

## FUNCTION FIELDS OF ONE VARIABLE OVER PAC FIELDS

MOSHE JARDEN AND FLORIAN POP

Received: July 8, 2009

Communicated by Peter Schneider

ABSTRACT. We give evidence for a conjecture of Serre and a conjecture of Bogomolov.

2000 Mathematics Subject Classification: 12E30, 20G15

Keywords and Phrases: Function fields of one variable, PAC fields, Conjecture II of Serre,

Conjecture II of Serre considers a field  $F$  of characteristic  $p$  with  $\text{cd}(\text{Gal}(F)) \leq 2$  such that either  $p = 0$  or  $p > 0$  and  $[F : F^p] \leq p$  and predicts that  $H^1(\text{Gal}(F), G) = 1$  (i.e. each principal homogeneous  $G$ -space has an  $F$ -rational point) for each simply connected semi-simple linear algebraic group  $G$  [Ser97, p. 139].

As Serre notes, the hypothesis of the conjecture holds in the case where  $F$  is a field of transcendence degree 1 over a perfect field  $K$  with  $\text{cd}(\text{Gal}(K)) \leq 1$ . Indeed, in this case  $\text{cd}(\text{Gal}(F)) \leq 2$  [Ser97, p. 83, Prop. 11] and  $[F : F^p] \leq p$  if  $p > 0$  (by the theory of  $p$ -bases [FrJ08, Lemma 2.7.2]). We prove the conjecture for  $F$  in the special case, where  $K$  is PAC of characteristic 0 that contains all roots of unity.

One of the main ingredients of the proof is the projectivity of  $\text{Gal}(K(x)_{\text{ab}})$  (where  $x$  is transcendental over  $K$  and  $K(x)_{\text{ab}}$  is the maximal Abelian extension of  $K(x)$ ). We also use the same ingredient to establish an analog to the wellknown open problem of Shafarevich that  $\text{Gal}(\mathbb{Q}_{\text{ab}})$  is free. Under the assumption that  $K$  is PAC and contains all roots of unity we prove that  $\text{Gal}(K(x)_{\text{ab}})$  is not only projective but even free. This proves a stronger version of a conjecture of Bogomolov for a function field of one variable  $F$  over a PAC field that contains all roots of unity [Pos05, Conjecture 1.1].

1. THE PROJECTIVITY OF  $\text{Gal}(K(x)_{\text{ab}})$

We denote the separable (resp. algebraic) closure of a field  $K$  by  $K_s$  (resp.  $\tilde{K}$ ) and its absolute Galois group by  $\text{Gal}(K)$ . The field  $K$  is said to be PAC if every absolutely irreducible variety defined over  $K$  has a  $K$ -rational point. The proof of the projectivity result applies a local-global principle for Brauer groups to reduce the statement to Henselian fields.

For a prime number  $p$  and an Abelian group  $A$ , we say that  $A$  is  $p'$ -DIVISIBLE, if for each  $a \in A$  and every positive integer  $n$  with  $p \nmid n$  there exists  $b \in A$  such that  $a = nb$ . Note that if  $p = 0$ , then “ $p'$ -divisible” is the same as “divisible”.

LEMMA 1.1: *Let  $p$  be 0 or a prime number,  $B$  a torsion free Abelian group, and  $A$  is a  $p'$ -divisible subgroup of  $B$  of finite index. Then  $B$  is also  $p'$ -divisible.*

*Proof:* First suppose  $p = 0$  and let  $m = (B : A)$ . Then, for each  $b \in B$  and a positive integer  $n$  there exists  $a \in A$  such that  $mb = mna$ . Since  $B$  is torsion free,  $m = na$ . Thus,  $B$  is divisible.

Now suppose  $p$  is a prime number, let  $mp^k = (B : A)$ , with  $p \nmid m$  and  $k \geq 0$ , and consider  $b \in B$ . Then  $mp^k b \in A$ . Hence, for each positive integer  $n$  with  $p \nmid n$  there exists  $a \in A$  with  $mp^k b = mna$ . Thus,  $p^k b = na$ . Since  $p \nmid n$ , there exist  $x, y \in \mathbb{Z}$  such that  $xp^k + yn = 1$ . It follows from  $xp^k b = xna$  that  $b = n(xa + yb)$ , as claimed. □

COROLLARY 1.2: *Let  $L/K$  be an algebraic field extension,  $v$  a valuation of  $L$ , and  $p = 0$  or  $p$  is a prime number. Suppose that  $v(K^\times)$  is  $p'$ -divisible. Then  $v(L^\times)$  is  $p'$ -divisible.*

*Proof:* Let  $x \in L^\times$  and  $n$  a positive integer with  $p \nmid n$ . Then  $v(K(x)^\times)$  is a torsion free Abelian group and  $v(K^\times)$  is a subgroup of index at most  $[K(x) : K]$ . Since  $v(K^\times)$  is  $p'$ -divisible, Lemma 1.1 gives  $y \in K(x)^\times$  such that  $v(x) = nv(y)$ . It follows that  $v(L^\times)$  is  $p'$ -divisible. □

Given a Henselian valued field  $(M, v)$  we use  $v$  also for its unique extension to  $M_s$ . We use a bar to denote the residue with respect to  $v$  of objects associated with  $M$ , let  $O_M$  be the valuation ring of  $M$ , and let  $\Gamma_M = v(M^\times)$  be the value group of  $M$ .

We write  $\text{cd}_l(K)$  and  $\text{cd}(K)$  for the  $l$ th cohomological dimension and the cohomological dimension of  $\text{Gal}(K)$  and note that  $\text{cd}(K) \leq 1$  if and only if  $\text{Gal}(K)$  is projective [Ser97, p. 58, Cor. 2].

LEMMA 1.3: *Let  $(M, v)$  be a Henselian valued field. Suppose  $p = \text{char}(M) = \text{char}(\bar{M})$ ,  $\text{Gal}(\bar{M})$  is projective, and  $\Gamma_M$  is  $p'$ -divisible. Then  $\text{Gal}(M)$  is projective.*

*Proof:* We denote the INERTIA FIELD of  $M$  by  $M_u$ . It is determined by its absolute Galois group:  $\text{Gal}(M_u) = \{\sigma \in \text{Gal}(M) \mid v(\sigma x - x) > 0 \text{ for all } x \in M_s \text{ with } v(x) \geq 0\}$ . The map  $\sigma \mapsto \bar{\sigma}$  of  $\text{Gal}(M)$  into  $\text{Gal}(\bar{M})$  such that  $\bar{\sigma} \bar{x} = \overline{\sigma x}$  for each  $x \in O_M$  is a well defined epimorphism [Efr06, Thm. 16.1.1] whose kernel is  $\text{Gal}(M_u)$ . It therefore defines an isomorphism

$$(1) \quad \text{Gal}(M_u/M) \cong \text{Gal}(\bar{M}).$$

CLAIM A:  $\bar{M}_u$  is separably closed. Let  $g \in \bar{M}[X]$  be a monic irreducible separable polynomial of degree  $n \geq 1$ . Then there exists a monic polynomial  $f \in O_{M_u}[X]$  of degree  $n$  such that  $\bar{f} = g$ . We observe that  $f$  is also irreducible and separable. Moreover, if  $f(X) = \prod_{i=1}^n (X - x_i)$  with  $x_1, \dots, x_n \in M_s$ , then  $g(X) = \prod_{i=1}^n (X - \bar{x}_i)$ . Given  $1 \leq i, j \leq n$  there exists  $\sigma \in \text{Gal}(M_u)$  such that  $\sigma x_i = x_j$ . By definition,  $\bar{x}_j = \overline{\sigma x_i} = \bar{\sigma x_i} = \bar{x}_i$ . Since  $g$  is separable,  $i = j$ , so  $n = 1$ . We conclude that  $\bar{M}_u$  is separably closed.

CLAIM B: Each  $l$ -Sylow group of  $\text{Gal}(M_u)$  with  $l \neq p$  is trivial. Indeed, let  $L$  be the fixed field of an  $l$ -Sylow group of  $\text{Gal}(M_u)$  in  $M_s$ . If  $l = 2$ , then  $\zeta_l = -1 \in L$ . If  $l \neq 2$ , then  $[L(\zeta_l) : L] | l - 1$  and  $[L(\zeta_l) : L]$  is a power of  $l$ , so  $\zeta_l \in L$ .

Assume that  $\text{Gal}(L) \neq 1$ . By the theory of finite  $l$ -groups,  $L$  has a cyclic extension  $L'$  of degree  $l$ . By the preceding paragraph and Kummer theory, there exists  $a \in L^\times$  such that  $L' = L(\sqrt[l]{a})$ . By Corollary 1.2, there exists  $b \in L^\times$  such that  $lv(b) = v(a)$ . Then  $c = \frac{a}{b^l}$  satisfies  $v(c) = 0$ . By Claim A,  $\bar{L}$  is separably closed. Therefore,  $\bar{c}$  has an  $l$ th root in  $\bar{L}$ . By Hensel's lemma,  $c$  has an  $l$ th root in  $L$ . It follows that  $a$  has an  $l$ -root in  $L$ . This contradiction implies that  $L = M_s$ , as claimed.

Having proved Claim B, we consider again a prime number  $l \neq p$  and let  $G_l$  be an  $l$ -Sylow subgroup of  $\text{Gal}(M)$ . By the Claim,  $G_l \cap \text{Gal}(M_u) = 1$ , hence the map  $\text{res} : \text{Gal}(M) \rightarrow \text{Gal}(M_u/M)$  maps  $G_l$  isomorphically onto an  $l$ -Sylow subgroup of  $\text{Gal}(M_u/M)$ . By (1),  $G_l$  is isomorphic to an  $l$ -Sylow subgroup of  $\text{Gal}(\bar{M})$ . Since the latter group is projective, so is  $G_l$ , i.e.  $\text{cd}_l(G) \leq 1$  [Ser97, p. 58, Cor. 2].

Finally, if  $p \neq 0$ , then  $\text{cd}_p(M) \leq 1$  [Ser97, p. 75, Prop. 3], because then  $\text{char}(M) = p$ . It follows that  $\text{cd}(M) \leq 1$  [Ser97, p. 58, Cor. 2]. □

LEMMA 1.4: Let  $F$  be an extension of a PAC field  $K$  of transcendence degree 1 and characteristic  $p$ . Suppose  $v(F^\times)$  is  $p'$ -divisible for each valuation  $v$  of  $F/K$ . Then  $\text{Gal}(F)$  is projective.

*Proof:* Let  $K_{\text{ins}}$  be the maximal purely inseparable algebraic extension of  $K$  and set  $F' = FK_{\text{ins}}$ . Then  $K_{\text{ins}}$  is PAC [FrJ08, Cor. 11.2.5],  $\text{trans.deg}(F'/K_{\text{ins}}) = 1$ , and  $v((F')^\times)$  is  $p'$ -divisible for every valuation  $v$  of  $F'$  (by Corollary 1.2). Moreover,  $\text{Gal}(F') = \text{Gal}(F)$ . Thus, we may replace  $K$  by  $K_{\text{ins}}$  and  $F$  by  $F'$ , if necessary, to assume that  $K$  is perfect.

Let  $V(F/K)$  be a system of representatives of the equivalence classes of valuations of  $F$  that are trivial on  $K$ . For each  $v \in V(F/K)$  we choose a Henselian closure  $F_v$  of  $F$  at  $v$ . By [Efr01, Thm. 3.4], there is an injection of Brauer groups,

$$(2) \quad \text{Br}(F) \rightarrow \prod_{v \in V(F/K)} \text{Br}(F_v).$$

For each  $v \in V(F/K)$  we have,  $v(F_v^\times) = v(F^\times)$  is  $p'$ -divisible. Also, the residue field  $\bar{F}_v$  is an algebraic extension of  $K$ . Since  $K$  is PAC, a theorem of Ax says

that  $\text{Gal}(K)$  is projective [FrJ08, Thm. 11.6.2], hence  $\text{Gal}(\bar{F}_v)$  is projective [FrJ08, Prop. 22.4.7]. Finally,  $\text{char}(F_v) = \text{char}(\bar{F}_v)$ . Therefore, by Lemma 1.3,  $\text{Gal}(F_v)$  is projective, hence  $\text{Br}(F_v) = 0$  [Ser97, p. 78, Prop. 5]. It follows from the injectivity of (2) that  $\text{Br}(F) = 0$ .

If  $F_1$  is a finite separable extension of  $F$ ,  $v_1 \in V(F_1/K)$ , and  $v = v_1|_F$ , then  $v(F^\times)$  is  $p'$ -divisible. Hence, by Corollary 1.2,  $v_1((F_1)^\times)$  is  $p'$ -divisible. It follows from the preceding paragraph that  $\text{Br}(F_1) = 0$ . Consequently, by [Ser97, p. 78, Prop. 5],  $\text{cd}(\text{Gal}(F)) \leq 1$ . □

LEMMA 1.5: *Let  $p$  be either 0 or a prime number and let  $\Gamma$  be an additive subgroup of  $\mathbb{Q}$ . Suppose  $\frac{1}{n} \in \Gamma$  for each positive integer  $n$  with  $p \nmid n$ . Then  $\Gamma$  is  $p'$ -divisible.*

*Proof:* We consider  $\gamma \in \Gamma$ . If  $p = 0$ , we write  $\gamma = \frac{a}{b}$ , with  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ . Given  $n \in \mathbb{N}$ , we have  $\frac{\gamma}{n} = a \cdot \frac{1}{nb} \in \Gamma$ .

If  $p > 0$ , we write  $\gamma = \frac{a}{bp^k}$ , where  $a \in \mathbb{Z}$ ,  $b \in \mathbb{N}$ ,  $k \in \mathbb{Z}$ , and  $p \nmid a, b$ . Let  $n \in \mathbb{N}$  with  $p \nmid n$ . If  $k \leq 0$ , then  $\frac{\gamma}{n} = ap^{-k} \cdot \frac{1}{nb} \in \Gamma$ . If  $k > 0$ , we may choose  $x, y \in \mathbb{Z}$  such that  $xp^k + ynb = 1$ . Then  $\frac{\gamma}{n} = \frac{a}{nbp^k} = \frac{axp^k + aynb}{nbp^k} = ax \cdot \frac{1}{nb} + by \cdot \frac{a}{bp^k} \in \Gamma$ , as claimed. □

PROPOSITION 1.6: *Let  $K$  be a PAC field that contains all roots of unity and let  $E$  be an extension of  $K$  of transcendence degree 1. Then  $\text{Gal}(E_{\text{ab}})$  is projective.*

*Proof:* First we consider the case where  $E = K(x)$ , where  $x$  is transcendental over  $K$ , and set  $F = E_{\text{ab}}$ . In the notation of Lemma 1.4 we consider a valuation  $v \in V(F/K)$  normalized in such a way that  $v(E^\times) = \mathbb{Z}$ . Then  $v(F^\times) \leq \mathbb{Q}$ . On the other hand, let  $p = \text{char}(K)$  and consider a positive integer  $n$  with  $p \nmid n$ . Let  $e \in E$  with  $v(e) = 1$ . Then  $e^{1/n} \in F$  (because  $K$  contains a root of 1 of order  $n$ ). Therefore,  $\frac{1}{n} = v(e^{1/n}) \in v(F^\times)$ . By Lemma 1.5,  $v(F^\times)$  is  $p'$ -divisible. We conclude from Lemma 1.4 that  $\text{Gal}(F)$  is projective.

In the general case we choose  $x \in E$  transcendental over  $K$ . By the preceding paragraph,  $\text{Gal}(K(x)_{\text{ab}})$  is projective. Since taking purely inseparable extensions of a field does not change its absolute Galois group,  $\text{Gal}(K(x)_{\text{ab,ins}})$  is projective. Now note that  $\text{Gal}(E_{\text{ab,ins}})$  as a subgroup of  $\text{Gal}(K(x)_{\text{ab,ins}})$  is also projective. Hence,  $\text{Gal}(E_{\text{ab}})$  is projective. □

Remark 1.7: Proposition 1.6 is false if  $K$  does not contain all roots of unity. Indeed, the authors will elsewhere provide an example of a prime number  $l$  and a PAC field  $K$  of characteristic 0 that contains all roots of unity of order  $n$  with  $l \nmid n$  but not  $\zeta_l$  such that  $\text{Gal}(K(x)_{\text{ab}})$  is not projective. □

2. SERRE AND SHAFAREVICH

We refer to a simply connected semi-simple linear algebraic group  $G$  as a SIMPLY CONNECTED GROUP. In this case  $H^1(\text{Gal}(K), G)$  will be also denoted by  $H^1(K, G)$ . Since each element of  $H^1(K, G)$  is represented by a principal homogeneous space  $V$  of  $G$  and  $V$  is an absolutely irreducible variety defined over

$K$ ,  $V$  has a  $K$ -rational point if  $K$  is PAC. Hence,  $V$  is equivalent to  $G$  [LaT58, Prop. 4]. Thus,  $H^1(K, G) = 1$ .

The proof of Serre’s Conjecture II in our case is based on the following consequence of a theorem of Colliot-Thélène, Gille, and Parimala:

**PROPOSITION 2.1:** *Let  $F$  be a field and  $G$  a simply connected group defined over  $F$ . Suppose  $F$  is a  $C_2$ -field of characteristic 0,  $\text{cd}(F) \leq 2$ , and  $\text{cd}(F_{\text{ab}}) \leq 1$ . Then  $H^1(F, G) = 1$ .*

*Proof:* Let  $F'$  be a finite extension of  $F$ . Since  $F$  is  $C_2$ , [CGP04, Thm. 1.1(vi)] implies that if the exponent  $e$  of a central simple algebra  $A$  over  $F'$  is a power of 2 or a power of 3, then  $e$  is equal to the index of  $A$ . Since  $\text{cd}(F) \leq 2$  and  $\text{cd}(F_{\text{ab}}) \leq 1$ , [CGP04, Thm. 1.2(v)] implies that  $H^1(F, G) = 1$ . □

*Remark 2.2:* By Merkuriev-Suslin, the assumption that  $F$  is a  $C_2$ -field implies that  $\text{cd}(F) \leq 2$  [Ser97, end of page 88]. However, we will be able to prove both properties of  $F$  directly in the application we have in mind. □

The following result establishes the first condition on  $F$ .

**LEMMA 2.3:** *Let  $F$  be an extension of transcendence degree 1 over a perfect PAC field  $K$ . Suppose either  $\text{char}(K) > 0$  and  $K$  contains all roots of unity or  $\text{char}(K) = 0$ . Then  $\text{cd}(F) \leq 2$  and  $F$  is a  $C_2$ -field.*

*Proof:* By Ax,  $\text{cd}(K) \leq 1$  [FrJ08, Thm. 11.6.2]. Hence, by [Ser97, p. 83, Prop. 11],  $\text{cd}(F) \leq 2$ .

A conjecture of Ax from 1968 says that every perfect PAC field  $K$  is  $C_1$  [FrJ08, Problem 21.2.5]. The conjecture holds if  $K$  contains an algebraically closed field [FrJ08, Lemma 21.3.6(a)]. In particular, if  $p = \text{char}(K) > 0$  and  $K$  contains all roots of unity, then  $\overline{\mathbb{F}}_p \subseteq K$ , so  $K$  is  $C_1$ . If  $\text{char}(K) = 0$ ,  $K$  is  $C_1$ , by [Kol07, Thm. 1]. It follows that in each case,  $F$  is  $C_2$  [FrJ08, Prop. 21.2.12]. □

**THEOREM 2.4:** *Let  $F$  be an extension of transcendence degree 1 of a PAC field  $K$  of characteristic 0. Suppose  $K$  contains all roots of unity. Then  $F$  satisfies Serre’s conjecture II. That is,  $H^1(F, G) = 1$  for each simply connected group  $G$  defined over  $F$ .*

*Proof:* By Lemma 2.3,  $\text{cd}(F) \leq 2$  and  $F$  is a  $C_2$ -field. By Proposition 1.6,  $\text{cd}(F_{\text{ab}}) \leq 1$ . It follows from Proposition 2.1 that  $H^1(F, G) = 1$  for each simply connected group  $G$ . □

*Remark 2.5:* All of the ingredients of the proof of Theorem 2.4 except possibly Proposition 2.1 work also when  $\text{char}(K) > 0$ . □

The proof of the freeness of  $\text{Gal}(K(x)_{\text{ab}})$  applies the notion of ”quasi-freeness” due to Harbater and Stevenson. To this end recall that a FINITE SPLIT EMBEDDING PROBLEM  $\mathcal{E}$  for a profinite group  $G$  is a pair  $(\varphi: G \rightarrow A, \alpha: B \rightarrow A)$ , where  $A, B$  are finite groups,  $\varphi, \alpha$  are epimorphisms, and  $\alpha$  has a group theoretic section. A SOLUTION of  $\mathcal{E}$  is an epimorphism  $\gamma: G \rightarrow B$  such that

$\alpha \circ \gamma = \varphi$ . We say that  $G$  is QUASI-FREE if its rank  $m$  is infinite and every finite split embedding problem for  $G$  has  $m$  distinct solutions.

**THEOREM 2.6:** *Let  $F$  be a function field of one variable over a PAC field  $K$  of cardinality  $m$  containing all roots of unity and let  $x$  be a variable. Then  $\text{Gal}(F_{\text{ab}})$  is isomorphic to the free profinite group of rank  $m$ .*

*Proof:* Since  $K$  is PAC,  $K$  is AMPLE, that is every absolutely irreducible curve defined over  $K$  with a  $K$ -rational simple point has infinitely many  $K$ -rational points. By [HaS05, Cor. 4.4],  $\text{Gal}(F)$  is quasi-free of rank  $m = \text{card}(K)$ . Hence, by [Har09, Thm. 2.4],  $\text{Gal}(F_{\text{ab}})$  is also quasi-free of rank  $m$ . Since by Proposition 1.6,  $\text{Gal}(F_{\text{ab}})$  is projective, it follows from a result of Chatzidakis and Melnikov [FrJ08, Lemma 25.1.8] that  $\text{Gal}(F_{\text{ab}})$  is free of rank  $m$ .  $\square$

*Acknowledgment:* The authors thank Jean-Louis Colliot-Thélène for stimulating discussions, in particular for pointing out Proposition 2.1 to them.  $\square$

#### REFERENCES

- [CGP04] J.-L. Colliot-Thélène, P. Gille, and R. Parimala, *Arithmetic of linear algebraic groups over two-dimensional geometric fields*, Duke Mathematical Journal 121 (2004), 285–341.
- [Efr01] I. Efrat, *A Hasse principle for function fields over PAC fields*, Israel Journal of Mathematics 122 (2001), 43–60.
- [Efr06] I. Efrat, *Valuations, Orderings, and Milnor  $K$ -Theory*, Mathematical surveys and monographs 124, American Mathematical Society, Providence, 2006.
- [FrJ08] M. D. Fried and M. Jarden, *Field Arithmetic, Third Edition, revised by Moshe Jarden*, Ergebnisse der Mathematik (3) 11, Springer, Heidelberg, 2008.
- [Har09] D. Harbater, *On function fields with free absolute Galois groups*, Journal für die reine und angewandte Mathematik 632 (2009), 85–103.
- [HaS05] D. Harbater and K. Stevenson, *Local Galois theory in dimension two*, Advances in Mathematics 198 (2005), 623–653.
- [Kol07] J. Kollár, *A conjecture of Ax and degenerations of Fano varieties*, Israel Journal of Mathematics 162 (2007), 235–251.
- [LaT58] S. Lang and J. Tate, *Principal homogeneous spaces over abelian varieties*, American Journal of Mathematics 80 (1958), 659–684.
- [Pos05] L. Positselski, *Koszul property and Bogomolov’s conjecture*, International Mathematics Research Notices 31 (2005), 1901–1936.
- [Ser97] J.-P. Serre, *Galois Cohomology*, Springer, Berlin, 1997.

Moshe Jarden  
School of Mathematics  
Tel Aviv University  
Ramat Aviv, Tel Aviv 69978, Israel  
jarden@post.tau.ac.il

Florian Pop  
Department of Mathematics  
University of Pennsylvania  
Philadelphia, Pennsylvania 19104,  
USA  
pop@math.upenn.edu

