

Large Fields & Galois Theory

I. Introduction/Motivation/(Hi)story:

G profinite group; Denote:

- $\text{f.q.}(G) = \left(\text{finite quotients of } G \right)$
- $\text{cd}(G)$ cohom.dim. of G

K field, $\overline{K}|K$ separable closure; Denote:

- $G_K := \text{Aut}(\overline{K}|K)$ abs. Galois group of K
- $\text{cd}(K) := \text{cd}(G_K)$ cohom.dim. of K

Two Questions:

Inverse Galois Problem IGP:

Is $\text{f.q.}(G_{\mathbb{Q}}) = \left(\text{all finite groups} \right)$?

Bogomolov's Freeness Conjecture:

Is $\text{cd}(K[\sqrt[\infty]{K^{\times}}]) \leq 1$?

Comments/Explanations... Serre's Conjecture II...

State of the Art/classical Facts:

- Hilbert: IGP for $\mathbb{Q}(t) \Rightarrow$ IGP for \mathbb{Q} .

[Follows from: “ \mathbb{Q} is Hilbertian.”]

Conjecture **IGP** $_{k(t)}$:

f.q. $(G_{k(t)}) =$ (all finite groups) for all k .

Evidence for **IGP** $_{k(t)}$: true for

- k alg.closed char = 0, e.g. $k = \mathbb{C}$ (RET).
- k alg.closed char > 0 (Harbater, etc).
- $k = \mathbb{R}, \mathbb{Q}_p$, or complete w.r.t. $||$
(Harbater, Serre, Liu, etc.)

Evidence for **BFC**: true for

- K global field (Tate, Shafarevich)
- $K = k(t)$ all k alg.closed (Tsen).
- **BFC** for $K \Rightarrow$ **BFC** for $K((t))$.

Embedding problems/Large fields

Def. Embedding problem EP_G for G is a diagram of surjective group hom's

$$\begin{array}{ccc} & & G \\ & & \downarrow \gamma \\ EP_G & & \\ & B \xrightarrow{\alpha} & A \end{array}$$

- EP_G is finite if B finite; split if α split.
- $\beta : G \rightarrow B$ is solution, if $\alpha \circ \beta = \gamma$;
proper solution, if surjective.

Two fundamental results:

Thm (Serre, Gruenberg,???) . *TFAE*

- $cd(G) \leq 1$.
- Every (finite) EP_G has solutions.*

Thm (Iwasawa). *For G countably generated TFAE*

- G is profinite free of infinite rank.*
- Every finite EP_G has proper solutions.*

Def. k is called *large field* if for all curves X/k :
 $X_{\text{reg}}(k) \neq \emptyset \Rightarrow X(k)$ infinite.

Examples:

- alg.closed fields, PAC fields.
- real/ p -adically closed fields.
- More general: Henselian valued fields.
- p -fields.

Recall: Embed.problem for $G_{k(t)}$ is a diagram:

$$\begin{array}{ccc}
 & & G_{k(t)} \\
 & & \downarrow \gamma \\
 \text{EP}_{k(t)} & & A \\
 & \xrightarrow{\alpha} &
 \end{array}$$

Thm (P 1996). *Given k large field, one has:*

- 1) \forall finite split $\text{EP}_{k(t)}$ has proper solutions.
- 2) Conjecture $\text{IGP}_{k(t)}$ holds.

Comments/Remarks... **Rational connectedness...**

What about non-large fields?

Thm (Koenigsmann's example 2003).

\exists non-large fields K s.t. Conjecture IGP_K holds.

Thm (Harbater–Stevenson 2004).

Given k alg.closed, set $K = k((t, u))$. Then

- 1) Every fin. split EP_K has $|K|$ proper sol.
- 2) Conjecture IGP_K holds.

Thm (Harbater 2006).

$k, K := k((t, u))$ as above. Then $G_{K^{\text{ab}}}$ is profinite free. In part., conjecture BFC holds for K .

Thm (E. Paran, Thesis 2006).

R local, Noetherian, complete, ... $K = \text{Quot}(R)$.

- 1) Every finite split EP_K has proper sol.
- 2) Conjecture IGP_K holds.

Comments/Remarks... **Proofs:** difficult & huge...

...It looks like we are beyond “large” at last...

U n f o r t u n a t e l y . . .

...continue on the blackboard...