, let a k-embedding $\jmath : F \to \kappa_{\mathcal{U}}$ be given.
   a) A field extension $F'|F$ is $\jmath$-*definable*, if $F' = \overline{F} \cap \kappa_{\mathcal{U}}$ as F-field extensions.
   b) $E|F$ is called $\jmath$-*pseudo-split*, if $\jmath$ prolongs to an F-embedding $\imath : E \hookrightarrow \kappa_{\mathcal{U}}$.
2) $E|F$ is *generalized* Σ_k-*pseudo-split*, for short *generalized* Σ_k-*p.s.*, if $E|F$ is separably generated and $\jmath$-pseudo-split for all ultrafilters $\mathcal{U} \supset \mathcal{P}$ on Σ_k and all k-embeddings $\jmath : F \hookrightarrow \kappa_{\mathcal{U}}$.

Remark 2.14. In the above notation, the *transitivity of* $\jmath$-*pseudo-splitness* holds as follows: Let $E_\alpha|F_\alpha$ be $\jmath_\alpha$-pseudo-split, say via $\jmath_\alpha : E_\alpha \to \kappa_{\mathcal{U}}$, $\alpha = 1, 2$. Then:

1) Suppose that $E_1|F_1 \hookrightarrow E_2|F_2$, and $(\jmath_2)|_{E_1} = \jmath_1$. Then $E_2|F_1$ is $\jmath_1$-pseudo-split.
2) In particular, if $\tilde{E}_1|F_1 \hookrightarrow E_1|F_1$ is a k-subextension, then $\tilde{E}_1|E_1$ is $\jmath_1$-pseudo-split.

In particular, the same holds correspondingly for generalized Σ_k-pseudo-splitness.

Proposition 2.15. *Let $E|F$ be an extension of function fields over k. Let $\jmath : F \hookrightarrow \kappa_{\mathcal{U}}$ be a k-embedding, and $F' = \overline{F} \cap \kappa_{\mathcal{U}}$ be the resulting $\jmath$-definable extension of F. One has:*

1) *Let $E = F(Z_F)$ with Z_F an integral F-variety. Then $E|F$ is $\jmath$-pseudo-split iff $Z_F \times_F F'$ is geometrically reduced and $Z_F(\kappa_{\mathcal{U}})$ is Zariski dense.*

2) *In particular, for $F' = \overline{F} \cap \kappa_{\mathcal{U}}$ as above, the following hold:*
   a) *If $\kappa_{\mathcal{U}}$ is PAC, then $E|F$ is $\jmath$-pseudo-split iff the F'-algebra $E \otimes_F F'$ has a factor $E'|F'$ which is regular field extension.*
   b) *If $\mathrm{char}(k) = 0$, $E|F$ is $\jmath$-split iff $\jmath : F \hookrightarrow \kappa_{\mathcal{U}}$ has a prolongation $E \hookrightarrow \kappa_{\mathcal{U}}$.*

Proof. To 1): The implication $\Rightarrow$ is simply a reformulation in terms of algebraic geometry of the fact that $E|F$ is pseudo-split above F'. For the converse implication, one has: First, $Z_{F'} := Z_F \times_F F'$ being reduced, its ring of rational functions is the product of the function fields $E'_\alpha := F'(Z'_\alpha)$ of the irreducible components Z'_α of $Z_{F'}$. Second, since $Z_F(\kappa_{\mathcal{U}})$ is Zariski dense, $Z'_\alpha(\kappa_{\mathcal{U}})$ is Zariski dense for some α. Finally, arguing as in the proof of assertion 2) from Proposition 2.12, $Z'_\alpha(\kappa_{\mathcal{U}})$ contains "generic points" of the F'-variety Z'_α. Finally, each such point defines an F'-embedding $E'_\alpha = F'(Z_\alpha) \hookrightarrow \kappa_{\mathcal{U}}$, which prolongs $\jmath : F \hookrightarrow \kappa_{\mathcal{U}}$.

To 2): First, the implication $\Rightarrow$ is the same as in assertion 1. The converse implication in case b) is clear, and in case a) it follows from assertion 1): Since $\kappa_{\mathcal{U}}$ is a PAC field, and $Z_{F'}$ is a geometrically integral F'-variety, it follows that $Z_{F'}(\kappa_{\mathcal{U}})$ is Zariski dense, etc. □

Corollary 2.16 (Fact 2.5 revisited). *Let k, Σ_k be as in Fact 2.5, Σ_k satisfy $(\mathcal{P})$, and $E|F$ be function fields over k. Then a k-embedding $F \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to an embedding $E \hookrightarrow \kappa_{\mathcal{U}}$ iff $E \otimes_F F'$ has a factor E' such that $E'|F'$ is a regular field extension, where $F' := \overline{F} \cap \kappa_{\mathcal{U}}$.*

2.3.2. *Setup for Theorem 3.1 and Theorem 4.1.*

Let k be an arbitrary field and for extensions $E|F$ of function fields over k, recall the canonical restriction map $\mathcal{D}(E|k) \to \mathcal{D}(F|k)$, $v \mapsto w := v|_F$. For a morphism of k-varieties $f : X \to Y$, $x \mapsto y$, let $K_x := \kappa(x) \hookleftarrow \kappa(y) =: L_y$ be the canonical k-embedding of the residue function fields. In particular, for the canonical restriction map $\mathcal{D}(K_x|k) \to \mathcal{D}(L_y|k)$, $v_x \mapsto w_y$, one has k-embeddings of the residue function fields $k_x := K_x v_x \hookleftarrow L_y w_y = l_y$.

Definition/Notations 2.17. Let k and $\Sigma_k \subset \mathrm{Val}(k)$ be as in Notations/Remarks 2.4, and recall the field extensions $\kappa_{\mathcal{U}}|k$ for each $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$. We define/consider the following:

1) Given $\mathcal{D}(E|k) \to \mathcal{D}(F|k)$ for some $E|F$, we say that $w \in \mathcal{D}(F|k)$ is Σ_k-g.p.s. in $\mathcal{D}(E|k)$ if for every $\mathcal{U}$ and every k-embedding $\jmath : Fw \hookrightarrow \kappa_{\mathcal{U}}$ there is $v \in \mathcal{D}(E|k)$ such that $w = v|_L$, $e(v|w) = 1$ if w is non-trivial, and $Ev|Fw$ is $\jmath$-pseudo-split, i.e., $Ev|Fw$ is separably generated and $\jmath : Fw \to \kappa_{\mathcal{U}}$ prolongs to a k-embedding $Ev \hookrightarrow \kappa_{\mathcal{U}}$.

   Further, we say that $\mathcal{D}(F|k)$ is Σ_k-g.p.s. in $\mathcal{D}(E|k)$ if all $w \in \mathcal{D}(F|k)$ are Σ_k-g.p.s..
2) For $f : X \to Y$, $x \mapsto y$, we say that $w_y \in \mathcal{D}(L_y|k)$ is Σ_k-g.p.s. under f, if for every $\mathcal{U}$ and every k-embedding $\jmath_y : l_y \to \kappa_{\mathcal{U}}$, there is $x \in X_y$ and $v_x \in \mathcal{D}(K_x|k)$ such that $v_y = (v_x)|_{L_y}$, $e(v_x|w_y) = 1$ if w_y is non-trivial, and $k_x|l_y$ is $\jmath_y$-pseudo-split.

   We say that f is Σ_k-g.p.s. if all $w_y \in \mathcal{D}(L_y|k)$, $y \in Y$, are Σ_k-g.p.s. under f.
3) Correspondingly, the natural generalization of $(\mathsf{Srj})_k$ from the Introduction is:

   $(\mathsf{Srj})_{\Sigma_k}$ *There is $A \subset k^\times$ such that $f^{k_v} : X(k_v) \to Y(k_v)$ is surjective for all $v \in U_A$.*
4) Finally consider the generalization of hypothesis $(\mathsf{p.s.})_k$ from the Introduction:

   $(\mathsf{g.p.s.})_{\Sigma_k}$ *$f : X \to Y$ is a Σ_k-generalized-pseudo-split morphism of k-varieties.*

Remarks 2.18. We notice the following:

1) Let k satisfying Hypothesis $(\mathsf{H})_k$ from the Introduction, and $\Sigma_k = \mathbb{P}_S(k)$ for some model S of k. Then for $f : X \to Y$ and $L = k(Y) \hookrightarrow k(X) = K$, one has: (i) $(\mathsf{Srj})_k$ and $(\mathsf{Srj})_{\Sigma_k}$ are equivalent; (ii) hypotheses $(\mathsf{p.s.})_k$ and $(\mathsf{p.s.})_{\Sigma_k}$ are equivalent; (iii) pseudo-splitness of $\mathcal{D}(L|k)$ in $\mathcal{D}(K|k)$ is equivalent to Σ_k-g.p.s. of $\mathcal{D}(L|k)$ in $\mathcal{D}(K|k)$.

$*$ In particular, this is so for k a number field. Further, if $\mathrm{char}(k) = 0$, AKE Principle holds for $k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}$ for each $\mathcal{U}$, hence the weaker $(\mathsf{qAKE})_{\Sigma_k}$ holds.

2) For k, Σ_k with property $(\mathcal{P})$ as in Notations/Remarks 2.4 and a morphism $f : X \to Y$ of k-varieties. Then by Fact 2.7, property $(\mathsf{Srj})_{\Sigma_k}$ is equivalent to $f^{\mathcal{U}} : X({}^*k_{\mathcal{U}}) \to Y({}^*k_{\mathcal{U}})$ being surjective for all ultrafiltes $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ on Σ_k. In particular, if $\mathrm{char}(k) = 0$, the AKE Principle holds for $k_{\mathcal{U}} \hookrightarrow {}^*k_{\mathcal{U}}$ for each $\mathcal{U}$, thus property $(\mathsf{Srj})_{\Sigma_k}$ is equivalent to:

$$f^{\mathcal{U}} : X(k_{\mathcal{U}}) \to Y(k_{\mathcal{U}}) \textit{ is surjective for all ultrafiltes } \mathcal{U} \supset \mathcal{P}_{\Sigma_k} \textit{ on } \Sigma_k.$$

3. Proof of (Generalizations of) Theorem 1.1

Taking into account the above discussion, Theorem 1.1 follows from the more general:

Theorem 3.1. *In the context of Notations/Remarks 2.4 and Definition/Notation 2.17, let* $\mathrm{char}(k) = 0$ *and* $f : X \to Y$ *be a morphism of arbitrary k-varieties. Then one has:*

$$f \textit{ is } \Sigma_k\textit{-generalized-pseudo-split iff } f \textit{ has property } (\mathsf{Srj})_{\Sigma_k}.$$

Proof. First, by Remark 2.18, 2) the property $(\mathsf{Srj})_{\Sigma_k}$ is equivalent to $f^{k_{\mathcal{U}}} : X(k_{\mathcal{U}}) \to Y(k_{\mathcal{U}})$ being surjective for all ultrafilters $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$. Hence Theorem 3.1 is follows from the following.

Key Lemma 3.2. *In the hypothesis of Theorem 3.1 above, one has the following:*

$$f \textit{ is } \Sigma_k\textit{-generalized-pseudo-split} \iff f^{k_{\mathcal{U}}}\colon X(k_{\mathcal{U}}) \to Y(k_{\mathcal{U}}) \textit{ is surjective for all } \mathcal{U} \supset \mathcal{P}_{\Sigma_k}.$$

Proof of Key Lemma 3.2 We show that the implication "$\Rightarrow$" holds unconditionally, but its proof is quite involved. The proof of "$\Leftarrow$" is relatively short, but uses that $\mathrm{char}(k) = 0$.

We also notice that in the realm of "conjectural math" the direct implication of Theorem 3.1 would hold in concrete situations in which the hypothesis $(\mathsf{qAKE})_{\Sigma_k}$ is satisfied.

We begin by recalling basic of valuation theory (well known to experts). In not otherwise explicitly stated, k, Σ_k, $\mathcal{P}$, $\mathcal{U} \supset \Sigma_k$, etc., are as in Notations/Remarks 2.4.

Fact 3.3. *Let Ω, w be a Henselian field with* $\mathrm{char}(\Omega w) = 0$*. Then every subfield $l \subset \Omega$ with $w|_l$ trivial is contained in a field of representatives $\kappa' \subset \Omega$ for Ωw.*

Proof. This is a well known consequence of the Hensel Lemma. □

We next recall basic facts about valuations without (transcendence) defect, see [BOU], Ch. VI, and [Ku], for some/more details on (special cases of) this. Let Ω, w be a valued field with $w|_{\kappa_0}$ trivial on the prime field κ_0 of Ω. One says that w has no *(transcendence) defect* if there exists a transcendence basis of $\Omega \,|\, \kappa_0$ of the form $\boldsymbol{t}_w \cup \boldsymbol{t}$ satisfying the following: First, $w\boldsymbol{t}_w$ is a *basis* of the $\mathbb{Q}$-vector space $w\Omega \otimes \mathbb{Q}$, and second, $\boldsymbol{t}$ consists of w-units such that its image in the residue field Ωw, which we denote again by $\boldsymbol{t}$, is a *transcendence basis* of $\Omega w \,|\, \kappa_0$. In particular, if $\kappa'_{\boldsymbol{t}} \subset \Omega$ is the relative algebraic closure of $\kappa_0(\boldsymbol{t})$ in Ω, then $\kappa'_{\boldsymbol{t}}$ is a maximal subfield of Ω such that w is trivial on $\kappa'_{\boldsymbol{t}}$, and further, Ωw is algebraic over $\kappa'_{\boldsymbol{t}}w$. Moreover, if w is Henselian, then Hensel Lemma implies that Ωw is purely inseparable over $\kappa'_{\boldsymbol{t}}w$.

One of the main properties of valuations w without defect is that for any subfield $F \subset \Omega$, the restriction of w to F is a valuation without defect as well, see [Ku]. In particular, if $l \subset \Omega$ is any subfield such that $w|_l$ is trivial, and $F \,|\, l$ is a function field, then $w|_F$ is a prime divisor of the function field $F|l$ if and only if $w|_F$ is a discrete valuation.

Hence for $k_{\mathcal{U}} = \kappa_{\mathcal{U}}(\pi_{\mathcal{U}})^h$ endowed with $v_{\mathcal{U}}$ as in Notations/Remarks 2.4, 5), one has:

Fact 3.4. *Let $l \subset k_{\mathcal{U}}$ be a subfield with $v_{\mathcal{U}}$ trivial on l. Let $F \,|\, l$ be a function field and $F \hookrightarrow k_{\mathcal{U}}$ be an l-embedding. Then $w := (v_{\mathcal{U}})|_F$ is either trivial, or a prime divisor of $F \,|\, l$.*

Proof. This is an immediate consequence of the discussion above. □

Fact 3.5. *Let F^h be the Henselization of a function field $F|l$ w.r.t. a prime divisor w. Let $\kappa' \subset \Omega$ be a field of representatives for Fw, and $\pi \in F$ have $w(\pi) = 1$. Then $F^h = \kappa'(\pi)^h$.*

Proof. The Henselian subfield $\tilde{F} := \kappa'(\pi)^h$ of F^h satisfies $\tilde{F}w = F^h w$ and $w\tilde{F} = wF$. Since w has no defect, the fundamental equality holds. Hence $[F^h\colon \tilde{F}] = e(F^h|\tilde{F})f(F^h|\tilde{F}) = 1$, thus finally implying $F^h = \tilde{F} = \kappa'(\pi)^h$. □

Coming back to the proof of Key Lemma 3.2, proceed as follows.

3.1. *The implication "$\Rightarrow$".*

Let $y_{\mathcal{U}} \in Y(k_{\mathcal{U}})$ be defined by a point $y \in Y$ and a k-embedding $\jmath_{\mathcal{U}} : L_y \hookrightarrow k_{\mathcal{U}}$. By Fact 3.4 above, $w := v_y := (v_{\mathcal{U}})|_{L_y} \in \mathcal{D}(L_y|k)$ is either trivial or a prime divisor of $L_y|k$, and let $\jmath : l_y \hookrightarrow \kappa_{\mathcal{U}}$ be the corresponding k-embedding of the residue fields. Since f is Σ_k-g.p.s., there is $x \in X_y$ and $v := v_x \in \mathcal{D}(K_x|k)$ on $K_x = \kappa(x)$ such that $w = v|_{L_y}$, the residue field embedding $k_x|l_y$ is $\jmath$-pseudo-split, and $e(v|w) = 1$ if w is non-trivial. Hence by definitions, $k_x|l_y$ is separably generated, and $\jmath : l_y \hookrightarrow \kappa_{\mathcal{U}}$ has a prolongation $\imath : k_x \hookrightarrow \kappa_{\mathcal{U}}$. Let $\boldsymbol{t}_0$ be a separable transcendence basis of k_x over l_y, and $\boldsymbol{t} \subset K_x$ be a preimage of $\boldsymbol{t}_0$ under the canonical residue field projection $\mathcal{O}_v \to K_x v_x$. One has the following:

- Setting $F := L_y$ and $E := K_x$, one has $Fw = l_y$, $k_x = Ev$, and further: $\boldsymbol{t}_0$ is a separable transcendence basis of Ev over Fw, and $\boldsymbol{t} \subset E$ is a preimage of $\boldsymbol{t}_0$ under $\mathcal{O}_v \to Ev$.
- Set $F_{\boldsymbol{t}} := F(\boldsymbol{t}) \subset E$. Since $w = v|_F$, it follows by mere definition that $w_{\boldsymbol{t}} := v|_{F_{\boldsymbol{t}}}$ is the Gauss valuation of $F_{\boldsymbol{t}}$ defined by w and $\boldsymbol{t}$.
- Setting $\kappa_F := \jmath(Fw) \hookrightarrow \imath(Ev) =: \kappa_E$, it follows that $\imath(\boldsymbol{t}_0)$ is a separable transcendence basis of κ_E over κ_F.
- Setting $F_{\mathcal{U}} := \jmath_{\mathcal{U}}(F) \subset k_{\mathcal{U}}$, let $\boldsymbol{t}_{\mathcal{U}} \subset k_{\mathcal{U}}$ be a preimage of $\imath(\boldsymbol{t}_0)$ under $\mathcal{O}_{\mathcal{U}} \to \kappa_{\mathcal{U}}$, and set $F_{\boldsymbol{t}_{\mathcal{U}}} := F_{\mathcal{U}}(\boldsymbol{t}_{\mathcal{U}})$. Then the restriction $w_{\boldsymbol{t}_{\mathcal{U}}}$ of $v_{\mathcal{U}}$ to $F_{\boldsymbol{t}_{\mathcal{U}}}$ is the Gauss valuation of $F_{\mathcal{U}}$ defined by $w_{\mathcal{U}} = (v_{\mathcal{U}})|_{F_{\mathcal{U}}}$ and $\boldsymbol{t}_{\mathcal{U}}$. Hence one has a k-isomorphism of valued fields
$$\jmath_{\boldsymbol{t}_{\mathcal{U}}} : F_{\boldsymbol{t}} \to F_{\boldsymbol{t}_{\mathcal{U}}} \subset k_{\mathcal{U}}.$$
- Let $F^h_{\mathcal{U}} \subset F^h_{\boldsymbol{t}_{\mathcal{U}}} \subset k_{\mathcal{U}}$ be the Henselizations of $F_{\mathcal{U}} \subset F_{\boldsymbol{t}_{\mathcal{U}}}$ in $k_{\mathcal{U}}$. Then since κ_E is finite separable over the residue field $F_{\boldsymbol{t}_{\mathcal{U}}} w_{\boldsymbol{t}_{\mathcal{U}}} = \kappa_F\big(\imath(\boldsymbol{t}_0)\big)$, one has: There exists a unique algebraic unramified subextension $E^0_{\mathcal{U}} \,|\, F^h_{\boldsymbol{t}_{\mathcal{U}}}$ of $k_{\mathcal{U}} \,|\, F^h_{\boldsymbol{t}_{\mathcal{U}}}$ with residue field $E^0_{\mathcal{U}} v_{\mathcal{U}} = \kappa_E$.

Finally, one has the following case-by-case discussion:

<u>Case 1</u>. v is trivial. Then w is trivial, hence $F = Fw \hookrightarrow Ev = E$, and $\tilde{y} \in Y(k_{\mathcal{U}})$ is defined by the k-embedding $\jmath_{\mathcal{U}} : \kappa(y) = F \to F_{\mathcal{U}} \subset k_{\mathcal{U}}$. In particular, in the above notation, the valuations $w_{\boldsymbol{t}}$ and $w_{\boldsymbol{t}_{\mathcal{U}}}$ are trivial, thus $F = F^h \hookrightarrow F^h_{\boldsymbol{t}_{\mathcal{U}}} = F_{\boldsymbol{t}_{\mathcal{U}}}$, and $E^0_{\mathcal{U}} \,|\, F_{\boldsymbol{t}_{\mathcal{U}}}$ is a finite

separable extension of $F_{\boldsymbol{t}_{\mathcal{U}}}$ such that the residue map $\mathcal{O}_{\mathcal{U}} \to \kappa_{\mathcal{U}}$ defines an isomorphism $E^0_{\mathcal{U}} \to \kappa_E$. Hence if $\imath_0 : \kappa_E \to E^0_{\mathcal{U}}$ is the inverse of the isomorphism $E^0_{\mathcal{U}} \to \kappa_E$, then

$$\imath_{\mathcal{U}} : E \xrightarrow{\imath} \kappa_E \xrightarrow{\imath_0} E^0_{\mathcal{U}} \subset k_{\mathcal{U}}$$

is an isomorphism prolonging $\jmath_{\mathcal{U}} : F \to k_{\mathcal{U}}$, thus defining $\tilde{x} \in X(k_{\mathcal{U}})$ such that $f^{k_{\mathcal{U}}}(\tilde{x}) = \tilde{y}$.

Case 2. v is non-trivial and w is trivial, hence $F = Fw$. Then we can view v as a prime divisor of $E|F$, and in the above notation one has: Let $\boldsymbol{t} \subset E$ be a preimage of a separable transcendence basis $\boldsymbol{t}_0 \subset Ev$ of $Ev|F$, and $F_{\boldsymbol{t}} = F(\boldsymbol{t})$. Then $w_{\boldsymbol{t}} := v|_{F_{\boldsymbol{t}}}$ is trivial, and the relative algebraic closure E^0 of $F(\boldsymbol{t})$ in E^h is a field of representatives for Ev. In particular, if $\pi \in E$ has $v(\pi) = 1$, then $E^h = E^0(\pi)^h$ by Fact 3.5.

Next, let $\boldsymbol{t}_{\mathcal{U}} \subset k_{\mathcal{U}}$ be a preimage of $\imath(\boldsymbol{t}_0) \subset \kappa_{\mathcal{U}}$ under the canonical residue map $\mathcal{O}_{\mathcal{U}} \to \kappa_{\mathcal{U}}$. Then $v_{\mathcal{U}}$ is trivial on $F_{\boldsymbol{t}_{\mathcal{U}}} = F_{\mathcal{U}}(\boldsymbol{t}_{\mathcal{U}})$, and $\kappa_E = \imath(Ev)$ has a unique preimage $E^0_{\mathcal{U}} \subset k_{\mathcal{U}}$ which is algebraic over $F_{\boldsymbol{t}_{\mathcal{U}}}$. Finally, the k-isomorphism $E^0 \to Ev \to E^0_{\mathcal{U}}$ together with $\pi \mapsto \pi_{\mathcal{U}}$ give rise to k-embeddings of fields

$$\imath_{\mathcal{U}} : E \hookrightarrow E^h = E^0(\pi)^h \to E^0_{\mathcal{U}}(\pi_{\mathcal{U}})^h \subset k_{\mathcal{U}},$$

with $E^0(\pi)^h \to E^0_{\mathcal{U}}(\pi_{\mathcal{U}})^h$ an isomorphism, and $\imath_{\mathcal{U}}$ prolonging $\jmath_{\mathcal{U}} : F \to k_{\mathcal{U}}$ to E. Hence the $k_{\mathcal{U}}$-rational point $\tilde{x} \in X(k_{\mathcal{U}})$ defined by $\imath_{\mathcal{U}} : E \hookrightarrow k_{\mathcal{U}}$ satisfies $f^{k_{\mathcal{U}}}(\tilde{x}) = \tilde{y}$.

Case 3. w is non-trivial. Let $\pi \in F$ be such that $w(\pi) = 1$, hence $v(\pi) = 1$ by the fact that $e(v|w) = 1$. Then $F_{\boldsymbol{t}} = F(\boldsymbol{t}) \hookrightarrow E$ gives rise to the embedding of the Henselizations $E^h \,|\, F^h_{\boldsymbol{t}}$. Reasoning as above, the unique unramified subextension $E_0 \,|\, F^h_{\boldsymbol{t}}$ of $E^h \,|\, F^h_{\boldsymbol{t}}$ satisfies $E^h = E_0$, and $F_{\boldsymbol{t}} \to F_{\boldsymbol{t}_{\mathcal{U}}}$ together with $\pi \mapsto \pi_{\mathcal{U}}$, gives rise to a k-embedding $\imath_{\mathcal{U}} : E \to k_{\mathcal{U}}$ prolonging $\jmath_{\mathcal{U}} : F \to k$, etc. One gets a point $\tilde{x} \in X(k_{\mathcal{U}})$ such that $f^{k_{\mathcal{U}}}(\tilde{x}) = \tilde{y}$.

3.2. *The implication "$\Leftarrow$".*

In the context and situation from Definition /Notation 2.17, for given $\mathcal{U} \supset \mathcal{P}$, let $y \in Y$, $F := L_y = \kappa(y)$, and $w := w_y \in \mathcal{D}(F|k)$ together with a k-embedding $\jmath : Fw = l_y \hookrightarrow \kappa_{\mathcal{U}}$ be given. We show that there is $x \in X_y$ such that setting $E := \kappa(x)$ there is $v \in \mathcal{D}(E|k)$ such that $w = v|_F$, $e(v|w) = 1$, and $Fw = l_y \hookrightarrow k_x = Ev$ is $\jmath$-pseudo-split.

Indeed, for given $\mathcal{U}$ and w, we define a $k_{\mathcal{U}}$-rational point $\tilde{y} = \tilde{y}_w \in Y(k_{\mathcal{U}})$ as follows: First, if w is trivial, let $\tilde{y}_w$ be defined by the k-embedding $\jmath : F = \kappa(y) \hookrightarrow \kappa_{\mathcal{U}} \subset k_{\mathcal{U}}$. Second, if w is non-trivial, hence a prime divisor of $F|k$, let $\kappa_w \subset F^h$ be a field of representatives for Fw. (**Note** *that since* $\mathrm{char}(k) = 0$*, such a field of representatives exists.*) Thus by Fact 3.5, one has $F^h = \kappa_w(\pi)^h$. Hence setting $\kappa'_w = \jmath(Fw) \subset \kappa_{\mathcal{U}} \subset k_{\mathcal{U}}$, one has that F^h has a canonical k-embedding $\jmath^h_{\mathcal{U}} : F^h = \kappa_w(\pi)^h \to \kappa'_w(\pi_{\mathcal{U}})^h \subset k_{\mathcal{U}}$ via $\jmath : \kappa_w \to Fw \to \kappa'_w \subset \kappa_{\mathcal{U}}$, $\pi \mapsto \pi_{\mathcal{U}}$.

Let $\tilde{y} \in Y(k_{\mathcal{U}})$ be defined by the k-embedding $\jmath_{\mathcal{U}} := \jmath^h_{\mathcal{U}}|_F : F \hookrightarrow F^h \hookrightarrow k_{\mathcal{U}}$. Since $f^{k_{\mathcal{U}}} : X(k_{\mathcal{U}}) \to Y(k_{\mathcal{U}})$ is surjective, there is some $\tilde{x} \in X(k_{\mathcal{U}})$ such that $f^{k_{\mathcal{U}}}(\tilde{x}) = \tilde{y}$. Let $\tilde{x}$ be defined by some point $x \in X$ and a k-embedding of the residue field $\imath_{\mathcal{U}} : E = \kappa(x) \hookrightarrow k_{\mathcal{U}}$. Then by mere definition, $f(x) = y$ and the canonical k-embedding $f_{xy} : F = \kappa(y) \hookrightarrow \kappa(x) = E$ satisfies $\imath_{\mathcal{U}} \circ f_{xy} = \jmath_{\mathcal{U}}$. Hence setting $v := (v_{\mathcal{U}})|_E$, one has $w = v|_F$, and the following hold: First, one has a canonical k-embedding $Fw \hookrightarrow Ev \hookrightarrow \kappa_{\mathcal{U}}$. Second, one has canonical embeddings of value groups $wF \hookrightarrow vE \hookrightarrow v_{\mathcal{U}}k_{\mathcal{U}}$; and if w is non-trivial, then by the definition of w one has: $w(\pi) = 1 = v_{\mathcal{U}}(\pi_{\mathcal{U}})$, hence $wF \hookrightarrow vE \hookrightarrow v_{\mathcal{U}}k_{\mathcal{U}}$ are isomorphisms, and $e(v|w) = 1$. Finally, since $\jmath : Fw \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to a k-embedding $Ev \hookrightarrow \kappa_{\mathcal{U}}$, it follows that $Ev|Fw$ is $\jmath$-pseudo-split.

This completes the proof of Key Lemma 3.2, hence of Theorem 3.1. $\square$

4. Proof of (Generalizations of) Theorem 1.2

Let Z be an integral k-variety, and $F = k(Z)$ be its function field. A point $z \in Z$ is called *valuation-regular-like* (*v.r.l.*), if there exist $\tilde{w} \in \mathrm{Val}_k(F)$ and $w \in \mathcal{D}(F|k)$ both having center $z \in Z$ such that $F\tilde{w} = \kappa(z)$, $Fw \,|\, \kappa(z)$ is a regular field extension, and $w(u) = 1$ for all $u \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$. We say that Z is *valuation-regular-like,* if all $z \in Z$ are v.r.l. points. We notice:

First, the *regular points* $z \in Z$ *are v.r.l.* Indeed, Let $(t_1, \dots, t_d)$ be a system of regular parameters of $\mathcal{O}_z$. Then the canonical k-embedding $F \hookrightarrow \kappa(z)((t_1))\dots((t_d))$ defines a valuation $\tilde{w} \in \mathrm{Val}_k(F)$ with $F\tilde{w} = \kappa(z)$. Further, the so called degree valuation w, defined by $w(t) = 1$ for $t \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$ has the rational function field $Fw = \kappa(z)(t_i/t_d)_{i<d}$ as residue field. In particular, regular k-varieties are valuation regular like. But the converse does not hold, because rationally double points and cusps are v.r.l. points, but not regular points.

Second, if $Z' \to Z$ is a proper birational morphism with Z' regular and Z valuation regular like, then $Z'(l) \to Z(l)$ is surjective for all field extensions $l|k$.

We define Σ_k-v.r.l. as follows. Let k, Σ_k be as in Notations/Remarks 2.4, Z be a k-variety, and $F = k(Z)$. We say that $z \in Z$ is Σ_k-*v.r.l.,* if z is v.r.l. point in the usual sense, and for every $\mathcal{U} \supset \mathcal{P}_{\Sigma_k}$ and k-embedding $\jmath_w : \kappa(z) \to \kappa_{\mathcal{U}}$, there is $w \in \mathcal{D}(F|k)$ with center z on Z such $Fw|\kappa(z)$ is $\jmath_z$-pseudo-split, i.e., $Fw|\kappa(z)$ is separably generated, and $\jmath_z$ prolongs to a k-embedding $Fw \hookrightarrow \kappa_{\mathcal{U}}$. Further, Z is Σ_k-*valuation-regular-like,* if all $z \in Z$ are Σ_k-v.r.l. Finally, a function field $F|k$ is Σ_k-*valuation-regular-like*, if $F|k$ has a co-final system $(Z_\alpha)_\alpha$ of *proper* Σ_k-valuation-regular-like models. We notice the following:

a) If $z \in Z$ is a regular point, then z is Σ_k-v.r.l. Indeed, if $\boldsymbol{t} = (t_1, \dots, t_d)$ is a regular system of parameters at z, there is $w_z \in \mathcal{D}(F|k)$ with center z on Z satisfying $w_z(t_i) = 1$ for all i and Fw_z the rational function field $Fw_z = \kappa(z)(t_i/t_d)_{1 \leqslant i<d}$. Hence since $\mathrm{td}(\kappa_{\mathcal{U}}|k)$ is infinite, every k-embedding $\kappa(z) \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to an embedding $Fw_z \hookrightarrow \kappa_{\mathcal{U}}$.

b) If k, Σ_k are as in Fact 2.5, by Corollary 2.16 one has: If $z \in Z$ is v.r.l., then z is Σ_k-v.r.l.

c) If $\mathrm{char}(k) = 0$, by Hironaka's Desingularization Theorem, every function field $F|k$ is valuation-regular-like. Hence the function fields $K = k(X)$, $L = k(Y)$ from Theorem 1.2 from the Introduction are Σ_k-regular like.

Finally, the main result of this section is the following.

Theorem 4.1. *With k, Σ_k satisfying $(\mathcal{P})$ as in Notations/Remarks 2.4 and Definition/Notation 2.17, suppose that* $\mathrm{char}(k) = 0$ *and $f : X \to Y$ is a dominant morphism of proper integral Σ_k-valuation-regular-like k-varieties. Then on has the following:*

f is Σ_k-generalized-pseudo-split iff $\mathcal{D}(L|k)$ is Σ_k-generalized-pseudo-split in $\mathcal{D}(K|k)$.

Proof. We begin by mentioning the following facts, all of which follow by mere definition.

Fact 4.2. Let $f : X \to Y$ be a dominant morphism of proper integral k-varieties with function fields $K|L$. Then there are "many" co-final systems $f_\alpha : X_\alpha \to Y_\alpha$, $\alpha \in I$ of dominant morphisms of proper k-varieties dominating $f : X \to Y$ defining $K|L$. The following hold:

1) If K and L are Σ_k-valuation-regular-like, one can choose X_α, Y_α to be so.
2) Let $v \in \mathrm{Val}_k(K)$, $w := v|_L$ have centers $x_\alpha \in X_\alpha$ on X_α, respectively $y_\alpha \in Y$ on Y_α. Then $f_\alpha(x_\alpha) = y_\alpha$, and $L \hookrightarrow K$ gives rise to canonical k-embeddings:

$\mathfrak{m}_w = \cup_\alpha \mathfrak{m}_{y_\alpha} \subset \cup_\alpha \mathcal{O}_{y_\alpha} = \mathcal{O}_w \hookrightarrow \mathcal{O}_v = \cup_\alpha \mathcal{O}_{x_\alpha} \supset \cup_\alpha \mathfrak{m}_{x_\alpha} = \mathfrak{m}_v$, $Lw = \cup_\alpha \kappa(y_\alpha) \hookrightarrow \cup_\alpha \kappa(x_\alpha) = Kv$.

3) For each $v \in \mathcal{D}(K|k)$ $\exists\, I_v \subset I$ cofinal such that $\mathcal{O}_v = \mathcal{O}_{x_\alpha}$, $\mathcal{O}_w = \mathcal{O}_{y_\alpha}$ for all $\alpha \in I_v$.

Back to the proof of Theorem 4.1, let $f_\alpha : X_\alpha \to Y_\alpha$, $\alpha \in I$ be a cofinal system of morphisms of proper Σ_k-valuation-regular-like k-varieties dominating $f : X \to Y$ and defining $K|L$. Since the proof is quite involved, we split it in the two subsections below.

4.1. *The implication "$\Rightarrow$".*

With k, Σ_k satisfying $(\mathcal{P})$ as in Notations /Remarks 2.4 and Definition /Notation 2.17, let $f : X \to Y$ be a dominant morphism of proper integral Σ_k-valuation-regular-like k-varieties and $L = k(Y)$, $K = k(X)$ be Σ_k-valuation-regular-like.

Lemma 4.3. *In the above notation, let* $\mathrm{char}(k)$ *be arbitrary, and suppose that* $f : X \to Y$ *id* Σ_k*-generalized-pseudo-split. Then* $\mathcal{D}(L|k)$ *is* Σ_k*-generalized-pseudo-split in* $\mathcal{D}(K|k)$.

Proof. We show that every $w \in \mathcal{D}(L|k)$ is Σ_k-generalized-pseudo-split in $\mathcal{D}(K|k)$.

Case 1. w is the trivial valuation of $L|k$. Then the center $y \in Y$ of w is the generic point $y = \eta_Y$ of Y, and $X_y = X_L$ is the generic fiber of $f : X \to Y$. Further, $w_y := w$ is the trivial valuation of $L_y = L$, hence $l_y = L_y = L$. Since $y = \eta_Y$ is Σ_k-pseudo-split under f, for every $\mathcal{U}$ and each k-embedding $\jmath_w : L \hookrightarrow \kappa_{\mathcal{U}}$ there is $x \in X_y = X_L$ and $v_x \in \mathcal{D}(K_x|k)$ such that $k_x|l_y$ is $\jmath_w$-pseudo-split, i.e., $k_x|l_y$ is separably generated and $\jmath_w$ prolongs to a k-embedding $\imath_x : k_x \hookrightarrow \kappa_{\mathcal{U}}$. Since X is proper, by the valuative criterion for properness, $v_x \in \mathcal{D}(K_x|k)$ has a center z on X, and actually, z lies in the closure $Z \subset X$ of x in X. Then $\mathfrak{m}_{Z,z} = \mathcal{O}_{Z,z} \cap \mathcal{O}_{v_x}$ inside $K_x = \kappa(x)$, hence one has a canonical k-embedding $\kappa(z) \hookrightarrow k_x$. Thus $\kappa(z)|k$ is separably generated (because $k_x|k$ was so), and $\imath_z := (\imath_x)|_{\kappa(z)}$ prolongs $\jmath_w$ to $\kappa(z)$. Since X is Σ_k-valuation-regular-like, there is $v \in \mathcal{D}(K|k)$ with center z on X such that $Kv|\kappa(z)$ is $\imath_z$-pseudo-split, i.e., $Kv|\kappa(z)$ is separably generated, and there is $\imath_v : Kv \to \kappa_{\mathcal{U}}$ prolonging $\imath_z$. Conclude that $Kv|l_y$ is separably generated (because $Kv|\kappa(z)$ and $\kappa(z)|l_y$ are so), and $\jmath_w = (\imath_z)|_{l_y} = (\imath_v)|_{l_y}$. Hence finally w is Σ_k-pseudo-split in $\mathcal{D}(K|k)$.

Case 2. w is non-trivial, hence $w \in \mathcal{D}(L|k)$ is a prime divisor of $L|k$. Letting $y = \eta_Y$ be the generic point of Y, one has $L_y = L$, $w_y := w \in \mathcal{D}(L_y)$, $l_y = Lw$, and $X_y = X_L$ is the generic fiber of $f : X \to Y$. Let $\jmath_y : l_y \hookrightarrow \kappa_{\mathcal{U}}$ be a k-embedding. Then $w_y \in \mathcal{D}(L_y|k)$ being Σ_k-pseudo-split under f implies that there is $x \in X_y = X_L$ and a prime divisor $v_x \in \mathcal{D}(K_x|k)$ with $w_y = (v_x)|_{L_y}$ under $L_y \hookrightarrow K_x$ such that $e(v_x|w) = 1$ and $k_x|l_y$ is $\jmath_y$-pseudo-split, i.e., $k_x|l_y$ is separably generated, and $\jmath_y$ prolongs to a k-embedding $\imath_x : k_x \hookrightarrow \kappa_{\mathcal{U}}$.

Let $\pi \in L$ satisfy $w(\pi) = 1$, hence in particular, $v_x(\pi) = 1$ under the k-embedding $L = L_y \hookrightarrow K_x$. Since $K|k$ is Σ_k-valuation-regular-like, there is $\tilde{v} \in \mathrm{Val}_L(K)$ with center $x \in X$ and $K\tilde{v} = \kappa(x) = K_x$. In particular, $\tilde{v}|_L$ is trivial on L under $L \hookrightarrow K$, and the valuation theoretical composition $v := v_x \circ \tilde{v} \in \mathrm{Val}_k(K)$ satisfies:

a) $Kv = K_x v_x = k_x$, and $w = v|_L$ under $L \hookrightarrow K$, thus $\mathcal{O}_w = \mathcal{O}_v \cap L$.

b) Since $wL = v_x K_x \hookrightarrow vK$, it follows that $v(\pi)$ is the minimal positive element of vK.

c) In particular, $\mathfrak{m}_v = \pi\mathcal{O}_v$, hence $\pi \in \mathfrak{m}_v \backslash \mathfrak{m}_v^2$.

Let $y_\alpha \in Y_\alpha$ be the center of w on Y_α. By Fact 4.2, there is a co-final segment $I_w \subset I$ such that $\mathcal{O}_w = \mathcal{O}_{y_\alpha}$, thus $\mathfrak{m}_w = \mathfrak{m}_{y_\alpha}$ and $Lw = \kappa(y_\alpha)$ for $\alpha \in I_w$. Recalling that $f_\alpha : X_\alpha \to Y_\alpha$ are proper morphisms, since $w = v|_L$ has the center $y_\alpha \in Y_\alpha$, it follows that v has a (unique) center $x_\alpha \in X_\alpha$, and $f(x_\alpha) = y_\alpha$. In particular, since $w = v|_L$, by Fact 4.2 one has: First, since $k_x = Kv$ is finitely generated over k, there is a cofinal segment $I_x \subset I$ such that $Kv = k_x = \kappa(x_\alpha)$ for all $\alpha \in I_x$. Recalling that $I_w \subset I$ is a cofinal segment such that $Lw = \kappa(y_\alpha)$ for $\alpha \in I_w$, it follows that $I' := I_w \cap I_x$ is a cofinal segment in I such that:

$$\mathcal{O}_w = \mathcal{O}_{y_\alpha} = \mathcal{O}_{x_\alpha} \cap L,\ \mathfrak{m}_w = \mathfrak{m}_{y_\alpha} = \mathfrak{m}_{x_\alpha} \cap L,\ Lw = \kappa(y_\alpha) \hookrightarrow \kappa(x_\alpha) = k_x \text{ for all } \alpha \in I'.$$

In particular, $\pi \in \mathfrak{m}_{x_\alpha}$, and $\pi \notin \mathfrak{m}_{x_\alpha}^2$ for $\alpha \in I'$ (by the fact that $\pi \notin \mathfrak{m}_v^2$).

Since X_α is Σ_k-valuation regular-like, there are $v_\alpha \in \mathcal{D}(K|k)$ with center $x_\alpha \in X_\alpha$ such that $v_\alpha(a) = 1$ for all $a \in \mathfrak{m}_{x_\alpha} \backslash \mathfrak{m}_{x_\alpha}^2$ and $k_x = \kappa(x_\alpha) \hookrightarrow Kv_\alpha$ is $\jmath_x$-pseudo-split, i.e., $Kv_\alpha|k_x$ is separably generated, and $\jmath_x : k_x \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to a k-embedding $\jmath_v : Kv_\alpha \hookrightarrow \kappa_{\mathcal{U}}$. Hence $v_\alpha(\pi) = 1$ by the fact that $\pi \notin \mathfrak{m}_{x_\alpha} \backslash \mathfrak{m}_{x_\alpha}^2$, thus $w_\alpha := (v_\alpha)|_L$ lies in $\mathcal{D}(L|k)$, and y_α is the center of w_α in Y_α (by the fact that $y_\alpha = f(x_\alpha)$, and x_α is the center of v_α in X_α). Hence $\mathcal{O}_w = \mathcal{O}_{y_\alpha} = \mathcal{O}_{w_\alpha}$, thus $w = w_\alpha$. Finally, $v_\alpha(\pi) = 1 = w_\alpha(\pi)$ implies $e(v_\alpha|w) = 1$, and $Kv_\alpha|Lw$ is $\jmath_w$-pseudo-split, because $k_x|Lw$ is so, and $Kv_\alpha|k_x$ is $\imath_x$-pseudo-split. Conclude that $w \in \mathcal{D}(L|k)$ is Σ_k-pseudo-split in $\mathcal{D}(K|k)$, as claimed. $\square$

4.2. *The implication "$\Leftarrow$".*

We show that if $\mathcal{D}(L)$ is Σ_k-pseudo-split in $\mathcal{D}(K)$, then $f : X \to Y$ satisfies hypothesis $(\mathsf{p.s.})_{\Sigma_k}$. Recalling that $X_y \subset X$ is the (reduced) fiber of f at $y \in Y$, and $L_y := \kappa(y)$, we show that every $w_y \in \mathcal{D}(L_y|k)$ is Σ_k-pseudo-split under f. First, if $y = \eta_Y$ is the generic point of Y, then the generic point $x = \eta_X$ of X is in X_y, and $L = L_y = K_x = K$ under f. Finally, $w := w_y \in \mathcal{D}(L|k)$ is Σ_k-pseudo-split in $\mathcal{D}(K|k) = \mathcal{D}(K_x|k)$ by hypothesis.

Hence w.l.o.g., $y \neq \eta_Y$, hence $X_y \subset X$ is a closed (proper) k-subvariety.

Case 1. w_y is the trivial valuation of L_y, i.e., $L_y = Lw_y = l_y$. Let $\jmath_y : l_y \hookrightarrow \kappa_{\mathcal{U}}$ be a k-embedding. First, since $L|k$ is Σ_k-regular-like, there exists $w \in \mathcal{D}(L|k)$ having center $y \in Y$ such that $Lw|l_y$ is $\jmath_y$-pseudo-split, and let $\jmath_w : Lw \hookrightarrow \kappa_{\mathcal{U}}$ prolong $\jmath_y$ to Lw. Since $\mathcal{D}(L|k)$ is Σ_k-pseudo-split in $\mathcal{D}(K|k)$, there is $v \in \mathcal{D}(K|k)$ such that $e(v|w) = 1$ and $Kv|Lw$ is $\jmath_w$-pseudo-split, i.e., j_w has a prolongation $i_v : Kv \to \kappa_{\mathcal{U}}$. Hence if $x \in X$ is the center of v on X, then $y = f(x)$ is the center of w on Y, hence $x \in X_y$, and $l_y = L_y = \kappa(y) \hookrightarrow \kappa(x) \subset Kv$ canonically. Therefore, the restriction $\imath_x : \kappa(x) \to \kappa_{\mathcal{U}}$ of $\imath_v : Kv \hookrightarrow \kappa_{\mathcal{U}}$ to $\kappa(x) \subset Kv$ prolongs $\jmath_y : l_y \hookrightarrow \kappa_{\mathcal{U}}$ to $\kappa(x)$. Thus setting $K_x := \kappa(x)$ and letting $v_x \in \mathcal{D}(K_x|k)$ be trivial, one has $k_x = \kappa(x) = K_x v_x$, and $l_y \hookrightarrow \kappa_{\mathcal{U}}$ prolongs to an embedding $k_x \hookrightarrow \kappa_{\mathcal{U}}$.

Case 2. $w_y \in \mathcal{D}(L_y|k)$ is non-trivial. The proof is a little bit involved, and takes place in two main steps: Namely let $\jmath_y : l_y \to \kappa_{\mathcal{U}}$ be given. In Step 1 we find the "right" point $x \in X_y$, and a *discrete k-valuation* v' of $K_x = \kappa(x)$ with $w_y = v'|_{L_y}$, $e(v'|w_y) = 1$, $(K_x v')|l_y$ separably generated. In Step 2 we use v' to finally find $v_x \in \mathcal{D}(K_x|k)$ with the desired properties.

Step 1. Since $L|k$ is Σ_k-valuation-regular-like, there is $\tilde{w} \in \mathrm{Val}_k(L)$ with center $y \in Y$ and $L\tilde{w} = L_y$. Then the valuation theoretical composition $w := w_y \circ \tilde{w}$ has $Lw = L_y w_y = l_y$, $\mathcal{O}_w \subset \mathcal{O}_{\tilde{w}}$, $\mathfrak{m}_w \supset \mathfrak{m}_{\tilde{w}}$, and $\mathcal{O}_{w_y} = \mathcal{O}_w/\mathfrak{m}_{\tilde{w}}$, hence $w_y L_y = wL/\tilde{w}L$ canonically. Let $\pi_y \in L_y$ have $w_y(\pi_y) = 1$, and $\pi \in \mathcal{O}_w$ be a fixed preimage of π_y under $\mathcal{O}_w \to \mathcal{O}_{w_y}$. Then $w(\pi) \in wL$ is the unique minimal positive element, and $\mathfrak{m}_w = \pi\mathcal{O}_w$, $\mathfrak{m}_{\tilde{w}} \subset \mathfrak{m}_w \subset \mathcal{O}_w$ is the unique maximal ideal not containing π, and $\mathcal{O}_{\tilde{w}} = \mathcal{O}_w[1/\pi]$. By Fact 4.2 one gets: Let $y_\alpha, \tilde{y}_\alpha \in Y_\alpha$ be the centers of w and $\tilde{w}$ on Y_α, and $\mathfrak{m}_{y_\alpha} \subset \mathcal{O}_{y_\alpha}$, $\mathfrak{m}_{\tilde{y}_\alpha} \subset \mathcal{O}_{\tilde{y}_\alpha}$ be the corresponding local rings, and $\mathfrak{p}_\alpha := \mathfrak{m}_{\tilde{w}} \cap \mathcal{O}_{y_\alpha} \in \mathrm{Spec}(\mathcal{O}_{y_\alpha})$ be the center of $\tilde{w}$ on $\mathrm{Spec}(\mathcal{O}_{y_\alpha}) \subset Y_\alpha$, hence $\mathcal{O}_{\tilde{y}_\alpha} = (\mathcal{O}_{y_\alpha})_{\mathfrak{p}_\alpha}$. One has canonical embeddings $\mathcal{O}_{y_\alpha}/\mathfrak{p}_\alpha \hookrightarrow \mathcal{O}_w/\mathfrak{m}_{\tilde{w}} = \mathcal{O}_{w_y}$, and $\mathcal{O}_w = \cup_\alpha \mathcal{O}_{y_\alpha} \subset \cup_\alpha \mathcal{O}_{\tilde{y}\alpha} = \mathcal{O}_{\tilde{w}}$,

$\mathfrak{m}_w = \cup_\alpha \mathfrak{m}_{y_\alpha} \supset \cup_\alpha \mathfrak{m}_{\tilde y_\alpha} = \mathfrak{m}_{\tilde w}$, $L_y = L\tilde w = \cup_\alpha \kappa(\tilde y_\alpha)$, $l_y = Lw = \cup_\alpha \kappa(y_\alpha)$, $\mathcal{O}_{w_y} = \cup_\alpha \mathcal{O}_{y_\alpha}/\mathfrak{p}_\alpha$. Since $L_y|k$, $l_y|k$ are finitely generated, and $\mathcal{O}_{w_y} = \mathcal{O}_w/\mathfrak{m}_{\tilde w}$, $\mathcal{O}_{w_y}/(\pi) = l_y$, there is $I_y \subset I$ cofinal segment such that for all $\alpha \in I_y$ the following hold:

$$(*) \qquad \kappa(\tilde y_\alpha) = L_y, \quad \kappa(y_\alpha) = l_y, \quad \mathcal{O}_{w_y} = \mathcal{O}_{y_\alpha}/\mathfrak{p}_\alpha, \quad \pi \notin \mathfrak{m}^2_{y_\alpha}$$

Hence replacing I by I_y, w.l.o.g., we can and <u>will</u> suppose that $(*)$ above hold for all $\alpha \in I$.

Next let $\jmath_y : l_y \hookrightarrow \kappa_{\mathcal{U}}$ be a fixed k-embedding. Since Y_α is Σ_k-valuation-regular-like, there is $w_\alpha \in \mathcal{D}(L|k)$ with center $y_\alpha \in Y_\alpha$ such that $w_\alpha(a) = 1$ for all $a \in \mathfrak{m}_{y_\alpha}\backslash\mathfrak{m}^2_{y_\alpha}$, and $Lw_\alpha|l_y$ is $\jmath_y$-pseudo-split, i.e., $Lw_\alpha|l_y$ is a separably generated, and $\jmath_y$ has a k-prolongation $\jmath_\alpha : Lw_\alpha \hookrightarrow \kappa_{\mathcal{U}}$. Since $\pi \in \mathfrak{m}_{y_\alpha}\backslash\mathfrak{m}^2_{y_\alpha}$, one has $w_\alpha(\pi) = 1$, hence $\mathfrak{m}_{w_\alpha} = (\pi)$. Since $\mathcal{D}(L|k)$ is Σ_k-pseudo-split in $\mathcal{D}(K|k)$, there is $v_\alpha \in \mathcal{D}(K|k)$ with $w_\alpha = (v_\alpha)|_L$, $e(v_\alpha|w_\alpha) = 1$, and $Kv_\alpha|Lw_\alpha$ is $\jmath_\alpha$-pseudo-split, i.e., $Kv_\alpha|Lw_\alpha$ is separably generated, and there is $\imath_\alpha : Kv_\alpha \hookrightarrow \kappa_{\mathcal{U}}$ prolonging $\jmath_\alpha$. Hence if $x_\alpha \in X_\alpha$ is the center of v_α, one has: $f_\alpha(x_\alpha) = y_\alpha$ together with canonical k-embeddings $l_y \to \kappa(x_\alpha) \to Kv_\alpha$. Since $Kv_\alpha|Lw_\alpha|l_y$ are separably generated, so is the k-subextension $\kappa(x_\alpha)|l_y$ of $Kv_\alpha|l_y$, and $(\imath_\alpha)|_{\kappa(x_\alpha)}$ prolongs $\jmath_y$. Hence $\kappa(x_\alpha)|l_y$ is $\jmath_y$-pseudo-split, and since $v_\alpha(\pi) = 1 = w_\alpha(\pi)$, one has $\pi \in \mathfrak{m}_{x_\alpha}\backslash\mathfrak{m}^2_{x_\alpha}$. Finally, the canonical projections $Y_{\alpha'} \to Y_\alpha$, $y_{\alpha'} \mapsto y_\alpha$ satisfy $\kappa(y_{\alpha'}) = l_y = \kappa(y_\alpha)$, and the projective system $(X_\alpha)_\alpha$ defining $\mathrm{Val}_k(K)$ has $(X_{y_\alpha})_\alpha$ as a projective subsystem, with projective limit $\mathrm{Val}_w(K) := \{v \in \mathrm{Val}_k(K) \,|\, v|_L = w\}$.

Let $X_{\alpha,\pi,\jmath_y} \subset X_{\alpha,\pi}$ be the set of all $x_\alpha \in X_\alpha$ satisfying the conditions

(i) $\pi \notin \mathfrak{m}^2_{x_\alpha}$; (ii) $\kappa(x_\alpha)\,|\,l_y$ is separably generated; (iii) $\jmath_y$ has a prolongation $\imath_\alpha : \kappa(x_\alpha) \hookrightarrow \kappa_{\mathcal{U}}$.

Lemma 4.4. $(X_{\alpha,\pi,\jmath_y})_\alpha$ *is a projective subsystem of* $(X_{y_\alpha})_\alpha$ *with non-empty projective limit* $\mathrm{Val}_{\jmath_y}(K)$ *consisting of all* $v \in \mathrm{Val}_w(K)$ *such that* $\pi \notin \mathfrak{m}^2_v$, $Kv\,|\,l_y$ *is separably generated, and* $\imath_y : l_y \to \kappa_{\mathcal{U}}$ *prolongs to a* k*-embedding* $\imath_v : Kv \hookrightarrow \kappa_{\mathcal{U}}$.

Proof of Lemma 4.4. First, $(X_{\alpha,\pi,\jmath_y})_\alpha$ is a projective system, because conditions (i), (ii), (iii) are compatible with specialization, i.e., if $x_{\alpha'}$ satisfies (i), (ii), (iii), and $x_{\alpha'} \mapsto x_\alpha$, then x_α obviously satisfies (i), (ii), (iii). Next let $(x_\alpha)_\alpha \in (X_{\alpha,\pi,\jmath_y})_\alpha$ be given, and $v \in \mathrm{Val}_w(K)$ be its limit. Then by Fact 4.2, it immediately follows that $\pi \in \mathfrak{m}_v\backslash\mathfrak{m}^2_v$, and further, $Kv = \cup_\alpha \kappa(x_\alpha)$ is separably generated over l_y, because each $\kappa(x_\alpha)|l_y$ is so. Finally, since $Kv = \cup_\alpha \kappa(x_\alpha)$, and there is a prolongation $\imath_\alpha : \kappa(x_\alpha) \hookrightarrow \kappa_{\mathcal{U}}$ of $\jmath_y$ to $\kappa(x_\alpha)$, by the saturation property of $\kappa_{\mathcal{U}}$, it follows that $\jmath_y : l_y \to \kappa_{\mathcal{U}}$ prolongs to a k-embedding $\imath_v : Kv \hookrightarrow \kappa_{\mathcal{U}}$. □

In the above notation, let $v \in \mathrm{Val}_{\jmath_y}(K)$ be given, hence $w = v|_L$. Since $\pi \in \mathfrak{m}_v\backslash\mathfrak{m}^2_v$, one has: $v(\pi)$ is the (unique) minimal positive element in vK. Hence $\mathfrak{m}_w = \pi\mathcal{O}_w \hookrightarrow \pi\mathcal{O}_v = \mathfrak{m}_v$, and therefore: $\mathcal{O}_{\tilde v} = \mathcal{O}_v[1/\pi]$, is a valuation ring such that $\tilde w = \tilde v|_L$, and $\mathcal{O}_{v_0} := \mathcal{O}_v/\mathfrak{m}_{\tilde v}$ is a DVR of $K_0 := K\tilde v$ with $\mathfrak{m}_{v_0} = \pi\mathcal{O}_{v_0} = \mathfrak{m}_v/\mathfrak{m}_{\tilde v}$. In particular, $e(v_0|w_y) = 1$, and $l_y = L_y w_y \hookrightarrow (K_0 v_0) = Kv$ is the residue field extension.

Let $x \in X$ be the center of $\tilde v$ on X, and $K_x := \kappa(x) \hookrightarrow K\tilde v = K_0$ be canonical embeddings. Then $\tilde w = \tilde v|_L$ implies $f(x) = y$, and let $L_y = \kappa(y) \hookrightarrow \kappa(x) = K_x \hookrightarrow K_0$ be the resulting residue field embeddings. Then $v'_x := v_0|_{K_x}$ satisfies: $v'_x|_{L_y} = v_0|_{L_y} = w_y$, hence $e(v'_x|v_y)$ divides $e(v_0|w_y) = 1$, thus $e(v'_x|w_y) = 1$, and let $l_y \hookrightarrow k'_x := K_x v'_x \hookrightarrow Kv$ be the residue field embeddings. Then one has: Since $Kv\,|\,l_y$ is separably generated, so is $k'_x\,|\,l_y$, and second, the restriction of $\imath_v : Kv \hookrightarrow \kappa_{\mathcal{U}}$ to $k'_x \subset Kv$ prolongs $\jmath_y$ to a k-embedding $\imath'_x : k'_x \hookrightarrow \kappa_{\mathcal{U}}$.[2]

[2] **Note** that we do not claim here that v_1 is a prime divisor of $K_x|k$, but rather a discrete valuation.

Step 2. One concludes the proof of Case 2) of assertion 2) by applying the Key Lemma below with $F = L_y$, $w = w_y$, $Fw = l_y$, $E = K_x$, and taking into account that in our situation $\mathrm{char}(k) = 0$, hence all transcendence bases are separable.

Key Lemma 4.5. *Let $F, w \hookrightarrow E, v'$ be an extension of separably generated discrete valued function fields over k such that $w \in \mathcal{D}(F|k)$, $v'E = wF$, and $\jmath : Fw \to \kappa_{\mathcal{U}}$ be a k-embedding which prolongs to a k-embedding $Ev' \hookrightarrow \kappa_{\mathcal{U}}$. Then there are $v \in \mathcal{D}(E|k)$ with $w = v|_F$, $e(v|w) = 1$, and $\imath : Ev \to \kappa_{\mathcal{U}}$ prolonging $\jmath$.*

Proof of Key Lemma 4.5. For transcendence bases $\boldsymbol{t} = (t_1, \dots, t_d)$ of $Ev|Fw$, we consider their preimages in $\mathcal{O}_v$ (which we again denote by $\boldsymbol{t}$), and set $F_{\boldsymbol{t}} := F(\boldsymbol{t})$. Then $w_{\boldsymbol{t}} := v'|_{F_{\boldsymbol{t}}}$ is the Gauss valuation on $F_{\boldsymbol{t}}$ defined by w and $\boldsymbol{t}$, hence $wF = w_{\boldsymbol{t}}F_{\boldsymbol{t}} = v'E$, $\mathrm{td}(F_{\boldsymbol{t}}|F) = \mathrm{td}(F_{\boldsymbol{t}}w_{\boldsymbol{t}}|Fw)$.

In particular, if $d = \mathrm{td}(E|F)$, then one has:

$$\mathrm{td}(Ev|k) = \mathrm{td}(Ev|Fw) + \mathrm{td}(Fw|k) = \mathrm{td}(E|F) + \mathrm{td}(F|k) - 1 = \mathrm{td}(E|k) - 1,$$

hence $v \in \mathcal{D}(E|k)$, and there is noting left to prove.

Next suppose that $e := \mathrm{td}(E|F_{\boldsymbol{t}}) > 0$. The proof in this case is more involved. Let namely $E^h|F^h_{\boldsymbol{t}}$ be the corresponding Henselizations, and $F_1 \subset E^h$ be the relative algebraic closure of $F^h_{\boldsymbol{t}}$ in E^h and set $w_1 := ({v'}^h)|_{F_1}$. Since $Ev'|F_{\boldsymbol{t}}w_{\boldsymbol{t}}$ is algebraic separable and $v'E = w_{\boldsymbol{t}}F_{\boldsymbol{t}} = wF$, one has $F_1w_1 = Ev'$ and $F_1|F^h_{\boldsymbol{t}}$ is unramified. Further, $E_1 := EF_1 \subset E^h$ and $v_1 := ({v'}^h)|_{E_1}$ satisfy: $E_1|F_1$ is separably generated (because $E|F$ is so), $v_1E_1 = w_1F_1$, $E_1v_1 = Ev' = F_1w_1$, hence $F_1, w_1 \hookrightarrow E_1, v_1$ is an immediate extension of valued fields, and $E_1|F_1$ is a separably generated function field.

Let $E_1 = F_1(\theta_0, \boldsymbol{\theta})$ with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_e)$ a separable transcendence basis of $E_1|F_1$. Replacing θ_i by $u\theta_i$ with $u \in F$, $w(u) \gg 0$, w.l.o.g., one has $v_1(\theta_i) \geqslant 0$ for all i, and there is an irreducible polynomial $p(\boldsymbol{T}) \in \mathcal{O}_{w_1}[\boldsymbol{T}]$, $\boldsymbol{T} = (T_0, \dots, T_e)$ such that $p(\theta_0, \boldsymbol{\theta}) = 0$, and $p' := \partial p/\partial T_0$ satisfies $p'(\theta_0, \boldsymbol{\theta}) \neq 0$. Since F_1 is dense in E_1, there are $\boldsymbol{x}' = (x'_0, \dots, x'_e)$ in F_1^{e+1} such that $v_1(\theta_i - x'_i) \gg 0$, hence $w_1(x'_i) \geqslant 0$ for all i, and $w_1\big(p(\boldsymbol{x}')\big) \gg 0$ and $w_1\big(p'(\boldsymbol{x}')\big) = v_1\big(p'(\theta_0, \boldsymbol{\theta})\big)$. By Hensel's Lemma over F_1, there are $\boldsymbol{x} = (x_0, \dots, x_e)$ in F_1^{e+1} such that $p(\boldsymbol{x}) = 0$, $w_1(x'_i - x_i) \gg 0$ for all i, hence $w_1\big(p'(\boldsymbol{x})\big) = v_1\big(p'(\theta_0, \boldsymbol{\theta})\big)$.

Let $F_0 \subset F_1$ be a finite extension of $F_{\boldsymbol{t}}$ such that $p(\boldsymbol{T})$ and $\boldsymbol{x}$ are defined over F_0, and set $w_0 = (w_1)|_{F_0}$, $E_0 = EF_0 = F_0(\theta_0, \boldsymbol{\theta})$. After the change of variables $T_i \leftrightarrow T_i - x_i$, w.l.o.g., one has $\boldsymbol{x} = (0, \dots, 0)$, hence if $p(\boldsymbol{T}) = p_1(\boldsymbol{T}) + p_2(\boldsymbol{T}) + \dots$ with $p_j(\boldsymbol{T})$ the degree j homogeneous part of $p(\boldsymbol{T})$, then $p_1(\boldsymbol{T}) = a_0T_0 + \dots + a_eT_e$ with $a_i \in \mathcal{O}_{w_0}$ and $a_0 \neq 0$. After the change of variable $T_0 \leftrightarrow T_0/a_0$ we can suppose that $a_0 = 1$, concluding that $\boldsymbol{x} = (0, \dots, 0)$ is a smooth point of $\mathcal{Z} := V\big(p(\boldsymbol{T})\big) \subset \mathbb{A}^{e+1}_{\mathcal{O}_{\tilde{w}}}$, and $(\pi, \boldsymbol{\theta})$ is a regular system of parameters at $\boldsymbol{x}$, where $\pi \in \mathcal{O}_w$ is any uniformizing parameter. Hence the completion of the local ring $\mathcal{O}_{\boldsymbol{x}}$ of $\boldsymbol{x} \in \mathcal{Z}$ is of the form $\widehat{\mathcal{O}}_{w_0}[[\boldsymbol{\theta}]]$, concluding that $\theta_0 = f(\boldsymbol{\theta})$ is a power series in $\boldsymbol{\theta}$ over $\widehat{\mathcal{O}}_{w_0}$. Hence setting $\boldsymbol{\eta} = \boldsymbol{\theta}/\pi$, one has $E_0 = F_0(\theta_0, \boldsymbol{\eta})$, and $\theta_0 = f(\pi\boldsymbol{\eta})$ is a power series in $\boldsymbol{\eta}$ which lies in the π-adic completion of $\mathcal{O}_{w_0}[\boldsymbol{\eta}]$, hence in the completion $F_0(\boldsymbol{\eta})_{w_{0,\boldsymbol{\eta}}}$ of $F_0(\boldsymbol{\theta})$ w.r.t. the Gauss valuation $w_{0,\boldsymbol{\eta}}$ defined by w_0 and $\boldsymbol{\eta}$ on $F_0(\boldsymbol{\eta})$, and so $E_0 \subset F_0(\boldsymbol{\eta})_{w_{0,\boldsymbol{\eta}}}$. Hence if v_0 is the prolongation of $w_{0,\boldsymbol{\eta}}$ to $E_0 \subset F_0(\boldsymbol{\theta})_{w_{0,\boldsymbol{\eta}}}$, then $E_0v_0 = (F_0w_0)(\boldsymbol{\eta})$, thus $\mathrm{td}(E_0v_0|F_0w_0) = e$. Therefore, $\mathrm{td}(E_0v_0|Fw) = e + \mathrm{td}(F_0w_0|Fw) = e + d = \mathrm{td}(E|F)$, implying that v_0 is a prime divisor of $E_0|k$. Hence finally $v := (v_0)|_E$ is a prime divisor of E with $v|_F = w$, $e(v|w) = e(v_0|w_0) = 1$, and $Ev \subset E_0v_0 = (F_0w_0)(\boldsymbol{\eta})$. Since $F_0|F_{\boldsymbol{t}}$ is finite, it follows that $F_0w_0|F_{\boldsymbol{t}}w_{\boldsymbol{t}}$ is finite, hence $F_0w_0 \subset Ev'$ is a finitely generated Fw-subextension of

$Ev'|Fw$. Thus by hypothesis, there is a prolongation $\jmath_0 : F_0w_0 \to \kappa_{\mathcal{U}}$ of $\jmath$. Finally, since $\kappa_{\mathcal{U}}|k$ has infinite transcendence degree, $\jmath_0$ prolongs to a k-embedding $\imath_0 : F_0w_0(\boldsymbol{\eta}) \to \kappa_{\mathcal{U}}$, which restricts to an embedding $\imath : Ev \to \kappa_{\mathcal{U}}$, prolonging $\jmath : Fw \to \kappa_{\mathcal{U}}$. □

5. Final Remarks

First, it is believed that the hypothesis $(\mathsf{qAKE})_{\Sigma_k}$ always holds, in particular, assertion 1) of Theorem 3.1 should hold unconditionally. Second, the question whether the conclusion of assertion 2) of Theorem 3.1 holds in positive characteristic, is related to subtle questions concerning the relationship between ramification index and purely inseparable non-liftable extensions of the residue field of prime divisors.

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Department of Mathematics, University of Pennsylvania
DRL, 209 S 33rd Street, Philadelphia, PA 19104, USA
E-mail address: `pop@math.upenn.edu`
URL: `https://www.math.upenn.edu/~pop/`