# The geometric case of a conjecture of Shafarevich

Florian Pop

**Forschungsschwerpunkt**

ARITHMETIK

Universität Heidelberg
Universität Mannheim

Heft Nr. 8 Oktober 1993

# The geometric case of a conjecture of Shafarevich

### — $G_{\tilde{\kappa}(t)}$ is profinite free —

by *Florian Pop** at Heidelberg

## Introduction and main results

We are interested in the following conjecture due to Shafarevich:

**(SC)** *The absolute Galois group of the maximal cyclotomic extension* $K^{\text{cycl}}$ *of a global field* $K$ *is* $\omega$*-free.*

The strongest evidence for this conjecture comes from two well known results: The first is a theorem of Iwasawa [I] which asserts that the Galois group of the maximal solvable extension of $K^{\text{cycl}}$ is prosolvable free. The second is the result of Tate, see for instance [S1], that the absolute Galois group of $K^{\text{cycl}}$ has cohomological dimension 1, and hence it is projective by a theorem of Gruenberg [G].

In this paper we will show that (SC) is true in the geometric case, ie for all global fields $K$ of positive characteristic. Such global fields $K$ are exactly the function fields of one variable over finite fields $\kappa$ and in particular, $K^{\text{cycl}}$ is the constant extension of $K$ by making the base field $\kappa$ algebraically closed. After this remark, the geometric case of (SC) is a special case of the following theorem which at the same time gives a positive answer to a long standing open question:

**Theorem.** *Let* $\kappa$ *be an algebraically closed field and* $K|\kappa$ *a function field of one variable over* $\kappa$*. Then the absolute Galois group* $G_K$ *of* $K$ *is a profinite free group.*

The result above is well known in characteristic zero and was proved by Douady [D] using the Riemann Existence Theorem, hence relying on analytical and topological methods. More precisely, Douady uses in his proof the precise structure theorem for the fundamental group $\pi_1(U)$ of small affine opens $U$ of the projective smooth

---

* The author would like to thank Q. Liu and M. Matignon at Bordeaux for many interesting and helpful discussions.

model $X$ of $K|\kappa$. It is namely known, that denoting by $g$ the genus of $X$ and by $s$ the number of closed points deleted from $X$ in order to get $U$, one has: $\pi_1(U)$ is the profinite free group on $2g+s-1$ generators in an "almost canonical" way. Unfortunately, the corresponding assertion in positive characteristik is false, see e.g. Abhyankar's conjecture, which is now proved by Harbater [H] using Raynaud's proof for the affine line case [R], see also results of Serre [S2]. Worse, the structure of $\pi_1(U)$ is not known, it depending first on the base field $\kappa$ and usually also on the deleted points from $X$ in order to get $U$. This makes the question concerning the structure of $G_K$ of independent interest and naturally more interesting then in the characteristic zero case.

We use in our approach methods belonging to the rigid analytic geometry. Hence our proof could be viewed as an algebraic one, and therefore it is interesting also in the characteristic zero case, where the result is already known. We attack the problem by considering special type opens of projective, normal, agebraically integral curves over a henselian field and studying the Galois action on large quotients of the fundamental group of such opens. The result we get enables us to solve embedding probles for $G_K$ and then we conclude by using Iwasawa's characterization of the profinite free groups. As a matter of fact, we generalize here both $\frac{1}{2}$ Riemann Existence Theorem with Galois Action from [P1] and Main Theorem from [P2]. After these explanations we can say that the catchy result announced in the title relies on mathematical phenomena which in our oppinion are much more important and interesting, namely the problem concerning the Galois action on the fundamental group of a curve. The paper is organized as follows:

The first section is of purely formal, ie axiomatic nature and describes the glueing procedure from [P1], section 1 and 2 in a more general setting. We hope that the results proved here could also be useful in understanding the fundamental group of an *affine* smooth curve in positive characteristic, relative to the fundamental group of the affine line (which itself is a big mistery). After presenting the context and indicating the new aspects of the problem the proofs from loc.cit. work usually just with minor changes. Therefore we will omit most of them.

In the second section we first give generalizations of both the main result from [P1] and that from [P2] and this is done by using the glueing procedure presented in the first section. We first give in subsection A) a generalization of $\frac{1}{2}$ Riemann Existence Theorem with Galois Action from [P1] by considering opens of a particular type of arbitrary projective, normal, geometrically integral curves over a henselian base field and studying the Galois action on large quotients of the fundamental group of such opens. Further we generalize in subsection B) the main result from [P2]. We hope that this more general result could be useful in understanding the structure of $G_K$ also for function fields of one variable $K|\kappa$ for some special non-algebraically closed fields $\kappa$, for instance for $\kappa$ a $p$-adic field. We end up with a proof of the theorem above. In the proof we use Iwasawa's characterization of the $\omega$-free groups [I], theorem 4, as well as a theorem of Mel'nicov [M] concerning projective limits of profinite free groups.

## 1. Glueing well behaved families of Galois covers of a projective curve

For definitions and basic facts we reffer to [B–G–R] or to the course notes [F].

Let $\kappa$ be a complete rank 1 valued field and $\lambda|\kappa$ some finite normal extension of $\kappa$. When speaking about projective varieties over $\lambda$ viewed as analytical $\lambda$-spaces we always mean the usual analytification of them, see eg [F], Ch 5. Admissible opens as well as admissible coverings and connectedness have to be understood in the same way. Every $\lambda$-analytical space can be viewed also as $\kappa$-analytical space in a canonical way.

### A) The glueing procedure

**(1.1)** Let $X$ be a projective, normal, geometrically integral curve over $\lambda$ and set $L = \lambda(X)$ the function field of $X$. Let $\mathcal{L} = (L_k|L)_k$ be a finite family of finite Galois extensions and $\mathfrak{G}_k = \mathrm{Gal}(L_k|L)$ the Galois group of $L_k|L$. Let $X_k \xrightarrow{\phi_k} X$ be the normalization of $X$ in the field extension $L_k|L$ (all $k$). Then $X_k$ are finite Galois coverings of $X$. By general formalism, the automorphism group $\mathrm{Aut}_X(X_k)$ operating on $X_k$ from the left is canonically isomorphic to the opposite group of $\mathfrak{G}_k$. Hence, denoting the opposite operator by bold face writing, we have: $\mathrm{Aut}_X(X_k) \cong \boldsymbol{\mathfrak{G}}_k$. By the analytification functor and GAGA principles, $\boldsymbol{\mathfrak{G}}_k$ is in a canonical way also the group of $X^{\mathrm{an}}$-automorphisms of $X_k^{\mathrm{an}}$, hence a group of analytical $\lambda$-automorphisms of $X_k^{\mathrm{an}}$. We will say that the family $\mathcal{L}$ is *a well behaved family* if it satisfies the following conditions i), ii) below:

- i) $X_k$ are projective, normal, geometrically integral curves over $\lambda$ (all $k$). Equivalently, $L_k|\lambda$ are regular field extension.
- ii) For every $k$ there exists an admissible covering $U'_k \cup U_k$ of $X^{\mathrm{an}}$ by affinoids defined over $\lambda$ such that the following conditions are satisfied:
  - a) $U_{k'} \cap U_{k''} = \phi$ (all $k' \neq k''$) and further, $U = \cap U'_k$ (all $k$) is connected and has $\lambda$-rational points, ie $\mathrm{Hom}(\lambda, U)$ is not empty.
  - b) The preimage $V'_k$ of $U'_k$ by $\phi_k$ (which is an admissible open of $X_k^{\mathrm{an}}$) contains an admissible open $V'_{k,\iota}$ which is mapped by $\phi_k$ isomorphically onto $U'_k$.
  - c) The preimage $V_k$ of $U_k$ by $\phi_k$ (which is an admissible open of $X_k^{\mathrm{an}}$) is connected, hence in particular $U_k$ is connected.

Setting $\partial U_k = U'_k \cap U_k$ it follows that $\partial U_k$ is non-empty, as $X^{\mathrm{an}}$ is connected. Further $\partial U_k$ are disjoint (all $k$), as $U_k$ themselves are disjoint by assumption ii) a). Hence, $U$ contains in a natural way an isomorphic copy of the disjoint union of all $\partial U_k$ which we denote by $\partial U$. This is an admissible open of $U$. One gets $X^{\mathrm{an}}$ canonically from $\coprod_k U_k$ and $U$ by identifying $\coprod_k \partial U_k$ with $\partial U$. We remark that $\partial V_k = V_k \cap V'_k$ also is an admissible open of $X_k^{\mathrm{an}}$ and hence of $V_k$ and $V'_k$. Clearly, $V_k$, $V'_k$ and $\partial V_k$ are $\mathfrak{G}_k$-invariant. We define $\partial V_{k,\iota} := V'_{k,\iota} \cap V_k$ and remark:

**(1.2)** Setting $V'_{g_k} = g_k(V'_{k,\iota})$ and $\partial V_{g_k} = g_k(\partial V_{k,\iota})$ for all $g_k \in \mathfrak{G}_k$ the following holds for every $k$:

(1) $V'_{g_k}$ ($g_k \in \mathfrak{G}_k$) are pairwise disjoint admissible opens of $V'_k$ and moreover, one has $g'_k(V_{g_k}) = V_{g'_k g_k}$ for all $g_k, g'_k \in \mathfrak{G}_k$. Hence $\left(V'_{g_k}\right)_{g_k}$ is a $\mathfrak{G}_k$-invariant admissible disjoint covering of $V'_k$. Therefore, $\phi_k$ gives rise to a $\mathfrak{G}_k$-isomorphism of $\lambda$-analytical spaces

$$V'_k \longrightarrow \mathfrak{G}_k \times U'_k$$

which is canonical up to the choice of $V_{k,\iota}$.

(2) Correspondingly, the same holds for $\partial V_{g_k}$ and $\partial V_k$. We shall call $\partial V_{g_k}$ the components of $\partial V_k$. These are exactly the connected components of $\partial V_k$ if and only if $\partial U_k$ is connected.

Finally, we remark that the fields $L_k$ are linearly disjoint over $L$. Indeed, from condition ii) it follows that the fibre product $X_0 = \times_X X_k$ (all $k$) of all $X_k$ over $X$ is geometrically integral and normal. Hence in particular, the field extensions $L_k|L$ are linearly disjoint over $L$ and $X_0$ is the normalization of $X$ in the field extension $L_0|L$, where $L_0$ is the compositum $L_0 = \cup L_k$ (all $k$) of all the fields $L_k$ in the algebraic closure of $L$. In particular, $L_0|L$ is a Galois extension and there exists a canonical isomorphism $\mathrm{Gal}(L_0|L) \cong \prod_k \mathfrak{G}_k$ by $g \mapsto (g_k)_k$ with $g_k$ the restriction of $g$ to $L_k$.

Let now $\mathcal{L} = \left(L_k|L\right)_k$ a well behaved family, and set as usual $\mathfrak{G}_k = \mathrm{Gal}(L_k|L)$. Let $F = \ast_k \mathfrak{G}_k$ be the profinite free product and $F_0 = \prod_k \mathfrak{G}_k$ the direct product of the groups $\mathfrak{G}_k$. There exists a canonical projection homomorphism

$$F \longrightarrow F_0$$

and let $F^0$ be its kernel. We are going to show that there exists a canonical "universal" Galois extension $L_{\mathcal{L}}|L$ with Galois group $\mathrm{Gal}(L_{\mathcal{L}}|L) \cong F$ which intuitively represents a kind of "universal compositum" of the family of Galois extensions $L_k|L$ (all $k$) in the same way as $L_0$ with $\mathrm{Gal}(L_0|L) \cong F_0$ is the usual compositum of all $L_k|L$. The field $L_{\mathcal{L}}$ will be obtained as function field of some"canonical" quotient of the universal covering (in rigide analytical sense) of $X_0$.

To do this we will use the idea from [P1], section 1. The construction and the proof is literary the same. For the sake of completeness we briefly indicate here, how it goes. Let $\mathfrak{F}_k$ denote the "canonical" isomorphic copy of $\mathfrak{G}_k$ in $F$. Further we denote by $F_k \subseteq F$ the kernel of $F \longrightarrow F_0 \xrightarrow{p_k} \mathfrak{G}_k$, where $p_k$ are the structural projections. Clearly, $F_k$ is a complement of $\mathfrak{F}_k$ viewed as subgroup of $F$. Let $D \subseteq F^0$ be an open normal subgroup of $F$ and let $F \xrightarrow{\varphi} C = F/D$ denote the canonical projection.

Further, we set $C_k = \varphi(F_k)$ and $\mathfrak{C}_k = \varphi(\mathfrak{F}_k)$. Then $\varphi$ maps every $\mathfrak{F}_k$ isomorphically onto $\mathfrak{C}_k$, $C_k$ is a complement of $\mathfrak{C}_k$ (all $k$) and $(\mathfrak{C}_k)_k$ generate $C$.

Recalling that bold face writing denotes the opposite operator we next consider the following analytical $\lambda$-spaces:

(1) $$\mathcal{U} = \boldsymbol{C} \times U$$

which is by definition the disjoint union of $|\boldsymbol{C}|$ copies of $U$.

(2) $$\mathcal{V} = \coprod_k \boldsymbol{C}_k \times V_k$$

where $\boldsymbol{C}_k \times V_k$ is by definition the disjoint union of $|\boldsymbol{C}_k|$ copies of $V_k$. We remark that $\boldsymbol{C}$ has a natural left action on both $\mathcal{U}$ and $\mathcal{V}$ defined as follows:

On $\mathcal{U}$: $\boldsymbol{h}(\boldsymbol{g}, (\cdot)) = (\boldsymbol{hg}, (\cdot))$ for all $\boldsymbol{g}, \boldsymbol{h}$.

On $\mathcal{V}$: $\boldsymbol{h}(\boldsymbol{h}_k, (\cdot)) = (\boldsymbol{h}'_k, \boldsymbol{g}'_k(\cdot))$ with $\boldsymbol{h}'_k \in \boldsymbol{C}_k$, $\boldsymbol{g}'_k \in \boldsymbol{\mathfrak{C}}_k$ defined by $\boldsymbol{h}\boldsymbol{h}_k = \boldsymbol{h}'_k \boldsymbol{g}'_k$.

We glue the analytical $\lambda$-spaces from (1) and (2) by identifying all over the $\lambda$-spaces $\boldsymbol{h}_k \times \partial V_{\boldsymbol{g}_k}$ and $\boldsymbol{g} \times \partial U_k$ by means of $\phi_k$ if $\boldsymbol{g} = \boldsymbol{h}_k \boldsymbol{g}_k$. (Recall that $\phi_k$ maps the analytical $\lambda$-space $\partial V_{\boldsymbol{g}_k}$ isomorphically onto the analytical $\lambda$-space $\partial U_k$.) We denote the resulting analytical space by $Y^{\text{an}}$. We further remark that the action of $\boldsymbol{C}$ on $\mathcal{U}$ and $\mathcal{V}$ is compatible with the glueing we have done. This is nothing but the commutativity of the diagrams

$$\begin{array}{ccc} \boldsymbol{h}'_k \times \partial V_{\boldsymbol{g}'_k \boldsymbol{g}_k} & \xleftarrow{\boldsymbol{h}} & \boldsymbol{h}_k \times \partial V_{\boldsymbol{g}_k} \\ \downarrow \phi_k & & \downarrow \phi_k \\ \boldsymbol{hg} \times \partial U_k & \xleftarrow{\boldsymbol{h}} & \boldsymbol{g} \times \partial U_k \end{array}$$

for all $\boldsymbol{g}, \boldsymbol{h} \in \boldsymbol{C}$. Hence $\boldsymbol{C}$ also acts in a canonical way on $Y^{\text{an}}$. As in [P1] we have the following: $\boldsymbol{C}\backslash Y^{\text{an}}$ is $\lambda$-isomorphic to $X^{\text{an}}$ and in particular $Y^{\text{an}}$ is finite over $X^{\text{an}}$, and further $Y^{\text{an}}$ is connected. Hence $Y^{\text{an}}$ is a finite connected (analytical) covering $Y^{\text{an}} \longrightarrow X^{\text{an}}$ and $\boldsymbol{C}\backslash Y^{\text{an}}$ is canonically isomorphic to $X^{\text{an}}$. Therefore, the canonical homomorphism $\boldsymbol{C} \xrightarrow{\text{can}} \text{Aut}_{X^{\text{an}}}(Y^{\text{an}})$ is an isomorphism. By the GAGA principles, see for instance [Ko], there exists a Galois covering $\phi : Y \longrightarrow X$ such that $Y^{\text{an}} \longrightarrow X^{\text{an}}$ is the analytification of $\phi$. Furthermore, $\text{Aut}_X(Y) = \boldsymbol{C}$ and $\boldsymbol{C}\backslash Y$ is canonically isomorphic to $X$. Next we remark that for points of $Y^{\text{an}}$ lying over points in $U$ the analytical $\lambda$-space $Y^{\text{an}}$ is locally isomorphic to $U$. In particular, $Y^{\text{an}}(\lambda) \neq \phi$. Again, by the GAGA principles it follows that $Y(\lambda) \neq \phi$, hence $Y$ viewed as curve over $\lambda$ has $\lambda$-rational points and in particular, it is geometrically integral. Further, the glueing procedure has good functorial behaviour, and one concludes as in loc.cit..

## B) The Galois Action

Let as usual $L|\lambda$ be a regular function field of one variable and $\mathfrak{G}$ a finite group of $\kappa$-automorphisms of $L$. Let $K$ denote the fixed field of $\mathfrak{G}$ in $L$. To fix the ideas we shall further suppose that $\kappa$ is the fixed field of $\mathfrak{G}$ in $\lambda$, hence $\lambda|\kappa$ is a finite Galois extension and $\kappa = K \cap \lambda$. Let $Z$ be a normal projective model for $K|\kappa$ and $X \longrightarrow Z$ the normalization of $Z$ in the field extension $L|K$. By the general formalism, the opposite group $\boldsymbol{\mathfrak{G}}$ of $\mathfrak{G}$ will be identified canonically with the group of $Z$-automorphisms of $X$. Clearly, every $\boldsymbol{\sigma} \in \boldsymbol{\mathfrak{G}}$ defines an analytical $\kappa$-automorphism of $X^{\text{an}}$ viewed as $\kappa$-analytical space. Hence we can speak about $\boldsymbol{\mathfrak{G}}$-invariant admissible coverings, etc.

**Definition.** *Let $\mathcal{L} = \left(L_k|L\right)_k$ be a well behaved family of finite Galois extensions.*

1) *We say that $\mathcal{L}$ is* Galois invariant (with respect to $K$) *if every $\sigma \in G_K$ maps $\mathcal{L}$ onto itself.*

2) *We say that $\mathcal{L}$ is Galois compatible well behaved if it is Galois invariant and there exist coverings $U_k \cup U_k'$ of $X$ satisfying the condition* ii) *from* **(1.1)** *such that $\{U\} \cup \{U_k\}_k$ is a $\boldsymbol{\mathfrak{G}}$-invariant covering of $X^{\text{an}}$.*

Exactly as in [P1] one has:

**(1.3)** Let $\mathcal{L} = \left(L_k|L\right)_k$ be Galois compatible well behaved. With the usual notations the following holds:

(1) The compositum $L_0$ of all the fields $L_k$ is a normal extension of $K$. In particular, there exists a canonical exact sequence given by Galois theory

$$1 \to \mathrm{Gal}\,(L_0|L) \longrightarrow \mathrm{Gal}\,(L_0|K) \longrightarrow \mathfrak{G} \to 1.$$

(2) Let $\sigma_0$ be a prolongation of some $\sigma \in \mathfrak{G}$ to $L_0$. Via this prolongation $\sigma_0$ to $L_0$ we can define an action of $\sigma$ on the subfields of $L_0$, etc. Using this convention we have: For fixed $k, k'$ such that $\sigma L_{k'} = L_k$ there exists (via $\sigma_0$) a commutative diagram of the form

$$\begin{array}{ccc} L_{k'} & \xrightarrow{\ \sigma\ } & L_k \\ \Big| & & \Big| \\ L & \xrightarrow{\ \sigma\ } & L \end{array}$$

and hence isomorphisms $\mathfrak{G}_{k'} \xleftarrow{(\cdot)^\sigma} \mathfrak{G}_k$ and $\mathfrak{G}_{k'} \xrightarrow{{}^\sigma(\cdot)} \mathfrak{G}_k$, the conjugation being the one defined by $\sigma_0$ in $\mathrm{Gal}\,(L_0|K)$.

With the usual notations, $\boldsymbol{\sigma}$ induces (via $\sigma_0$) isomorphisms of $Z$-schemes (which we also denote by $\boldsymbol{\sigma}$) making the following diagrams commutative:

$$\begin{array}{ccc} X_{k'} & \xleftarrow{\boldsymbol{\sigma}} & X_k \\ \big\downarrow \phi_{k'} & & \big\downarrow \phi_k \\ X & \xleftarrow{\boldsymbol{\sigma}} & X \end{array}$$

and $\boldsymbol{\sigma}$ defines (via $\sigma_0$ ) group isomorphisms $\mathfrak{G}_{k'} \xrightarrow{(\cdot)^{\sigma}} \mathfrak{G}_k$ and $\mathfrak{G}_{k'} \xleftarrow{{}^{\sigma}(\cdot)} \mathfrak{G}_k$ the conjugation being considered in $\mathrm{Aut}_z(X_0)$.

(3) Every $\boldsymbol{\sigma} \in \mathfrak{G}$ maps $U$ $\kappa$-isomorphically onto itself and permutes the other members $U_k$ of the admissible covering. More precisely, if $\sigma L_{k'} = L_k$ then $\boldsymbol{\sigma}(U_k) = U_{k'}$ and moreover, $\boldsymbol{\sigma}$ maps $\partial U_k$ isomorphically onto $\partial U_{k'}$.
Furthermore, by the commutativity of the diagrams above it follows that the $\kappa$-isomorphisms $X_{k'} \xleftarrow{\boldsymbol{\sigma}} X_k$ map $V_k$ and $\partial V_k$ isomorphically onto $V_{k'}$, respectively $\partial V_{k'}$.

**Definition.** *Let $\mathcal{L} = (L_k|L)_k$ be a Galois compatible well behaved family and $\alpha_0$ a group theoretical section of the canonical projection* $\mathrm{Gal}(L_0|K) \longrightarrow \mathfrak{G}$. *For every $k$ let $H_k$ be the stabilizer of $L_k$ in $\alpha_0(\mathfrak{G})$. We say that $\alpha_0$ is a good section for $\mathcal{L}$ if for every $k$ there exists an admissible open $\partial V_{k,\iota}$ of $\partial V_k$ which is mapped by $\phi_k$ isomorphically onto $\partial U_k$ and is stabilized by $\boldsymbol{H}_k$.*

**Convention.** For a given Galois compatible well behaved family $\mathcal{L} = (L_k|L)_k$ and a good section $\alpha_0$ for $\mathcal{L}$, when speaking about the action of $\mathfrak{G}$ on $L_0$ and its subfields or about the action of $\mathfrak{G}$ on $X_0$ and its quotients we mean the action defined via $\alpha_0$ as at **(1.3)** (2) above.

Going on with the remarks from above we have as in [P1], section 2:

(4) Let $\alpha_0$ be a good section for $\mathcal{L}$. Then there exists a labeling $\partial V_{\boldsymbol{g}_k}$ of the components of $\partial V_k$ as indicated at **(1.2)** such that for all $\boldsymbol{\sigma} \in \mathfrak{G}$ one has $\boldsymbol{\sigma}(\partial V_{\boldsymbol{g}_k}) = \partial V_{\boldsymbol{g}_{k'}}$, where $\boldsymbol{g}_{k'} = {}^{\boldsymbol{\sigma}}\boldsymbol{g}_k$ (all $k$).

Let now $\mathcal{L} = (L_k)_k$ be a Galois compatible well behaved family and $\alpha_0$ a good section for $\mathcal{L}$. Via $\alpha_0$ we have an action of $\mathfrak{G}$ on the family of groups $(\mathfrak{G}_k)_k$ as indicated at **(1.3)** (2) above. In particular, via $\alpha_0$ we can define a right action of $\mathfrak{G}$ both on $F_0 = \prod_k \mathfrak{G}_k$ and on $F = \ast_k \mathfrak{G}_k$ and the canonical projection $F \longrightarrow F_0$ is compatible with this action. With respect to this action we further consider the semi-direct products $\mathfrak{G} \ltimes F_0$ and $\mathfrak{G} \ltimes F$. By the definition of the action if follows that the two canonical exact sequences

$$1 \to F_0 \longrightarrow \mathfrak{G} \ltimes F_0 \longrightarrow \mathfrak{G} \to 1$$

and

$$1 \to \mathrm{Gal}(L_0|L) \longrightarrow \mathrm{Gal}(L_0|K) = \alpha_0(\mathfrak{G})\,\mathrm{Gal}(L_0|L) \longrightarrow \mathfrak{G} \to 1$$

are canonically isomorphic. Now the main result of this subsection is the following theorem. Its proof is literarlry the same as the one of the main result from section 2, loc.cit. and therefore we omit it.

**(1.4) Theorem.** *Let $\mathcal{L} = \left(L_k\right)_k$ be a Galois compatible well behaved family and $\alpha_0$ a good section for $\mathcal{L}$. Then with the above notations it holds:*

(1) *The universal Galois extension $L_{\mathcal{L}}|L$ constructed in the previous subsection is a normal extension of $K$.*

(2) *The section $\alpha_0$ can be prolongued to a section $\alpha_{\mathcal{L}}$ of $\mathrm{Gal}(L_{\mathcal{L}}|K) \longrightarrow \mathfrak{G}$ in such a way that the canonical exact sequence coming from the Galois theory*

$$1 \to \mathrm{Gal}(L_{\mathcal{L}}|L) \longrightarrow \mathrm{Gal}(L_{\mathcal{L}}|K) = \alpha_{\mathcal{L}}(\mathfrak{G})\,\mathrm{Gal}(L_{\mathcal{L}}|L) \longrightarrow \mathfrak{G} \to 1$$

*is canonically isomorphic to the exact sequence*

$$1 \to F \longrightarrow \mathfrak{G} \ltimes F \longrightarrow \mathfrak{G} \to 1.$$

## 2. Applications

In this section we first prove a Riemann existence type theorem with Galois action which generalizes the main result from [P1]. Secondly we show that every split embedding problem for a function field of one variable over a field with a universal local-global principle has proper, regular solutions and this is a generalization of the main result from [P2]. We finish by giving a proof of the theorem from introduction.

### A) Riemann existence type theorem with Galois action

In this subsection we shall give an example where the conditions from section 1 above are satisfied. This example reduces in the case $X = \mathbb{P}^1$ to the situation considered in [P1]. It goes about the following: Let $\lambda|\kappa$ be a finite normal field extension and $L$ a function filed of one variable over $\lambda$ such that $L|\lambda$ is a regular extension. Let $\mathfrak{G}$ be a finite group of $\kappa$-automorphisms of $L$ and denote by $K$ the fixed field of $\mathfrak{G}$ in $L$. Hence, $L|K$ is a finite Galois extension, $\mathfrak{G} = \mathrm{Gal}(L|K)$ and to fix the ideas we suppose that $\kappa$ is the fixed field of $\lambda$ under the action of $\mathfrak{G}$. In particular, $K$ is a function field of one variable and $K|\kappa$ is a regular field extension. If $\mathfrak{g} = \mathrm{Gal}(\lambda|\kappa)$ then there exists a canonical projection map $\mathfrak{G} \longrightarrow \mathfrak{g}$ which is surjective. Its kernel $\mathfrak{g}_0$ is the group of geometrical isomorphisms of $L|K$ and the fixed field of $\mathfrak{g}_0$ in $L$ is exactly $K\lambda$. Let $Z$ be a projective normal geometrically integral model for $K$ and $X \longrightarrow Z$ the normalization of $Z$ in field extension $L|K$. Then $K = \kappa(Z)$, $L = \lambda(X)$ and $\mathfrak{G} = \mathrm{Aut}_Z(X)$ canonically and $Z$ is the quotient of $X$ by $\mathfrak{G}$.

Let further $\lambda'$ be a normal extension of $\kappa$ which contains $\lambda$. We denote by $L' = L\lambda'$ the compositum of $L$ and $\lambda'$ in the algebraic closure of $K$ and remark that $L'|K$ is a normal extension containing $L$. We denote $\mathfrak{G}' = \mathrm{Gal}(L'|K)$ and remark that the canonical projection $\mathfrak{G}' \longrightarrow \mathfrak{G}$ maps $\mathfrak{g}'_0$ isomorphically onto $\mathfrak{g}_0$ and

$\mathfrak{g}'$ onto $\mathfrak{g}$. Furthermore, denoting $X' = X \times_\lambda \lambda'$ the base change to $\lambda'$ it follows that $L' = \lambda'(X')$ is the function field of $X'$. We further remark that $X(\lambda') \cong X'(\lambda')$ canonically, and $\mathbf{\mathfrak{G}}'$ acts in a canonical way (from the left) on $X'(\lambda')$, so on $X(\lambda')$. If $\lambda'$ is the algebraic closure $\overline{\lambda}$ of $\lambda$ hence of $\kappa$, then we use the notations $X' = \overline{X}$ and $\mathfrak{G}' = \mathfrak{G}_\kappa$, $\mathbf{\mathfrak{G}}' = \mathbf{\mathfrak{G}}_\kappa$. In this case one has $\mathfrak{g}' = G_\kappa$.

**Definition.** *Let $\lambda'|\kappa$ be a finite normal extension containing $\lambda$ and $x$ a smooth $\lambda'$-rational point of $X'$. A special neighbourhood of $x$ is by definition an admissible affinoid $U$ of $X'^{\text{an}}$ which satisfies the following conditions:*

i) *$U$ is invariant by the decomposition group of $x$ in $\mathbf{\mathfrak{G}}'$, ie if $\sigma' x = x$ then $\sigma' U = U$. Equivalently, there exist rational functions $u_1, \cdots, u_m$ on $X'$ which lie in the decomposition field $L'_x$ of $x$ in $L'|K$ such that*
$$U = \{x' \mid 0 \leq v(u_l(x')) \ \text{ for all } 1 \leq l \leq m \ \}.$$
ii) *There exists an analytical $\lambda'$-isomorphism $\theta : U \longrightarrow U_{\lambda'}$ of $U$ onto the unit ball $U_{\lambda'}$ of $\lambda'$ such that $\theta(x) = 0$ and $\theta$ is equivariant with respect to the elements in the decomposition group of $x$, ie $\theta\sigma' = \sigma'_\kappa\theta$ for all $\sigma'$ in the decomposition group of $x$, where $\sigma'_\kappa$ is the image of $\sigma'$ by the canonical projection $\mathbf{\mathfrak{G}}' \longrightarrow \mathfrak{g}'$.*

iii) *Let $\partial U$ denote the preimage by $\theta$ of the "boundary" $\partial U_{\lambda'}$ of the unit ball. Then there exists an admissible affinoid $U'$ of $X'$ such that $U' \cap U = \partial U$. In particular, $U'$ is invariant with respect to the decomposition group of $x$ too and is connected. To fix notations we shall say that $\partial U$ is the "boundary" of the special neighbourhood $U$. Further, we will call the preimage of the "interior" of $U_{\lambda'}$ by $\theta$ the "interior" of $U$.*

We remark that every smooth $\lambda'$-rational point of $X'$ which is not ramified in $L'|K$ has a fundamental system of $v$-adic neighbourhoods which are special. This is just the non-archimedean implicit function theorem.

Let $S$ be a finite set of closed smooth points of $X$ *which are not ramified in* $L|K$. Equivalently, the decomposition group of any $s \in S$ in $\mathbf{\mathfrak{G}}$ does not meet the group of geometrical automorphisms. Let further $\mathbb{U}$ denote the complement of $S$ in $X$ and as usual, let $\overline{S} = \{s_1, \ldots, s_{n_S}\}$ and $\overline{\mathbb{U}}$ be the base change to the algebraic closure $\overline{\lambda}$ of $\lambda$.

**Definition.** *With the above notations we say that $S$ is pairwise adjusted if $n_S = 2n$ is even and the following conditions are satisfied:*

j) *The elements of $\overline{S}$ can be organized in pairs $(x_k, y_k)$ which are permuted by $\mathbf{\mathfrak{G}}_\kappa$ between themselves.*

jj) *For every $k$ there exists a special neighbourhood $U_k$ of $x_k$ such that $y_k$ lies in the interior of $U_k$.*

jjj) *The affinoids $U_k$ are pairwise disjoint.*

**Convention.** Let $S$ be pairwise adjusted and let $\lambda'|\kappa$ be some finite normal extension which contains $\lambda$ such that $\overline{S}$ consists of $\lambda'$-rational points (which are smooth, as $S$ consists of smooth points). Choosing in every orbit of $\overline{S}$ under the action of $\mathfrak{G}'$ a fixed generator $x_k$ we can suppose that for a conjugate, say $x_{k'} = \sigma' x_k$ of $x_k$, the data for the special neighbourhood $U_{k'}$ of $x_{k'}$ are defined as being the image of the ones for $U_k$ by $\sigma'$, ie we have $U_{k'} = \sigma' U_k$, $U'_{k'} = \sigma' U'_k$ and $\theta_{k'}\sigma' = \sigma'_\kappa \theta_k$.

Clearly, if the condition holds for some $\lambda'$ then it holds in the same form for any larger normal extension $\lambda''|\kappa$ of $\kappa$.

Now suppose that $S$ is pairwise adjusted and let $\mathbb{U}$ be the complement of $S$ in $X$. We denote as usual by $\overline{X}$ and $\overline{\mathbb{U}}$ the base change to the algebraic closure of $\kappa$ and remark that $\overline{S}$ is invariant with respect to the action of the group $\mathfrak{G}_\kappa = \mathrm{Aut}_z(\overline{X})$ which is the opposite of $\mathfrak{G}_\kappa$. Therefore, there exists an exact sequence of the form

$$1 \to \pi_1(\overline{\mathbb{U}}) \longrightarrow \pi_1(\mathbb{U}) \longrightarrow \mathfrak{G}_\kappa \to 1.$$

As in loc.cit. we are going to show that this exact sequence has an "interesting" quotient, whose structure can be explicitely given. Let us describe first this quotient.

Let $\boldsymbol{m} = (m_k)$ be a system of positive integers such that $m_k = m_{k'}$ if the pairs $(x_k, y_k)$ and $(x_{k'}, y_{k'})$ are conjugated under $\mathfrak{G}_\kappa$. We say that $S$ is pairwise $\boldsymbol{m}$-adjusted if one has $v(\theta_k(y_k)) > \epsilon_{m_k}$ (all $k$), where for a natural number $m$ we set $\epsilon_m = 0$ if either $v(m) = \infty$ or $v(m) = 0$ and $\epsilon_m = v(p)/(p-1) + v(m)$ otherwise, where $p$ is the residual characteristic. Clearly, the system $\mathcal{M}$ of all $\boldsymbol{m}$ for which $S$ is pairwise $\boldsymbol{m}$-adjusted is inductive and we can define the "Steinitz numbers" $\mathfrak{s}_k$ of $S$ to be the projective limit over the $k^{\text{th}}$ component of all $\boldsymbol{m} \in \mathcal{M}$. In particular, $S$ is pairwise $\boldsymbol{m}$-adjusted if and only if $m_k$ divides the $k^{\text{th}}$ Steinitz number $\mathfrak{s}_k$ of $S$. As in [P1] we associate to $S$ the profinite group $\overline{\Pi}$ on $2n = n_S$ generators and relations as follows:

$$\overline{\Pi} = \langle\, g_{x_1}, h_{y_1}, \ldots, g_{x_n}, h_{y_n} \mid g_{x_k} h_{y_k} = 1,\ g_{x_k}^{\mathfrak{s}_k} = 1 \text{ all } k \,\rangle$$

endowed with the right $\mathfrak{G}_\kappa$-action defined by $(g_{x_k})\sigma = (g_{\sigma x_k})^{\chi(\sigma)}$ and correspondingly for $h_{y_k}$, where $\chi$ is the cyclotomic character of $\mathfrak{G}_\kappa$. We call $\overline{\Pi}$ endowed with $\mathfrak{G}_\kappa$ the canonical $v$-quotient associated to $S$.

**(2.1) Theorem** (Riemann existence type theorem with Galois action).
*In the context above the canonical exact sequence:*

$$1 \to \pi_1(\overline{\mathbb{U}}) \longrightarrow \pi_1(\mathbb{U}) \longrightarrow \mathfrak{G}_\kappa \to 1$$

*has in a canonical way the following quotient:*

$$1 \to \overline{\Pi} \longrightarrow \mathfrak{G}_\kappa \ltimes \overline{\Pi} \longrightarrow \mathfrak{G}_\kappa \to 1,$$

*the semi-direct product being considered with the canonical action of $\mathfrak{G}_\kappa$ on $\overline{\Pi}$. Moreover, the generators $g_{x_k}$, $h_{y_k}$ are inertia elements associated to $x_k$, $y_k$ (all $k$).*

*Proof.* The proof follows the same main lines as the proof of the main result from [P1]. The only problem is to construct for every $\boldsymbol{m} \in \mathcal{M}$ a family $\mathcal{L} = \mathcal{L}^{\boldsymbol{m}}$ of cyclic Galois extensions which has good functorial properties as asked for in loc.cit. section 3. We shall do this using Kummer theory and Artin–Schreier theory via Witt vectors.

Thus, let $\boldsymbol{m} \in \mathcal{M}$ be fixed. We consider some finite normal extension $\lambda'|\kappa$ containing $\lambda$ such that $\overline{S}$ consists of (smooth) $\lambda'$-rational points and further $\lambda'$ contains the roots of unity of order $m_k$ (all $k$). Replacing $\lambda$ by $\lambda'$ (and so, $L$ by $L'$, etc) we associate to $\boldsymbol{m}$ a Galois compatible well behaved family $\mathcal{L} = \left(L_k|L\right)_k$ of cyclic Galois extensions of $L$ with $\mathrm{Gal}\,(L_k|L)$ isomorphic to the cyclic group $\mathfrak{G}_k$ of order $m_k$. Moreover, there will exist good sections for $\mathcal{L}$ and the way we do this has good functorial properties. We begin by recalling the following facts, see [P1], section 3, (3.6) for notations and definitions.

**(2.2)** Let $U_\lambda$ and $\partial U_\lambda$ denote the unit ball, respectively its "boundary" in $\lambda$ viewed as admissible affinoids of $\mathbb{P}^1_\lambda$. For $c = \theta_k(y_k)$ we set $u^\bullet = 1 + c/z$, $u^{\mathrm{a}} = c^2/[z(z+c)]$. Let $M$ be the cyclic field extension of $\lambda(z)$ defined by the character $\delta^\bullet_{m_k}(u^\bullet) + \delta^{\mathrm{a}}_{m_k}(u^{\mathrm{a}})$ and $Y \xrightarrow{\varphi_k} \mathbb{P}^1_\lambda$ the normalization of $\mathbb{P}^1_\lambda$ in the field extension $M\,|\,\lambda(z)$. Let $V_{\lambda,k}$ and $\partial V_{\lambda,k}$ denote the preimages of $U_\lambda$, respectively $\partial U_\lambda$ by $\varphi_k$. Then $V_{\lambda,k}$ and $\partial V_{\lambda,k}$ are admissible opens of $Y^{\mathrm{an}}$ and moreover, $\partial V_{\lambda,k}$ is isomorphic as an analytical $\lambda$-space to $\mathfrak{C}_k \times \partial U_{\lambda,k}$, where $\mathfrak{C}_k$ is the opposite of $\mathfrak{C}_k = \mathrm{Gal}\,(M|\lambda(z))$.

Using the observation above we get:

**(2.3) Lemma.** *For every $k$ there exists a unique cyclic extension $L_k|L$ such that denoting by $X_k \xrightarrow{\phi_k} X$ the normalization of $X$ in the field extension $L_k|L$ the following holds:*

1) $\mathfrak{G}_k = \mathrm{Gal}\,(L_k|L)$ *is isomorphic to the cyclic group of order $m_k$.*
2) *The preimage $V'_k$ of $U'_k$ by $\phi_k$ is analytically $\lambda$-isomorphic to $\mathfrak{G}_k \times \partial U'$.*
3) *The preimage $V_k$ of $U_k$ by $\phi_k$ is analytically $\lambda$-isomorphic to the above $V_{\lambda,k}$, say by $\psi_k : V_k \longrightarrow V_{\lambda,k}$, such that $\varphi_k\psi_k = \theta_k\phi_k$.*

*In particular, $\phi_k$ is ramified only in $x_k, y_k$ and $\mathfrak{G}_k$ is generated by inertia elements $g_k, h_k$ of $x_k$, respectively $y_k$ such that $g_k h_k = 1$.*

*Proof.* The existence follows immediately by glueing the $\lambda$-analytical spaces $V_{\lambda,k}$ and $\mathfrak{G}_k \times U'_k$ along the "boundaries" $\mathfrak{G}_k \times \partial U_k$ and $\partial V_{\lambda_k}$ via $\theta_k$. The resulting analytical $\lambda$-space is a finite separated connected Galois covering of $X^{\mathrm{an}}$ with $\mathrm{Aut}_{X^{\mathrm{an}}}(X_k^{\mathrm{an}}) \cong \mathfrak{G}_k$. By the GAGA principles it follows that the above covering is the analytification of a unique Galois covering $X_k \xrightarrow{\phi_k} X$ of $X$ with $\mathrm{Aut}_X(X_k) \cong \mathfrak{G}_k$. The unicity up to $X$-isomorphism follows immediately by conditions 2) and 3) which imply the existence of an analytical $X^{\mathrm{an}}$-isomorphism of any two such coverings, hence an $X$-isomorphism by the GAGA principles. The remaining assertions follow by the special construction from **(2.2)**. Next we remark:

**(2.4)** The family $\mathcal{L} = \big(L_k|L\big)_k$ constructed above is Galois compatible well behaved in the sense of the previous section.

*Proof.* By our convention it follows that the admissible covering $\{U\} \cup \{U_k\}_k$ is $\mathfrak{G}$-invariant. Hence the only problem is to prove that $\mathcal{L}$ is Galois invariant. Let $\tilde{\sigma} \in G_K$ be arbitrary and $\sigma$ denote its restriction to $L$. For a fixed $k$ let $L'_k$ be the preimage of $L_k$ by $\tilde{\sigma}$ and $X'_k$ the normalization of $X$ in the cyclic field extension $L'_k|L$. Let further $k'$ be such that $\boldsymbol{\sigma} x_k = x_{k'}$. We are going to show that $L'_k = L_{k'}$ and so $X'_k = X_{k'}$. To do this we shall use the unicity part from **(2.3)** above. We namely remark that the canonical commutative diagram

$$\begin{array}{ccc} L'_k & \xrightarrow{\tilde{\sigma}} & L_k \\ | & & | \\ L & \xrightarrow{\sigma} & L \end{array}$$

of $K$-morphisms of fields induces in a canonical way a commutative diagram of morphisms of $Z$-schemes

$$\begin{array}{ccc} X'_k & \xleftarrow{\tilde{\boldsymbol{\sigma}}} & X_k \\ \big\downarrow{\scriptstyle \phi'_k} & & \big\downarrow{\scriptstyle \phi_k} \\ X & \xleftarrow{\boldsymbol{\sigma}} & X \end{array}$$

The $\lambda$-analytical structure of $X'_k$ is exactly the image of the $\lambda$-structure of $X_k$ by $\tilde{\boldsymbol{\sigma}}$. As the morphisms $\theta_k$ are $\mathfrak{G}$-equivariant, it follows using the unicity part from **(2.3)** above that $X'_k$ is isomorphic over $X$ to $X_{k'}$. Hence $L'_k$ is $L$-isomorphic to $L_{k'}$ and so, $L'_k = L_{k'}$ because $L_k|L$ is a Galois extension. Thus we also have $X'_k = X_{k'}$.

Our next step is to show that for the Galois well behaved family $\mathcal{L}$ there exist good sections in the sense of B) above.[1]) With the usual notations we remark that by construction there exist generators $g_{x_k}$ and $h_{x_k}$ of $\mathfrak{G}_k$ which are inertia elements associated to $x_k$, respectively $y_k$ such $g_{x_k} h_{x_k} = 1$ and the action of $\mathfrak{G}$ on them is given by $(g_{x_k})\sigma = \big(g_{\boldsymbol{\sigma} x_k}\big)^{\chi(\boldsymbol{\sigma})}$ and correspondingly for $h_{y_k}$, where $\chi$ is the cyclotomic character of $\mathfrak{G}$. Hence, endowing $\prod_k \mathfrak{G}_k$ with this $\mathfrak{G}$ right action it follows that $\mathrm{Gal}\,(L_0|L) \cong \prod_k \mathfrak{G}_k$ as right $\mathfrak{G}$-modules. In particular, using Shapiro's Lemma it follows that the 2-cohomology class describing $\mathrm{Gal}\,(L_0|K)$ as an extension of $\mathfrak{G}$ by $\mathrm{Gal}\,(L_0|L)$ is trivial. Thus

$$1 \to \mathrm{Gal}\,(L_0|L) \longrightarrow \mathrm{Gal}\,(L_0|K) \longrightarrow \mathfrak{G} \to 1$$

splits and is isomorphic to

$$1 \to \prod_k \mathfrak{G}_k \longrightarrow \mathfrak{G} \ltimes \big(\prod_k \mathfrak{G}_k\big) \longrightarrow \mathfrak{G} \to 1,$$

[1]) We remark that the method from [P1] does not work in this new context.

where the semi-direct product is considered with respect to the action defined above. The existence of good sections is now of purely formal nature.[2]) Let namely $\alpha_0'$ be some section of the canonical projection $\mathrm{Gal}(L_0|K) \longrightarrow \mathfrak{G}$. Then $\boldsymbol{\mathfrak{G}}$ acts via $\alpha_0'$ on the set of all boundaries $\partial V_{\boldsymbol{g}_k}$ (all $k$ and $\boldsymbol{g}_k \in \boldsymbol{\mathfrak{G}}_k$). Indeed, this follows by the fact that in our special situations the boundaries $\partial V_{\boldsymbol{g}_k}$ are the connected components of the corresponding $\partial V_k$ and clearly, every $\boldsymbol{\sigma}$ via $\alpha_0'$ maps connected, admissible affinoids of $X_k^{\mathrm{an}}$ onto connected, admissible affinoids of the corresponding $X_{k'}^{\mathrm{an}}$. In particular, for every $\boldsymbol{\sigma}$ viewed via $\alpha_0'$ there exists a unique $\boldsymbol{g}_{\boldsymbol{\sigma}} = (\boldsymbol{g}_k)_k$ such that $\boldsymbol{\sigma}(\partial V_{\epsilon_k}) = \partial V_{\boldsymbol{g}_{k'}}$ if $\sigma(L_{k'}) = L_k$ (all $k$). One verifies without difficulties that the system $(\boldsymbol{g}_{\boldsymbol{\sigma}})_{\boldsymbol{\sigma}}$ is a 1-cocycle of $\boldsymbol{\mathfrak{G}}$ with values in $\mathrm{Aut}_X(X_0)$. In particular, the mapping $\alpha_0 : \mathfrak{G} \longrightarrow \mathrm{Gal}(L_0|K)$ defined by $\alpha_0(\sigma) = \alpha_0'(\sigma) g_\sigma^{-1}$ is again a section of the canonical projection. Clearly, via this new section every $\boldsymbol{\sigma}$ permutes the boundaries indexed by the neutral elements $\boldsymbol{\epsilon}_k \in \boldsymbol{\mathfrak{G}}_k$. In particular, $\alpha_0$ is a good section for $\mathcal{L}$.

The rest of the proof is identical with the one in [P1].

## B) Embedding problems for function fields of one variable

Our aim here is to give a natural generalization of the main result from [P2]. For the convenience of the reader we begin by recalling the following facts from [P2], section 1.

Let $G$ be an arbitrary profinite group. A (finite) embedding problem $\mathrm{EP} = (\gamma, \alpha)$ for $G$ consists of a diagram of profinite group homomorphisms of the form

$$\begin{array}{ccc} & & G \\ & & \big\downarrow{\scriptstyle\gamma} \\ B & \xrightarrow{\ \alpha\ } & A \end{array}$$

where $\alpha$ and $\gamma$ are surjective (and $B$ is finite). EP is called split if $\alpha$ has a section, or equivalently, if $\ker(\alpha)$ has a complement. A (proper) solution of EP is a (surjective) homomorphism $\beta : G \longrightarrow B$ with $\alpha\beta = \gamma$.

**(2.5)** Let the following situation be given:

- A homomorphic image $G \xrightarrow{\gamma} A$ of $G$ with $A$ finite.
- A finite family $\Sigma_0 = (\mathfrak{C}_k)_k$ of finite cyclic groups and a finite family $\Sigma = (\mathfrak{C}_{k,\sigma})_{k,\sigma}$ of isomorphic copies $\iota_{k,\sigma} : \mathfrak{C}_{k,\sigma} \longrightarrow \mathfrak{C}_k$ $(\sigma \in A)$ of $\mathfrak{C}_k$ (all $k$). For $g_k \in \mathfrak{C}_k$ we let $g_{k,\sigma}$ denote its preimage by $\iota_{k,\sigma}$.

Let $F$ be the profinite free product on $\Sigma$. There exists a natural right action of $A$ on $F$ which is defined by $(g_{k,\tau\sigma})\tau = g_{k,\sigma}$ for all $k, \sigma, \tau$. With respect to this action we denote by $A \ltimes F$ the semi-direct product of $F$ by $A$. We identify $A$ and $F$ with the

[2]) We believe that the "canonical" section defined by the semi-direct product is a good section. Unfortunately we cannot prove it.

subgroups $A \ltimes 1$ and respectively $1 \ltimes F$ of $A \ltimes F$. We further use $\Sigma$ also to denote the family of all $\mathfrak{C}_{k,\sigma}$ viewed as subgroups of $F$.

For normal subgroups $D$ of $A \ltimes F$ which are contained in $F$ we set $C = F/D$ and identify $(A \ltimes F)/D$ with $A \ltimes C$. Let $\alpha_D : A \ltimes F \longrightarrow A \ltimes C$, $\alpha^D : A \ltimes C \longrightarrow A$ denote the canonical projections and $\Sigma^D$ (or simply $\Sigma$ if no confusion is possible) the image of $\Sigma$ by $\alpha_D$. Clearly, $\mathrm{EP}^D = (\gamma, \alpha^D)$ is an embedding problem for $G$. We call such an embedding problem an *adjusted embedding problem* for $G$. We say that $\mathrm{EP}^D$ is an adjusted finite embedding problem for $G$ if $D$ is an open normal subgroup of $A \ltimes F$.

Let $\mathrm{EP} = (\gamma, \alpha)$ and $\mathrm{EP}' = (\gamma', \alpha')$ be embedding problems for $G$. We say that $\mathrm{EP}'$ dominates EP, if there exist surjective group homomorphisms $\gamma_0 : A' \longrightarrow A$ and $\beta_0 : B' \longrightarrow B$ such that $\gamma = \gamma_0\gamma'$, $\alpha\beta_0 = \gamma_0\alpha'$ and $\beta_0(\ker(\alpha')) = \ker(\alpha)$. In this context we have: If $\beta'$ is a (proper) solution of $\mathrm{EP}'$ then $\beta_0\beta'$ is a (proper) solution of EP.

We remark that for a given split embedding problem $\mathrm{EP} = (\gamma, \alpha)$ for $G$ and a homomorphic image $\gamma' : G \longrightarrow A'$ of $G$ such that $\ker(\gamma)$ contains $\ker(\gamma')$ there exists a "canonical" embedding problem $\mathrm{EP}' = (\gamma', \alpha')$ for $G$ which dominates EP. Indeed, let $\gamma_0 : A' \longrightarrow A$ be the canonical projection. We set $B' = B \times_A A'$ and denote by $\beta_0 : B' \longrightarrow B$ and $\alpha' : B' \longrightarrow A'$ the structural projections. The embedding problem we are looking for is exactly $\mathrm{EP}' = (\gamma', \alpha')$. Moreover, $\beta_0$ maps $\ker(\alpha')$ isomorphically onto $\ker(\alpha)$.

**(2.6)** The class of all adjusted finite embedding problems for $G$ is cofinal with respect to the domination relation in the class of all finite *split* embedding problems for $G$.

More precisely, let $\mathrm{EP} = (\gamma, \alpha)$ be a given finite split embedding problem for $G$ and $\overline{\alpha}$ a section of $\alpha$. Further let $\Sigma_0 = (\mathfrak{C}_k)_k$ be a finite set of cyclic subgroups of $\ker(\alpha)$ such that $\Sigma_0$ generates $\ker(\alpha)$ and is closed by conjugation with elements from $\overline{A} = \overline{\alpha}(A)$. Then with $\gamma$ and $\Sigma_0$ as in the hypothesis for **(2.5)** there exist open $D$ and surjective group homomorphisms $\beta_0 : A \ltimes C \longrightarrow B$ such that:

1) $\overline{\alpha}(\sigma) = \beta_0(\sigma \ltimes 1)$ for all $\sigma \in A$.
2) $\beta_0(\Sigma^D) = \Sigma_0$ and $\beta_0$ maps every $\mathfrak{C}_{k,\sigma}$ isomorphically onto $\overline{\alpha}(\sigma)\, \mathfrak{C}_k\, \overline{\alpha}(\sigma)^{-1}$.

In particular, $\mathrm{EP}^D$ dominates EP.

Let now $\kappa$ be an arbitrary field and $K|\kappa$ a regular field extension. For a homomorphism $\psi$ of $G_K$ into some profinite group we denote by $K_\psi$ and $\kappa_\psi$ the fixed field of $\ker(\psi)$ in the separable closure of $K$, respectively $\kappa$. Let $\mathrm{EP} = (\gamma, \alpha)$ be an embedding problem for $K$ and $\beta$ a solution of it. We say that $\beta$ is a *regular solution of* EP if $K_\beta|\kappa_\gamma$ is a regular field extension, or equivalently, $\kappa_\gamma = \kappa_\beta$. We remark that if we restrict ourselves to embedding problems for $K$ which are induced by embedding problems for $G_\kappa$ through the canonical projection $G_K \longrightarrow G_\kappa$ then the notion of regular solution from here coincides with the one defined in [P2], section 1.

The generalization of Main Theorem from loc.cit. we have in mind is the following:

**(2.7) Theorem.** *Let $\kappa$ be a field with a universal local-global principle and $K$ a function field of one variable such that $K|\kappa$ is a regular field extension. Then every finite, split embedding problem for $G_K$ has a proper, regular solution.*

*Proof.* The technical core of the proof consists of the following lemma, which is a special case of **(2.7)** and is a consequence of main result of the previous subsection.

**(2.8) Lemma.** *For an arbitrary field $\kappa$ we let $\hat{\kappa} = \kappa((u))$ be the power series field in one variable over $\kappa$. Let $K$ be a function field of one variable over $\hat{\kappa}$ such that $K|\hat{\kappa}$ is a regular field extension. Then every finite, split embedding problem for $G_K$ has a proper, regular solution.*

*Proof of the lemma.* Let $\mathrm{EP} = (\gamma, \alpha)$ be a finite embedding problem for $G_K$ and to simplify notations set $L := K_\gamma$ and $\lambda := \hat{\kappa}_\gamma$. Let $Z$ be a projective normal geometrically integral model for $K|\hat{\kappa}$ and $X \longrightarrow Z$ the normalisation of $Z$ in the field extension $L|K$. Then $X$ is a projective, normal, geometrically integral curve over $\lambda$. Let $\lambda'|\hat{\kappa}$ be some finite Galois extension of $\hat{\kappa}$ containing $\lambda$ and $L' = L\lambda'$ denote the compositum of $L$ and $\lambda'$ in the separable closure of $K$. Then $L'|K$ is a Galois extension and setting $A' = \mathrm{Gal}(L'|K)$ and denoting by $\gamma' : G_K \longrightarrow A'$ the canonical projection, we consider the "canonical" embedding problem $\mathrm{EP}'$ dominating EP as above. An easy verification shows that if $\beta'$ is a proper, regular solution for $\mathrm{EP}'$ then $\beta := \beta_0\beta'$ is a proper, regular solution for EP. Hence, it is sufficient to prove that there exists some $\lambda'$ as above such that the corresponding embedding problem $\mathrm{EP}'$ has proper, regular solutions.

With the above notations let $X'$ be the normalization of $X$ in the field extension $L'|L$, and hence of $Z$ in the field extension $L'|K$. As usual, we denote $\mathfrak{G}' = \mathrm{Gal}(L'|K)$. Clearly, $\boldsymbol{\mathfrak{G}}' = \mathrm{Aut}_Z(X')$ acts on the set of $\lambda'$-rational points $X'(\lambda') = X(\lambda') = \mathrm{Hom}(\lambda', X')$ of $X'$ by $\boldsymbol{\sigma}(x') = \boldsymbol{\sigma} \circ x'$. Next we have:

Claim. *Let $\lambda'|\hat{\kappa}$ be a finite Galois extension containing $\lambda$ and $x'$ a smooth $\lambda'$-rational point of $X'$. Then every $v$-adic neighbourhood of $x'$ in $X'(\lambda')$ contains points which have trivial stabilizer in $\boldsymbol{\mathfrak{G}}'$.*

*Proof.* Indeed, as the ramification locus of $L'|K\lambda'$ consists of only finitely many points, it follows that in every $v$-adic neighbourhood of $x'$ there exist smooth points $x$ which have trivial stabilizer in the $K\lambda'$-automorphisms group of $L'$. Hence it is sufficient to prove the claim for such a point $x$. On the other hand, for such a point $x$ there exists a non-constant function $t$ in $K$ such that $t - t(x)$ is a local uniformizer at $x$. For such a $t$ let $X \xrightarrow{\phi} \mathbb{P}^1_{\lambda'}$ be the morphism defined by $t$. Then for every $y \in X'(\lambda')$ and $\boldsymbol{\sigma} \in \boldsymbol{\mathfrak{G}}$ one has $\phi(\boldsymbol{\sigma} y) = \boldsymbol{\sigma}\phi(y)$, where the action of $\boldsymbol{\mathfrak{G}}$ on $\mathbb{P}^1_{\lambda'}$ is defined via the canonical projection $\boldsymbol{\mathfrak{G}}' \longrightarrow \mathbf{g}'$. On the other hand, since $x$ is smooth, it follows by the implicite function theorem that there exists a $v$-adic neighbourhood

$U_x$ of $x$ which is maped by $\phi$ isomorphically onto a $v$-adic neighbourhood of $\phi(x)$. Now take some $y_0 \in \phi(U_x)$ which has trivial stabilizer in $\mathfrak{g}'$ and is non-ramified in $L'|K\lambda'$. Then clearly, its preimage $x_0$ in $U_x$ has trivial stabilizer in $\mathfrak{G}'$. The claim is proved.

Let now EP be some finite split embedding problem for $G_K$ and suppose we have the situation from **(2.6)**. Let $m_{\Sigma_0}$ be the lowest common multiple of the orders of all the groups in $\Sigma_0$. Replacing $L$ by a constant extension $L' = L\lambda'$ and considering the corresponding finite quotient $\gamma' : G_K \longrightarrow A'$ of $G_K$ we can suppose that the following holds: The fixed field $L'$ of $\ker(\gamma')$ contains the roots of unity of order $m_{\Sigma_0}$ and the projective normal geometrically integral model $X'$ of $L'|\lambda'$ satisfies the conclusion of the claim above. Starting with these data we consider the "canonical" split embedding problem $\text{EP}' = (\gamma', \alpha')$ which dominates EP as above. As $\beta_0$ maps $\ker(\alpha')$ bijectively onto $\ker(\alpha)$ it follows that denoting by $\Sigma_0'$ the preimage of $\Sigma_0$ by $\beta_0$ we have $m_{\Sigma_0'} = m_{\Sigma_0}$. Hence working with $\text{EP}'$ instead of EP we can suppose that $L$ contains the roots of unity of order $m_{\Sigma_0}$ and the projective normal model $X$ of $L|\lambda$ contains (many) smooth $\lambda$-rational points $x$ which have trivial stabilizer in $\mathfrak{G} = \text{Aut}_z(X)$. We proceed by associating to EP an adjusted finite embedding problem by the procedure from **(2.6)**. Therefore, it is sufficient to prove the lemma for adjusted finite embedding problems $\text{EP} = \text{EP}^D = (\gamma, \alpha^D)$ such the fixed field $L := K_\gamma$ of $\ker(\gamma)$ contains the roots of unity of order $m_\Sigma$ and the model $X$ of $L|\lambda$ contains (many) smooth $\lambda$-rational points $x$ which have trivial stabilizer in $\mathfrak{G} = \text{Aut}_z(X)$. To prove the lemma in this special case we choose for every $\mathfrak{G}_{k,\sigma} \in \Sigma^D$ a smooth point $x_{k,\sigma}$ in $X(\lambda)$ which has trivial stabilizer in $\mathfrak{G}$ and moreover, we do this in such a way that the following condition is satisfied: $\tau x_{k,\tau\sigma} = x_{k,\sigma}$ for all $k, \sigma, \tau$. We further choose smooth $\lambda$-rational points $y_{k,\sigma}$ which are sufficiently close to $x_{k,\sigma}$ and satisfy $\tau y_{k,\tau\sigma} = y_{k,\sigma}$. We proceed as in [P2], proof of (1.6). The proof of **(2.8)** is finished.

To finish the proof of **(2.7)** one applies the lemma above together with the lemma (1.8) from [P2] and the following one:

**Lemma.** *Let $\hat{\kappa}|\kappa$ be a field extension with $\kappa$ existentially closed in $\hat{\kappa}$.* [2]) *For a function field of one variable $K|\kappa$ which is regular over $\kappa$ we denote by $\pi : G_{K\hat{\kappa}} \longrightarrow G_K$ the canonical projection. Let $\text{EP} = (\gamma, \alpha)$ be a finite, split embedding problem for $G_K$. Then EP has proper, regular solutions, provided the corresponding embedding problem $\widehat{\text{EP}} = (\gamma\pi, \alpha)$ for $G_{K\hat{\kappa}}$ has proper, regular solutions.*

The proof is, after the obvious necessary changes, identical with the proof of lemma (1.7) from [P2], and therefore we omit it.

---

[2]) See for instance [B–S] for definitions and basic facts.

### C) Proof of the theorem from the Introduction

Let $\kappa$ be an arbitrary algebraically closed field. Then $\kappa$ is the inductive limit of all its algebraically closed, countably subfileds $\kappa_i$. In particular, $\kappa(t)$ is the inductive limit of all $\kappa_i(t)$, hence $G_{\kappa(t)}$ is the projective limit of $G_{\kappa_i(t)}$ in a canonical way. By the result of Mel'nicov [M] it follows that $G_{\kappa(t)}$ is profinite free if every factor $G_{\kappa_i(t)}$ is a profinite free group. Hence it is sufficient to prove the assertion of the main theorem in the case where $\kappa$ is countable. Then $\kappa(t)$ is countable too, and $G_{\kappa(t)}$ is countably generated. On the other hand, in this case we can apply Iwasawa's theorem 4 from [I]. Indeed, for a given embedding problem EP for $G_{\kappa(t)}$ there always exist solutions, as $G_{\kappa(t)}$ has cohomological dimension 1. Starting with such a solution we construct as in [P2], section 2, proof of Theorem A, a finite, split embedding problem EP$'$ for $G_{\kappa(t)}$ which dominates EP. By **(2.7)** it follows that EP$'$ has proper solutions. Hence EP has proper solutions too. We conclude by applying Iwasawa's theorem.

#### References


[B–G–R] *Bosch, S. — Güntzer, U. — Remmert, R* Non-archimedean analysis, Grundl. Math. Wiss., Bd. 261, Berlin Heidelberg New York Springer 1984.

*Chevalley, C.* Introduction to the theory of algebraic functions of one variable, AMS, New York 1951.

*Deligne, P.* Le groupe fondamental de la droite projective mois trois points, Galois groups over **Q**, Math. Sci. Res. Inst. Publ. **16**, 79–297, Springer 1989.

[D] *Douady, A.* Détermination d´un groupe de Galois, C. R. Acad. Sci. Paris **258** (1964), 5305–5308.

*van den Dries, L. — Ribenboim, P.* Application de la théorie des modèle aux groupes de Galois de corps de fonctions, C. R. Acad. Sci. Paris **288**, A789–A792.

[F] *Fresnel, J.* Geometrie Analytique Rigide, Notes, Bordeaux 1984.

[G] *Gruenberg, K. W.* Projective profinite groups, J. London Math. Soc. **42** (1967), 155–165.

[F–J] *Fried, M. — Jarden, M.* Field Arithmetic, Springer Verlag Berlin Heidelberg 1986.

*Harbater, D.*:

Galois covers of the arithmetic line, in LNM 1240, Springer Verlag 1987, 185–195.

[H] A proof of Abhyankar's conjecture(?), manuscript 1993.

*Gerritzen, L. — van der Put, M.* Schottky groups and Mumford curves, LNM 817, Springer-Verlag Berlin Heidelberg New York 1980.

*Grothendieck, A.* SGA 1. Revètements Étale et Groupe Fundamental, LNM 224, Springer-Verlag Berlin Heidelberg New York 1971.

[I] *Iwasawa, K.* On solvable extensions of number fields, Ann. of Math. **58** (1953), 548–572.

[Ko] *Köpf, U.* Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen, Schriftenreihe des Math. Inst. Univ. Münster, 2. Serie, Heft 7 (1974).

*Krull, W. — Neukirch, J.* Die Struktur der absoluten Galoisgruppe über dem Körper $\mathbb{R}(t)$, Math. Ann. **193** (1971), 197–209.

*Liu, Q.* Tout groupe fini est un groupe de Galois sur $\mathbb{Q}_p(t)$, manuscript, Bordeaux 1991.

*Lütkebohmert, W.* Riemann's existence problem for a $p$-adic field, Invent. math. **111** (1993), 309–330.

*Matzat, B.-H.* Der Kenntnisstand in der Konstruktiven Galoisschen Theorie, in: Progress in Mathematics, Vol. 95, Birkhäuser Verlag Basel.

[M] *Mel'nicov, O.V.* Subgroups and homologies of free products of profinite groups, Izv. AN SSSR, Ser. Math. **52** (989), 97–120 (russian).

*Mumford, D.* An analytic construction of degenerating curves over complete local rings, Compositio Math., Vol **24**, Fasc. 2 (1972), 129–174.

*Pop, F.* :

[P1] $\frac{1}{2}$ Riemann Existence Theorem with Galois Action, to appear in Birkhäuser PM Series 1993(?).

[P2] Hilbertian fields with a universal local-global principle, Preprint, Heidelberg 1993.

*Popp, H.* Fundamentalgruppen algebraischer Mannigfaltigkeiten, LNM 176, Springer-Verlag Berlin–Heidelberg–New York 1970.

*Raynaud, M.* :

Géométrie analytique rigide d'apres Tate, Kiehl, ..., Mémoire de Soc. Math. de France, t. **39–40** (1974), 319–327.

[R] Revetements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar, manuscript 1992.

*Roquette, P.* :

Analytic theory of elliptic functions over a local field, Hamburger Mathematische Einzelschriften, Neue Folge, Heft 1 1970.

Some tendences in contemporary algebra, in: Perspectives in Mathematics, Anniversary of Oberwolfach 1984, Basel 1984, 393–422.

*Serre, J. P.* :

[S1] Cohomologie galoisienne, LNM 5, Springer Verlag 1973.

[S2] Revetements de courbes algébriques, Sém. Bourbaki n$^0$ 745 (1991–1992), to appear.

Topics in Galois Theory, Research notes in Math., Jones and Bartlett 1992.

*Tate, J.* Rigid analytic spaces, Inv. Math. **12** (1971), 257–289.

Mathematisches Institut der Universität, Im Neuenheimer Feld 288, D-6900 Heidelberg