Hilbertian Fields with a universal Local–Global Principle

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Introduction and main results

The motivation for this paper is at least threefold. First we give a positive answer to a long standing conjecture which originates in an unpublished note of Roquette and asserts that the absolute Galois group of a countable PAC hilbertian field is ω-free, see [F–J], Problem 24.41. This conjecture was proved in the characteristic zero case by Fried–Völklein [F–V] using the theory of Hurwitz spaces. Their proof relies on the Riemann Existence Theorem and this is the reason why they are only able to prove the conjecture in the characteristic zero case. Secondly we give new evidence for the conjecture of Shafarevich which asserts that the Galois group of $\mathbb{Q}^{ab}$ is ω-free. Finally, one of the most interesting applications of the theory we develop here is the insight into the Galois structure of the field of all totally $\mathfrak{S}$-adic numbers.

Let κ be an arbitrary field and κ^s its separable closure. A locality of κ is an algebraic separable extension Λ of κ which is either real closed or Henselian with respect to a non-trivial valuation of κ. We say that κ satisfies a *universal local-global principle* if there exists a set $\mathcal{L}$ (which could be empty) of localities of κ such that for every smooth variety V over κ the following holds:

$$V(\kappa) \neq \emptyset \text{ provided } V(\Lambda) \neq \emptyset \text{ for all } \Lambda \in \mathcal{L}.$$

The fields with a universal local-global principle represent a natural generalization of the PAC, PRC and PpC fields, see e.g. the literature list from [P2]. Concerning the terminology one has:

1) If $\mathcal{L}$ is empty then κ is a PAC field in the usual sense.
2) If $\mathcal{L}$ consists of all real closures, respectively p-adic closures of κ then one says that κ is a PRC, respectively a PpC field.
3) A natural generalization of the classes of fields above is obtained in the following way: A locality Λ of some field is said to be quasi-local if either Λ is a real

closed field or Λ is henselian with respect to a non-trivial valuation v which has finite residue field Λv and value group $v\Lambda$ not q-divisible for all rational prime numbers $q \neq \mathrm{char}(\Lambda v)$. We say that a set $\mathcal{L}$ of localities of κ is quasi-local if every $\Lambda \in \mathcal{L}$ is a quasi-local locality of κ and furthermore, $\mathcal{L}$ has the following compactness properties:

i) The cardinality of the residue fields Λv ($\Lambda \in \mathcal{L}$) is bounded.

ii) There exists a finite set $\{t_1, \ldots, t_s\}$ of elements of κ such that for every $\Lambda \in \mathcal{L}$ and $q \neq \mathrm{char}(\Lambda v)$ there exists k such that vt_k is not divisible by q in $v\Lambda$.

iii) The set $\mathcal{L}$ is quasi-compact in the étale topology.

Examples: *Fields of totally $\mathfrak{S}$-adic numbers and their algebraic extensions*

Interesting examples of PAC, PRC and PpC fields, or more generally, of fields satisfying a universal local-global principle are the fields of totally $\mathfrak{S}$-adic elements and their algebraic extensions. These can be described as follows: Let κ be a field and $\tilde{\kappa}$ its algebraic closure. We say that a place $\mathfrak{p}$ of κ is of local type if the completion $\kappa_{\mathfrak{p}}$ of κ with respect to $\mathfrak{p}$ is a locally compact field and $\kappa_{\mathfrak{p}}$ is separable over κ. Let $\mathbb{C}_{\mathfrak{p}}$ be the completion of the algebraic closure of $\kappa_{\mathfrak{p}}$. We say that an element $a \in \tilde{\kappa}$ is totally $\mathfrak{p}$-adic if for all κ-embeddings $\iota : \tilde{\kappa} \hookrightarrow \mathbb{C}_{\mathfrak{p}}$ one has $\iota(a) \in \kappa_{\mathfrak{p}}$.

We remark that the notion of a totally $\mathfrak{p}$-adic element generalizes the notion of a totally real or totally p-adic algebraic number in a natural way.

Let $\mathfrak{S}$ be a finite set of places of local type of κ. We say that an algebraic element a over κ is totally $\mathfrak{S}$-adic if a is totally $\mathfrak{p}$-adic for all $\mathfrak{p} \in \mathfrak{S}$. We denote by $\kappa^{\mathfrak{S}}$ the set of all totally $\mathfrak{S}$-adic elements in the algebraic closure of κ. By general valuation theory it follows that $\kappa^{\mathfrak{S}}$ is a field and moreover, $\kappa^{\mathfrak{S}}|\kappa$ is a Galois extension. Further we denote by $\mathcal{L}^{\mathfrak{p}}$ the set of the decomposition fields of all the prolongations of $\mathfrak{p}$ to $\kappa^{\mathfrak{s}}$ and by $\mathcal{L}^{\mathfrak{S}}$ the the union of all $\mathcal{L}^{\mathfrak{p}}$ ($\mathfrak{p} \in \mathfrak{S}$). Clearly, $\mathcal{L}^{\mathfrak{S}}$ is a quasi-local set of localities of κ and $\kappa^{\mathfrak{S}}$ is exactly the intersection of all $\Lambda \in \mathcal{L}^{\mathfrak{S}}$. One of the basic properties of the algebraic extensions of $\kappa^{\mathfrak{S}}$ is given by the following theorem, see e.g. [G–P–R], the semilocal theory, combined with [P2], the overfield theorem (2.8):

Theorem ($\mathfrak{S}$). *Let κ' be some algebraic extension of $\kappa^{\mathfrak{S}}$ and $\mathcal{L}' = \{\Lambda\kappa' \mid \Lambda \in \mathcal{L}^{\mathfrak{S}}\}$ the prolongation of $\mathcal{L}^{\mathfrak{S}}$ to κ'. Then κ' satisfies a universal local-global principle with respect to $\mathcal{L}'$.*

In particular it holds:

(1) *If all $\Lambda' \in \mathcal{L}'$ are separably closed, then κ' is a PAC field.*

(2) *The field of all totally real, respectively totally p-adic algebraic numbers is a PRC, respectively a PpC field.*

We now can announce one of the main results of the paper. For definitions we refer to the beginning of section 1 of the paper and to section 2, proof of Theorem B.

Main Theorem. *Let κ be a hilbertian field satisfying a universal local-global principle. Then every split embedding problem for the absolute Galois group G_κ of κ has a proper solution.*

The main theorem will be proved in section 2. The proof relies on **(1.5)**, which itself is a consequence of the $\frac{1}{2}$ Riemann Existence Theorem with Galois Action, see [P4]. As applications of the main theorem we derive the following Theorem A and Theorem B from which the results announced above easily follow.

Theorem A. *Let κ be a hilbertian countable field satisfying a universal local-global principle. If κ has cohomological dimension 1 then the absolute Galois group G_κ of κ is ω-free.*

As corollaries one has:

Theorem 1. *Let κ be a hilbertian countable PAC field. Then G_κ is ω-free.*

The proof follows immediately from Theorem A, taking into account that the PAC fields have cohomological dimension 1 by Ax [A].

Theorem 2. *Let κ be a hilbertian countable field and $\mathfrak{S}$ a finite set of places of local type of κ. Let $\kappa^{\mathfrak{S},\mathrm{cycl}}$ denote the maximal cyclotomic extension of $\kappa^{\mathfrak{S}}$. Then the absolute Galois group of $\kappa^{\mathfrak{S},\mathrm{cycl}}$ is ω-free.*

Proof. It is clear that $\kappa' = \kappa^{\mathfrak{S},\mathrm{cycl}}$ is the compositum of $\kappa^{\mathfrak{S}}$ with the maximal cyclotomic extension κ^{cycl} of κ.

We first remark that κ' is a hilbertian field. Indeed, if $X \neq \phi$ is a set of prime numbers which are $\neq \mathrm{char}(\kappa)$ and $\kappa^{\mathfrak{S}}(\mu_X)$ denotes the extension of $\kappa^{\mathfrak{S}}$ obtained by adjoining all the roots of unity which have order divisible only by the prime numbers in X, then one has:

1) $\kappa^{\mathfrak{S}}(\mu_X)$ is not contained in $\kappa^{\mathfrak{S}}$, as the latter field has localizations isomorphic to local fields.
2) If X and Y are not comparable sets of rational primes as above (with respect to the inclusion relation), then $\kappa^{\mathfrak{S}}(\mu_X)$ and $\kappa^{\mathfrak{S}}(\mu_Y)$ are not comparable (with respect to the inclusion relation).

Now choose non-comparable X and Y such that $X \cup Y$ consists of all rational prime numbers $\neq \mathrm{char}(\kappa)$. Then $\kappa^{\mathfrak{S}}(\mu_X)$ and $\kappa^{\mathfrak{S}}(\mu_Y)$ are Galois extensions of the hilbertian field κ, they are not comparable with respect to the inclusion and their compositum is κ'. By the result of Haran–Jarden [H–J] it follows that κ' is a hilbertian field.

Next we remark that κ' has cohomological dimension equal to 1. Indeed, let $\mathcal{L}'$ be the prolongation of $\mathcal{L}^{\mathfrak{S}}$ to κ'. By [P2], theorem (3.3) the absolute Galois group of κ' is relatively projective with respect to $\{ G_{\Lambda'} \mid \Lambda' \in \mathcal{L}' \}$ viewed as set of closed subgroups of $G_{\kappa'}$. On the other hand, if such a locality Λ' prolongs some $\Lambda \in \mathcal{L}^{\mathfrak{S}}$ to κ' we have $\Lambda' = \Lambda\kappa^{\text{cycl}} = \Lambda^{\text{cycl}}$ and, as the completion of Λ is a local field, it follows that Λ^{cycl} has cohomological dimension 1. Thus every Λ' has cohomological dimension 1 and consequently, every extension of a finite group by $G_{\Lambda'}$ is split. Therefore, by the definition of the relative projectivity, see [P2], Ch 1, it follows that every extension of a finite group by the absolute Galois group of κ' is split. Equivalently, κ' has cohomological dimension equal to 1. To finish the proof of the theorem one applies Theorem A.

The theorem above gives us new evidence for Shafarevich's conjecture which asserts that the absolute Galois group of κ^{cycl} is ω-free for every global field κ. Indeed, taking into account that every non-trivial place of a global field is of local type, it follows that in a precise sense the theorem above is the *semi-local version* of Shafarevich's conjecture. The result is interesting because the fields $\kappa^{\mathfrak{S},\text{cycl}}$ approximate κ^{cycl} "from the top" as $\mathfrak{S}$ runs over all finite sets of places of κ.

Theorem B. *Let κ be a hilbertian countable field which satisfies a universal local-global principle with respect to a quasi-local set of localities $\mathcal{L}_\kappa$ which is closed by conjugation with elements from G_κ. Then there exists an étale compact subset $\mathcal{G}$ of $\mathcal{G}_\kappa = \{ G_\Lambda \mid \Lambda \in \mathcal{L}_\kappa \}$ such that G_κ is $\mathcal{G}$-free.*

In particular, the subgroup $G_{\mathcal{L}_\kappa}$ of G_κ generated by all G_Λ $(\Lambda \in \mathcal{L}_\kappa)$ is the generalized profinite free product on some fundamental domain of $\mathcal{G}_\kappa$ with respect to the action of $G_{\kappa^{\mathfrak{S}}}$.

For the proof one uses the main theorem above together with a generalization of a theorem of Iwasawa from [I]. As a consequence of Theorem B we have the following description of the absolute Galois group of the field of all totally $\mathfrak{S}$-adic elements:

Theorem 3. *Let κ be a finitely generated field and $\mathfrak{S}$ a finite set of places of local type of κ. Then the absolute Galois group of the field of totally $\mathfrak{S}$-adic elements $\kappa^{\mathfrak{S}}$ is isomorphic as profinite group to the profinite free product*

$$G_{\kappa^{\mathfrak{S}}} \cong \mathop{*}_{\mathfrak{p}} F_{G_\mathfrak{p}, X_\mathfrak{p}} \quad (\mathfrak{p} \in \mathfrak{S})$$

where $F_{G_\mathfrak{p}, X_\mathfrak{p}}$ is the generalized profinite free product of $G_\mathfrak{p} = G_{\kappa_\mathfrak{p}}$ on the space $X_\mathfrak{p}$ of all the prolongations of $\mathfrak{p}$ to $\kappa^{\mathfrak{S}}$ (all $\mathfrak{p} \in \mathfrak{S}$).

Proof. We begin with the following obvious remark: Let $\mathfrak{S}$ be a finite set of independent places $\mathfrak{p}$ of a field κ. Let $\mathcal{L}^\mathfrak{p}$ be the set of the decomposition fields of all the prolongations of $\mathfrak{p}$ to κ^{s}. We suppose that these decomposition fields are not separably closed. Further let $\mathcal{L}^{\mathfrak{S}}$ be the union of all $\mathcal{L}^\mathfrak{p}$ $(\mathfrak{p} \in \mathfrak{S})$. By general valuation theory the following holds:

1) $\mathcal{L}^\mathfrak{p}$ is étale compact and homeomorphic as a G_κ-space to $G_\kappa/Z_\mathfrak{p}$, where $Z_\mathfrak{p}$ is the decomposition group of some prolongation of $\mathfrak{p}$ to κ^{s}.

2) $\mathcal{L}^{\mathfrak{S}}$ is the disjoint union of all $\mathcal{L}^{\mathfrak{p}}$ ($\mathfrak{p} \in \mathfrak{S}$) and it is étale compact.

Let $\mathfrak{S}', \mathfrak{S}''$ be finite sets of places of local type of κ which strictly contain $\mathfrak{S}$ and satisfy $\mathfrak{S}' \cap \mathfrak{S}'' = \mathfrak{S}$. Let λ denote the compositum $\lambda = \kappa^{\mathfrak{S}'} \kappa^{\mathfrak{S}''}$. Clearly, λ is contained in the field $\kappa^{\mathfrak{S}}$ of totally $\mathfrak{S}$-adic elements, as both $\kappa^{\mathfrak{S}'}$ and $\kappa^{\mathfrak{S}''}$ are contained in $\kappa^{\mathfrak{S}}$. Let further $\mathcal{L}'$ and $\mathcal{L}''$ denote the prolongations of $\mathcal{L}^{\mathfrak{S}'}$ and respectively $\mathcal{L}^{\mathfrak{S}''}$ to λ. By [P2], theorem (2.8) it follows that λ satisfies a universal local-global principle both with respect to $\mathcal{L}'$ as well as with respect to $\mathcal{L}''$. Using the density and unicity theorem (2.6) from loc.cit. it follows that λ satisfies a universal local-global principle with respect to $\mathcal{L} = \mathcal{L}' \cap \mathcal{L}''$. On the other hand, every $\Lambda_0 \in \mathcal{L}$ is of the form $\Lambda\lambda$ for some $\Lambda \in \mathcal{L}^{\mathfrak{S}}$. Hence $\Lambda_0 = \Lambda$, as both $\kappa^{\mathfrak{S}'}$ and $\kappa^{\mathfrak{S}''}$ are contained in Λ. Thus $\mathcal{L} = \mathcal{L}^{\mathfrak{S}}$. Next we remark that λ is hilbertian by the result of Haran–Jarden [H–J]. In particular we can apply Theorem B to λ endowed with $\mathcal{L}_\lambda := \mathcal{L}$, which clearly is a quasi-local set of localities of λ. With the notations from Theorem B it follows that $G_{\kappa^{\mathfrak{S}}}$ is exactly the subgroup $G_{\mathcal{L}_\lambda}$ of G_λ. Hence $G_{\kappa^{\mathfrak{S}}}$ is the generalized profinite free product on some fundamental domain, say $\mathcal{G}$ of $\mathcal{G}_\lambda$ with respect to the action of $G_{\kappa^{\mathfrak{S}}}$. By the remark at the beginning of the proof and using the notations from there it follows that setting $\mathcal{G}^{\mathfrak{p}} = \mathcal{G} \cap \{G_\Lambda \mid \Lambda \in \mathcal{L}^{\mathfrak{p}}\}$ one has: $\mathcal{G}^{\mathfrak{p}}$ is a fundamental domain for $\{G_\Lambda \mid \Lambda \in \mathcal{L}^{\mathfrak{p}}\}$ with respect to the action of $G_{\kappa^{\mathfrak{S}}}$ and clearly, $\mathcal{G}$ is the disjoint union of all $\mathcal{G}^{\mathfrak{p}}$ ($\mathfrak{p} \in \mathfrak{S}$). By the properties of the generalized free products, see remark (2.2), it follows that $G_{\kappa^{\mathfrak{S}}}$ is isomorphic to the profinite free product of the generalized profinite free products $F_{\mathcal{G}^{\mathfrak{p}}}$ (all $\mathfrak{p} \in \mathfrak{S}$). On the other hand, $\mathcal{G}^{\mathfrak{p}}$ is canonically homeomorphic (by the Galois correspondence) to a fundamental domain of $\mathcal{L}^{\mathfrak{p}}$ with respect to the action of $G_{\kappa^{\mathfrak{S}}}$. Hence, it is homeomorphic to the space $X_{\mathfrak{p}}$ of all prolongations of $\mathfrak{p}$ to $\kappa^{\mathfrak{S}}$. To proceed we again apply the remark (2.2).

1. Regular resolvents for fields satisfying a universal local-global principle

Let G be an arbitrary profinite group. An embedding problem $\mathrm{EP} = (\gamma, \alpha)$ for G consists of a diagram of profinite group homomorphisms of the form

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \gamma} \\ B & \xrightarrow{\alpha} & A \end{array}$$

where α and γ are surjective.

EP is called finite if B is a finite group.

EP is called split if α has a section, or equivalently, if $\ker(\alpha)$ has a complement.

A solution of EP is a group homomorphism $\beta : G \longrightarrow B$ with $\alpha\beta = \gamma$. Obviously, every split embedding problem has a solution.

A proper solution of EP is a *surjective* group homomorphism $\beta : G \longrightarrow B$ with $\alpha\beta = \gamma$.

(1.1) Example and **Definition.** Let the following situation be given:

- A homomorphic image $G \xrightarrow{\gamma} A$ of G with A finite.
- A finite family $\Sigma_0 = \left(\mathfrak{C}_k\right)_k$ of finite cyclic groups and a finite family $\Sigma = \left(\mathfrak{C}_{k,\sigma}\right)_{k,\sigma}$ of isomorphic copies $\iota_{k,\sigma} : \mathfrak{C}_{k,\sigma} \longrightarrow \mathfrak{C}_k$ $(\sigma \in A)$ of $\mathfrak{C}_k$ (all k). For $g_k \in \mathfrak{C}_k$ we let $g_{k,\sigma}$ denote its preimage by $\iota_{k,\sigma}$.

Let F be the profinite free product on Σ. There exists a natural right action of A on F which is defined by $(g_{k,\tau\sigma})\tau = g_{k,\sigma}$ for all k, σ, τ. With respect to this action we denote by $A \ltimes F$ the semi-direct product of F by A. We identify A and F with the subgroups $A \ltimes 1$ and respectively $1 \ltimes F$ of $A \ltimes F$. We further use Σ also to denote the family of all $\mathfrak{C}_{k,\sigma}$ viewed as subgroups of F.

For normal subgroups D of $A \ltimes F$ which are contained in F we set $C = F/D$ and identify $(A \ltimes F)/D$ with $A \ltimes C$. Let $\alpha_D : A \ltimes F \longrightarrow A \ltimes C$ and $\alpha^D : A \ltimes C \longrightarrow A$ denote the canonical projections and Σ^D (or simply Σ if no confusion is possible) the image of Σ by α_D.

Clearly, $\mathrm{EP}^D = (\gamma, \alpha^D)$ is an embedding problem for G. We call such an embedding problem an *adjusted embedding problem* for G. We say that EP^D is an adjusted finite embedding problem for G if D is an open normal subgroup of $A \ltimes F$.

Let $\mathrm{EP} = (\gamma, \alpha)$ and $\mathrm{EP}' = (\gamma', \alpha')$ be embedding problems for G. We say that EP' dominates EP, if there exist surjective group homomorphisms $\gamma_0 : A' \longrightarrow A$ and $\beta_0 : B' \longrightarrow B$ such that $\gamma = \gamma_0\gamma'$, $\alpha\beta_0 = \gamma_0\alpha'$ and $\beta_0\left(\ker(\alpha')\right) = \ker(\alpha)$. In this context we have: If β' is a (proper) solution of EP' then $\beta_0\beta'$ is a (proper) solution of EP.

(1.2) *The class of all adjusted finite embedding problems for* G *is cofinal with respect to the domination relation in the class of all finite* split *embedding problems for* G.

More precisely, let $\mathrm{EP} = (\gamma, \alpha)$ *be a given finite split embedding problem for* G *and* $\overline{\alpha}$ *a section of* α. *Further let* $\Sigma_0 = \left(\mathfrak{C}_k\right)_k$ *be a finite set of cyclic subgroups of* $\ker(\alpha)$ *such that* Σ_0 *generates* $\ker(\alpha)$ *and is closed by conjugation with elements from* $\overline{A} = \overline{\alpha}(A)$. *Then with* γ *and* Σ_0 *as in the hypothesis for* **(1.1)** *there exist open* D *and surjective group homomorphisms* $\beta_0 : A \ltimes C \longrightarrow B$ *such that:*

1) $\overline{\alpha}(\sigma) = \beta_0(\sigma \ltimes 1)$ *for all* $\sigma \in A$.
2) $\beta_0(\Sigma^D) = \Sigma_0$ *and* β_0 *maps every* $\mathfrak{C}_{k,\sigma}$ *isomorphically onto* $\overline{\alpha}(\sigma)\,\mathfrak{C}_k\,\overline{\alpha}(\sigma)^{-1}$.

In particular, EP^D *dominates* EP.

Proof. First we remark that A acts from the right on $\ker(\alpha)$ and its subsets by means of $\overline{\alpha}$ as follows: $(\cdot)\tau = (\cdot)^{\overline{\alpha}(\tau)}$ the conjugation being considered in B. Using the construction from **(1.1)** there exists a surjective homomorphism

$$\varphi : F \longrightarrow \ker(\alpha) \quad \text{by} \quad \varphi(g_{k,\sigma}) = (g_k)\sigma^{-1}$$

which is compatible with the action of A on F and C. Hence φ canonically induces a group homomorphism

$$\overline{\alpha} \ltimes \varphi : A \ltimes F \longrightarrow B \quad \text{by} \quad (\tau, g) \mapsto \overline{\alpha}(\tau)\varphi(g).$$

One clearly has $(\overline{\alpha} \ltimes \varphi)|_A = \overline{\alpha}$. Further $\overline{\alpha} \ltimes \varphi$ maps every $\mathfrak{C}_{k,\sigma}$ isomorphically onto $(\mathfrak{C}_k)\sigma^{-1} = \overline{\alpha}(\sigma)\, \mathfrak{C}_k\, \overline{\alpha}(\sigma)^{-1}$ and hence $(\overline{\alpha} \ltimes \varphi)(\Sigma) = \Sigma_0$. By a standard limit argument it follows that there exist open normal subgroups D of $A \ltimes F$ as in **(1.1)** such that $\overline{\alpha} \ltimes \varphi$ factorizes through α_D. $\square$

Let H be another profinite group and $\pi : H \longrightarrow G$ a surjective group homomorphism. Then every embedding problem $\mathrm{EP} = (\gamma, \alpha)$ gives rise in a canonical way to an embedding problem $\mathrm{EP}_H = (\gamma_H, \alpha)$ for H by setting $\gamma_H = \gamma\pi$. We say that a solution β of EP_H is regular if $\beta(\ker(\pi)) = \ker(\alpha)$. One has: Suppose that EP and EP$'$ are embedding problems for G and that EP$'$ dominates EP. Then for every regular solution β'_H of EP'_H it follows that $\beta_0\beta'_H$ is a regular solution for EP_H.

We say that H is a *regular resolvent* for G if for every finite split embedding problem EP for G, the corresponding embedding problem EP_H has a proper regular solution. Using **(1.2)** we get:

(1.3) *H is a regular resolvent for G if and only if for every adjusted finite embedding problem* EP *for G the corresponding embedding problem* EP_H *for H has a proper regular solution.*

For an arbitrary field κ and a regular field extension $K|\kappa$ let $\pi_{K\kappa} : G_K \longrightarrow G_\kappa$ denote the canonical projection homomorphism. As $K|\kappa$ is regular it follows that $\pi_{K\kappa}$ is surjective. By the observations above, every embedding problem EP for G_κ gives rise by means of $\pi_{K\kappa}$ canonically to an embedding problem EP_K for G_K.

We say that K is a *regular resolvent* for κ if G_K is a regular resolvent for G_κ.

We remark that by **(1.3)**, to show that K is a regular resolvent for κ it is sufficient to prove that for every adjusted embedding problem EP for G_κ the corresponding embedding problem EP_K for G_K has a proper regular solution.

We make a further restriction on the embedding problems which is necessary to be considered in order to verify that K is a regular resolvent for κ. For this we introduce the following notations:

- For a given finite family Σ of finite groups, let m_Σ denote the lowest common multiple of the orders of all the groups in Σ.
- For a given embedding problem $\mathrm{EP} = (\gamma, \alpha)$ for G_κ let λ_{EP} denote the fixed field of $\ker(\gamma)$ in the separable closure of κ.

We say that an embedding problem EP for G_κ is rationally adjusted if it is an adjusted embedding problem and λ_{EP} contains the roots of unity μ_{m_Σ} of order m_Σ.

(1.4) *Every finite split embedding problem* EP *for* G_κ *is dominated by a rationally adjusted finite embedding problem.*

Indeed, for a given split finite embedding problem $\mathrm{EP} = (\gamma, \alpha)$ for G_κ and given $\overline{\alpha}$ and Σ_0 as in **(1.2)** let $D \subseteq G_\kappa$ be an open normal subgroup which is contained in $\ker(\gamma)$ and such that the fixed field κ' of D contains μ_{m_Σ}. We set $A' = G_\kappa/D$ and denote by $\gamma' : G_\kappa \longrightarrow A'$ and $\gamma_0 : A' \longrightarrow A$ the canonical projections. Let us further set $B' = B \times_A A'$ and let $\alpha' : B' \longrightarrow A'$ and $\beta_0 : B' \longrightarrow B$ be the canonical projections. Then it follows that $\mathrm{EP}' = (\gamma', \alpha')$ dominates EP. Moreover, by the universality property there exists a (unique) section $\overline{\alpha}' : A' \to B'$ of α' such that $\beta_0\overline{\alpha}' = \overline{\alpha}$. Further, β_0 maps $C = \ker(\alpha')$ isomorphically onto $C = \ker(\alpha)$. Denoting by Σ_0' the preimage of Σ_0 in C one has: β_0 maps Σ_0' bijectively onto Σ_0 and every $\mathfrak{C}' \in \Sigma_0'$ isomorphically onto the corresponding $\mathfrak{C} \in \Sigma_0$. In particular, Σ_0' is a set of cyclic subgroups of C which generate C and Σ_0' is closed by conjugation with respect to elements of $\overline{A}' = \overline{\alpha}'(A')$. Moreover, $m_{\Sigma_0'} = m_{\Sigma_0}$.

Our claim now follows by **(1.2)** and **(1.1)** applied to EP'.

The main result of this section is:

(1.5) Theorem. *Suppose that* κ *satisfies a universal local-global principle. Then the rational function field in one variable* $\kappa(t)$ *is a regular resolvent for* κ. *Equivalently, if* EP *is a finite split embedding problem for* G_κ *then the corresponding embedding problem* $\mathrm{EP}_{\kappa(t)}$ *for* $G_{\kappa(t)}$ *has a proper regular solution.*

Proof. The technical core of the proof consists of the following lemma, which is a special case of **(1.5)** and is a consequence of $\frac{1}{2}$ Riemann Existence Theorem with Galois action, see [P4].

(1.6) Lemma. *For an arbitrary field* κ *we let* $\hat{\kappa} = \kappa((u))$ *be the power series field in one variable over* κ. *Then the rational function field in one variable* $K = \hat{\kappa}(t)$ *over* $\hat{\kappa}$ *is a regular resolvent for* $\hat{\kappa}$.

Proof (of the lemma). We first recall the content of $\frac{1}{2}$ Riemann Existence Theorem with Galois action for opens of the 1-dimensional projective space $\mathbb{P}^1_{\hat{\kappa}}$ over $\hat{\kappa}$.

Let S be a finite set of closed points in $\mathbb{P}^1_{\hat{\kappa}}$ and $\mathbb{U}$ the complement of S. We denote by $\overline{S} = \{s_1, \ldots, s_{n_S}\}$ and $\overline{\mathbb{U}}$ the base change of S and $\mathbb{U}$ to the algebraic closure of $\hat{\kappa}$. We fix a function $t \in \hat{\kappa}(\mathbb{P}^1_{\hat{\kappa}})$ such that $\hat{\kappa}(\mathbb{P}^1_{\hat{\kappa}}) = \hat{\kappa}(t)$ and set $K = \hat{\kappa}(t)$. By means of t we identify $\mathbb{P}^1_{\hat{\kappa}}(\Omega)$ with $\Omega \cup \infty$ for every overfield Ω of $\hat{\kappa}$. We shall suppose that S does not meet the pole of t. There exists an exact sequence

$$1 \to \pi_1(\overline{\mathbb{U}}) \longrightarrow \pi_1(\mathbb{U}) \longrightarrow G_{\hat{\kappa}} \to 1$$

which is split, as $\mathbb{P}^1_{\hat{\kappa}}$ has $\hat{\kappa}$-rational points. We suppose that $n_S = 2n$ is even and by the conventions above we identify $\overline{S}$ with a finite subset of the algebraic closure of

$\hat{\kappa}$. We say that S is pairwise adjusted if the elements of $\overline{S}$ can be organized in pairs $p_k = (x_k, y_k)$ $(1 \le k \le n)$ which are permuted by $G_{\hat{\kappa}}$ between themselves and satisfy $v(x_k - y_k) > v(x_k - x_{k'})$ for all $k \neq k'$, where v denotes the canonical valuation of $\hat{\kappa}$ and also its unique prolongation to the algebraic closure of $\hat{\kappa}$.

Now suppose that S is pairwise adjusted. We consider the profinite group $\overline{\Pi}$ on n_S generators and relations as follows:

$$\overline{\Pi} = \langle\, g_{x_1}, h_{y_1}, \ldots, g_{x_n}, h_{y_n} \mid g_{x_k} h_{y_k} = 1 \quad \text{for all } k \,\rangle$$

endowed with the right $G_{\hat{\kappa}}$-action defined by $(g_{x_k})\hat{\tau} = \left(g_{\hat{\tau}^{-1}x_k}\right)^{\chi(\hat{\tau}^{-1})}$ and correspondingly for h_{y_k}, where χ is the cyclotomic character of $G_{\hat{\kappa}}$ and $\hat{\tau} \in G_{\hat{\kappa}}$ is arbitrary. By the main result from [P4] one has the following (n.b., we are here in the equal characteristic case):

The canonical exact sequence

$$1 \to \pi_1(\overline{\mathbb{U}}) \longrightarrow \pi_1(\mathbb{U}) \longrightarrow G_{\hat{\kappa}} \to 1$$

has in a canonical way the following quotient:

$$1 \to \overline{\Pi} \longrightarrow G_{\hat{\kappa}} \ltimes \overline{\Pi} \longrightarrow G_{\hat{\kappa}} \to 1,$$

the semi-direct product being considered with the canonical right action of $G_{\hat{\kappa}}$ on $\overline{\Pi}$. Moreover, the generators g_{x_k} and h_{y_k} are inertia elements associated to x_k, respectively y_k (all k).

We return to the proof of **(1.6)**. By **(1.4)** and **(1.3)** it is sufficient to show that for a given rationally adjusted finite embedding problem $\mathrm{EP} = \mathrm{EP}^D$ for $G_{\hat{\kappa}}$, say of the form

$$\begin{array}{ccc} & & G \\ & & \big\downarrow \gamma \\ A \ltimes (F/D) & \xrightarrow{\alpha^D} & A \end{array}$$

the corresponding embedding problem EP_K for G_K has a proper regular solution. Let $\hat{\lambda} = \hat{\lambda}_{\mathrm{EP}}$ be the fixed field of $\ker(\gamma)$ in the separable closure of $\hat{\kappa}$. For a fixed normal basis $\mathcal{X} = \{x_\sigma\}_{\sigma \in A}$ of $\hat{\lambda}|\hat{\kappa}$ we choose elements a_k of $\hat{\kappa}$ such that all $x_{k,\sigma} = a_k x_\sigma$ (all k, σ) are distinct. We further take for every k some $b_k \neq a_k$ in $\hat{\kappa}$ (which is close enough to a_k in the valuation topology) and set $y_{k,\sigma} = b_k x_\sigma$ (all k, σ). Let $S' \subset \mathbb{P}^1_{\hat{\kappa}}$ be the finite set of closed points defined by $x_{k,\sigma}$ and $y_{k,\sigma}$ (all k, σ). With these notations $\overline{S}'$ is exactly the set of all $x_{k,\sigma}$ and $y_{k,\sigma}$ (all k, σ). Moreover, if b_k are sufficiently close to a_k, then S' is pairwise adjusted, the pairs being $p_{k,\sigma} = (x_{k,\sigma}, y_{k,\sigma})$ for all k, σ. Let $\mathbb{U}'$ denote the complement of S' in $\mathbb{P}^1_{\hat{\kappa}}$ and let $\overline{\Pi}'$ be the profinite group on generators $g_{x_{k,\sigma}}, h_{y_{k,\sigma}}$ (all k, σ) and relations as follows:

$$\overline{\Pi}' = \langle\, g_{x_{k,\sigma}}, h_{y_{k,\sigma}} \mid g_{x_{k,\sigma}} h_{y_{k,\sigma}} = 1 \quad \text{for all } k, \sigma \,\rangle$$

endowed with the right $G_{\hat{\kappa}}$-action defined by $(g_{x_{k,\sigma}})\hat{\tau} = (g_{\hat{\tau}^{-1}x_{k,\sigma}})^{\chi(\hat{\tau}^{-1})}$ and correspondingly for $h_{x_{k,\sigma}}$, where χ is the cyclotomic character of $G_{\hat{\kappa}}$ and $\hat{\tau} \in G_{\hat{\kappa}}$ is arbitrary. As remarked above, the canonical exact sequence

$$1 \to \pi_1(\overline{\mathbb{U}}') \longrightarrow \pi_1(\mathbb{U}') \longrightarrow G_{\hat{\kappa}} \to 1$$

has in a canonical way as a quotient the exact sequence

$$1 \to \overline{\Pi}' \longrightarrow G_{\hat{\kappa}} \ltimes \overline{\Pi}' \longrightarrow G_{\hat{\kappa}} \to 1,$$

the semi-direct product being considered with respect to the canonical action of $G_{\hat{\kappa}}$ on $\overline{\Pi}'$. Moreover, the generators $g_{x_{k,\sigma}}$ and $h_{y_{k,\sigma}}$ are inertia elements associated to $x_{k,\sigma}$, respectively $y_{k,\sigma}$ (all k, σ).

In the context from **(1.1)** for every k we fix a generator g_k of $\mathfrak{C}_k$ and denote by $g_{k,\sigma}$ the corresponding generator of $\mathfrak{C}_{k,\sigma}$ (all k, σ). In particular, $(g_{h,\tau\sigma})\tau = g_{k,\sigma}$ for all k and $\sigma, \tau \in A$. We next define $\varphi : \overline{\Pi}' \longrightarrow F$ by setting $\varphi(g_{x_{k,\sigma}}) = (g_{k,\sigma})^{\chi_{\mathrm{EP}}(\sigma^{-1})}$, where χ_{EP} is the cyclotomic character of $\mathrm{Gal}\left(\hat{\lambda}|\hat{\kappa}\right) = A$. Taking into account that $\hat{\lambda}$ contains the roots of unity of order m_k (all k), an immediate verification shows that φ is compatible with the action of $G_{\hat{\kappa}}$ on $\overline{\Pi}'$ and F via γ. Hence there exists a commutative diagram of exact sequences of the form:

$$\begin{array}{ccccccc}
1 \to & \overline{\Pi}' & \longrightarrow & G_{\hat{\kappa}} \ltimes \overline{\Pi}' & \longrightarrow & G_{\hat{\kappa}} & \to 1 \\
 & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \gamma \ltimes \varphi} & & \downarrow{\scriptstyle \gamma} & \\
1 \to & F & \longrightarrow & A \ltimes F & \longrightarrow & A & \to 1
\end{array}$$

Therefore, $\beta_K : G_K \longrightarrow \pi_1(\mathbb{U}) \longrightarrow G_{\hat{\kappa}} \ltimes \overline{\Pi}' \longrightarrow A \ltimes F \longrightarrow A \ltimes (F/D)$ is a proper regular solution of the embedding problem EP_K. The proof of **(1.6)** is finished.

We now return to the proof of **(1.5)**. The proof is based on the following field theoretical assertions.

(1.7) Lemma. *Let $\hat{\kappa}|\kappa$ be a field extension with κ existentially closed in $\hat{\kappa}$.* [2]) *Let $K|\kappa$ be a regular function field over κ and suppose that K and $\hat{\kappa}$ are linearly disjoint over κ. Let $K\hat{\kappa}$ be the compositum of K and $\hat{\kappa}$ (in some universe containing both of them). Then, if $K\hat{\kappa}$ is a regular resolvent for $\hat{\kappa}$, so is K for κ.*

Proof. The proof follows immediately using non-standard type arguments concerning function fields, see for instance [vdD–S], [R–R], [P1] and others. Therefore

[2]) See for instance [B–S] for definitions and basic facts.

we will only sketch the proof here. First, as κ is existentially closed in $\hat{\kappa}$, there exists an ultrapower $^*\kappa = \kappa^I/\mathcal{U}$ of κ and a κ-embedding $\hat{\kappa} \hookrightarrow {}^*\kappa$, see [B–S]. Setting $^*K = K^I/\mathcal{U}$ it follows that $^*K$ is linearly disjoint over K to the algebraic closure of K. We further consider the composita $K\hat{\kappa}$ and $K^*\kappa$ as subfields of $^*K$. Let $\mathrm{EP} = (\gamma, \alpha)$ be a split embedding problem for G_κ. We remark that $\hat{\kappa}|\kappa$ and $^*\kappa|\kappa$ are regular field extensions, hence by means of the canonical projections $G_{^*\kappa} \longrightarrow G_\kappa$ and $G_{\hat{\kappa}} \longrightarrow G_\kappa$ we can consider the induced embedding problems $\mathrm{EP}_{\hat{\kappa}}$ and $\mathrm{EP}_{^*\kappa}$ which are split. As $K\hat{\kappa}$ is by hypothesis a regular resolvent for $\hat{\kappa}$ it follows that $\mathrm{EP}_{K\hat{\kappa}}$ has a proper regular solution $\beta_{K\hat{\kappa}}$. Therefore, if $\pi : G_{K^*\kappa} \longrightarrow G_{K\hat{\kappa}}$ denotes the canonical projection, then $\beta_{K^*\kappa} = \beta_{K\hat{\kappa}}\pi$ is a proper regular solution for $\mathrm{EP}_{K^*\kappa}$. Let N be the fixed field of $\ker(\beta_{K^*\kappa})$ in the separable closure of $K^*\kappa$. Then $N\,|\,K^*\kappa$ is a finite Galois extension and in particular, $N|^*\kappa$ is a function field. Moreover, denoting by λ the fixed field of $\ker(\gamma)$ in the separable closure of κ and $^*\lambda = \lambda^I/\mathcal{U}$, it follows that $^*\lambda = {}^*\kappa\lambda$ and $N|^*\lambda$ is a regular field extension. Using the terminology from [P4], p.164 and especially the facts (7)–(9), it follows that $K^i = K$ are local representatives for $K^*\kappa$. Taking local representatives $N^i|\kappa$ for $N|^*\kappa$ it follows that locally $N^i|K$ are Galois extensions which give rise to proper regular solutions of EP_K. □

(1.8) Lemma. *Let κ be a field with a universal local-global principle. Then κ is existentially closed in the power series field in one variable $\hat{\kappa} = \kappa((x))$ over κ.*

Proof. It is well known that $\hat{\kappa}$ is separable over κ. Hence our assertion follows if we are able to show that κ is existentially closed in every finitely separably generated subextension $R|\kappa$ of $\hat{\kappa}|\kappa$. For such a field extension $R|\kappa$ one has the following general fact:

κ is existentially closed in R if and only if for every smooth model V of R which is defined over κ one has $V(\kappa) \neq \phi$.

To prove that in the context above we have $V(\kappa) \neq \phi$ we use the fact that κ satisfies a universal local-global principle with respect to some family of localities $\mathcal{L}$ of κ.

Case 1) $\mathcal{L} = \phi$.

Then κ is a PAC field and so $V(\kappa) \neq \phi$.

Case 2) $\mathcal{L} \neq \phi$.

Let $\Lambda \in \mathcal{L}$ be arbitrary. Although we cannot give a precise reference, a proof will appear in [K], it is a well known fact for the specialist in model theoretic algebra that every non-trivial valued henselian field Λ is existentially closed (as a field and a valued field) in its power series field in one variable $\hat{\Lambda} = \Lambda((x))$. As $V(R) \neq \phi$ and since $R \subseteq \hat{\kappa} \subseteq \hat{\Lambda}$ it follows that $V(\hat{\Lambda}) \neq \phi$. Hence $V(\Lambda) \neq \phi$ as Λ is existentially closed in $\hat{\Lambda}$. As κ satisfies a universal local-global principle with respect to $\mathcal{L}$ and $\Lambda \in \mathcal{L}$ was arbitrary, it follows that $V(\kappa) \neq \phi$. □

To finish the proof of **(1.5)** one applies **(1.6)**, **(1.7)** and **(1.8)**.

2. Proof of Main Theorem, Theorem A and Theorem B (Introduction)

Proof of Main Theorem

We set $K = \kappa(t)$ and let $\beta : G_K \longrightarrow B$ be a proper regular solution for EP_K which exists by **(1.5)**. Equivalently, there exists a commutative diagram of surjective group homomorphisms of the form:

$$\begin{array}{ccc} G_K & \xrightarrow{\pi_{K\kappa}} & G_\kappa \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow{\alpha} & A \end{array}$$

We denote by λ and N the fixed fields of $\ker(\gamma)$ in κ^s and respectively, of $\ker(\beta)$ in K^s. Then $N \cap \kappa^s = \lambda$ and the exact sequence

$$1 \to \ker(\alpha) \hookrightarrow B \xrightarrow{\alpha} A \to 1$$

is canonically isomorphic to

$$1 \to \mathrm{Gal}(N|L) \hookrightarrow \mathrm{Gal}(N|K) \xrightarrow{\pi} \mathrm{Gal}(\lambda|\kappa) \to 1,$$

where $L = \lambda(t) = K\lambda$ and π is the canonical projection. Let $\overline{\alpha}$ be a section of α and $\overline{\pi}$ the section of π corresponding to $\overline{\alpha}$. Further let M denote the fixed field of the image of $\overline{\pi}$. Then $M|K$ is a finite separable extension, $[N : M] = [\lambda : \kappa]$ and $N = M\lambda$. Hence $M|\kappa$ is a regular function field of one variable. As κ is hilbertian there exist (many) κ-rational places $\mathfrak{p}$ of K, ie specializations $t \mapsto a$ for some $a \in \kappa$, such that:

- $\mathfrak{p}$ is inert in M. Equivalently, $M\mathfrak{q}\,|\,K\mathfrak{p}$ is separable and $\mathfrak{p}$ has a unique prolongation $\mathfrak{q}$ to M and so, $[M : K] = [M\mathfrak{q} : K\mathfrak{p}]$.
- $M\mathfrak{q}$ and λ are linearly disjoint over κ.

If now $\mathfrak{p}$ is such a place of K it follows that $\mathfrak{p}$ is inert not only in M but also in N. Equivalently, $\mathfrak{p}$ has a unique prolongation $\mathfrak{r}$ to N, $\mathrm{Gal}(N|K) = Z(\mathfrak{r}\,|\,\mathfrak{p})$ is the decomposition group of $\mathfrak{r}\,|\,\mathfrak{p}$ and the inertia group $T(\mathfrak{r}\,|\,\mathfrak{p}) = 1$ is trivial. Taking some prolongation $\mathfrak{r}^s$ of $\mathfrak{r}$ to the separable closure of K we denote by Z and T the decomposition, respectively the inertia group of $\mathfrak{r}^s|\,\mathfrak{p}$. By general valuation theory it follows that the images of Z and T in $\mathrm{Gal}(N|K)$ are exactly $Z(\mathfrak{r}\,|\,\mathfrak{p})$, respectively $T(\mathfrak{r}\,|\,\mathfrak{p})$. Hence these images are respectively $\mathrm{Gal}(N|K)$ and 1. Therefore we have $\beta(Z) = B$ and $\beta(T) = 1$. Hence the restriction of β to Z factorizes as follows

$$\beta : Z \longrightarrow Z/T \xrightarrow{\hat{\beta}} B,$$

where the first map is the canonical projection and $\hat{\beta}$ is surjective.

Next we remark that the restriction of the canonical projection $\pi_{K\kappa} : G_K \longrightarrow G_\kappa$ to Z defines the canonical exact sequence for $Z = Z(\mathfrak{r}^s|\,\mathfrak{p})$:

$$1 \to T \hookrightarrow Z \xrightarrow{\pi_{\mathfrak{p}}} G_{K\mathfrak{p}} \to 1.$$

As $\mathfrak{p}$ is κ rational, ie $K\mathfrak{p} = \kappa$, we have $G_{K\mathfrak{p}} = G_\kappa$. Hence the restriction of $\pi_{K\kappa}$ to Z factorizes as follows

$$\pi_{K\kappa} : Z \longrightarrow Z/T \xrightarrow{\hat{\pi}} G_\kappa,$$

where the first map is the canonical projection and $\hat{\pi}$ is an isomorphism. Therefore, the above commutative diagram gives rise to a commutative diagram of the form

$$\begin{array}{ccc} Z/T & \xrightarrow{\hat{\pi}} & G_\kappa \\ \downarrow{\scriptstyle \hat{\beta}} & & \downarrow{\scriptstyle \gamma} \\ B & \xrightarrow{\alpha} & A \end{array}$$

with $\hat{\pi}$ an isomorphism and $\hat{\beta}$ surjective. Hence, $\hat{\beta}\,\hat{\pi}^{-1}$ is a proper solution of EP. The proof of Main Theorem is finished. □

Proof of Theorem A

We consider the situation of Theorem A and prove that G_κ is ω-free. By Iwasawa's theorem 4 from [I] it is sufficient to prove that every finite embedding problem EP

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \gamma} \\ B & \xrightarrow{\alpha} & A \end{array}$$

for G_κ has a proper solution. First, as G_κ has cohomological dimension 1, it follows that EP has solutions. Starting with such a solution β there exists a general procedure to construct a split finite embedding problem $\mathrm{EP}' = (\gamma, \alpha')$ which dominates EP. Namely we set $A' = G_\kappa/\ker(\beta)$ and $G_\kappa \xrightarrow{\gamma'} A'$ for the canonical projection. There exists a unique isomorphism $A' \xrightarrow{\iota} \beta(G_\kappa)$ such that $\beta = \iota\gamma'$ and a unique surjective homomorphism $A' \xrightarrow{\gamma_0} A$ such that $\gamma = \gamma_0\gamma'$. With respect to the surjective homomorphisms $B \xrightarrow{\alpha} A$ and $A' \xrightarrow{\gamma_0} A$ we consider $B' = B \times_A A'$ and denote by $B' \xrightarrow{\alpha'} A'$ and $B' \xrightarrow{\beta_0} B$ the structural projections. Clearly, the embedding problem $\mathrm{EP}' = (\gamma', \alpha')$ dominates EP. Moreover, by the universality property of the fibre product it follows that there exists a unique group homomorphism $\overline{\alpha}' : A' \longrightarrow B'$ such that $\alpha'\overline{\alpha}' = \mathrm{id}_{A'}$ and $\beta_0\overline{\alpha}' = \iota$. In particular, EP' is a split embedding problem which dominates EP. Hence, without loss of generality we can suppose that EP is a split embedding problem for G_κ. One proceeds by applying the main theorem. □

Proof of Theorem B

We first briefly recall basic facts about relatively projective groups and generalized profinite free products, see Haran [H], section 3, Mel′nicov [M], and [P2], Ch 1.

Let G be a profinite group. We endow the set of all closed subgroups of G with the étale topology, which by definition has as basis the sets of the form

$$\mathcal{U}_{\Gamma,N} = \{\Gamma' \mid \Gamma' \subseteq N\Gamma\},$$

where Γ and N run over the sets of all closed subgroups of G and the set of all open normal subgroups of G, respectively. When speaking about the action of G on spaces of subgroups of G we always mean the inner conjugation. This action is continuous in the étale topology.

Let G be a profinite group and $\mathcal{G}$ an étale quasi-compact set of its closed subgroups. We denote by $\mathrm{con}(\mathcal{G})$ the smallest set of closed subgroups of G which is G-invariant and closed by taking subgroups and remark that $\mathrm{con}(\mathcal{G})$ is étale quasi-compact. Further we denote by $\mathcal{G}_{\max}$ the set of all maximal elements of $\mathcal{G}$ with respect to the inclusion relation and remark that $\mathcal{G}_{\max}$ is étale quasi-compact and $\mathrm{con}(\mathcal{G}_{\max}) = \mathrm{con}(\mathcal{G})$. Finally, a fundamental domain for $\mathcal{G}$ is by definition an étale quasi-compact subset of $\left(\mathrm{con}(\mathcal{G})\right)_{\max}$ containing exactly one representative from each conjugacy class. The following fact is an easy exercise with directed projective systems of finite sets:

(2.1) If G is countably generated then every étale quasi-compact set $\mathcal{G}$ of closed subgroups has a fundamental domain.

A (finite) $\mathcal{G}$-embedding problem $\mathrm{EP}_{\mathcal{G}} = (\gamma, \alpha, \mathcal{B})$ for G consists of:

1) A diagram of the form

$$\begin{array}{ccc} & & G \\ & & \big\downarrow{\scriptstyle\gamma} \\ B & \xrightarrow{\alpha} & A \end{array}$$

where α and γ are surjective (and B is finite).

2) A set $\mathcal{B}$ of subgroups Δ such that every $\gamma|_{\Gamma}$ factorizes through $\alpha|_{\Delta}$ for some $\Delta \in \mathcal{B}$.

We remark that condition 1) asserts that $\mathrm{EP} = (\gamma, \alpha)$ is a (finite) embedding problem for G as described at the beginning of section 1. The second point can be viewed as "local splitting conditions" in the following way: For every $\Gamma \in \mathcal{G}$ we can consider the induced embedding problem EP_{Γ} as follows:

$$\begin{array}{ccc} & & \Gamma \\ & & \big\downarrow{\scriptstyle\gamma} \\ \alpha^{-1}(\gamma(\Gamma)) & \xrightarrow{\alpha} & \gamma(\Gamma) \end{array}$$

The condition 2) above asserts that EP_{Γ} has solutions for every $\Gamma \in \mathcal{G}$.

A solution of $\mathrm{EP}_{\mathcal{G}}$ is any homomorphism $\beta : G \to B$ such that $\gamma = \alpha\beta$. A strong solution of $\mathrm{EP}_{\mathcal{G}}$ is a solution β satisfying $\beta(\mathcal{G}) \subseteq \mathrm{con}(\mathcal{B})$.

We say that G is (strongly) $\mathcal{G}$-projective if every finite $\mathcal{G}$-embedding problem for G has a (strong) solution.

A profinite group junk is by definition a totaly disconnected compact space X endowed with a set $\mathcal{X}$ of distinguished profinite groups Γ (which are topological subspaces of X) such that $X = \bigcup_{\Gamma} \Gamma$ ($\Gamma \in \mathcal{X}$). A morphism of profinite group junks $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ is a continuous mapping $f : X \longrightarrow Y$ having the property that for every $\Gamma \in \mathcal{X}$ there exists some $\Delta \in \mathcal{Y}$ such that $f(\Gamma) \subseteq \Delta$ and $f : \Gamma \longrightarrow \Delta$ is a group homomorphism.

For G a profinite group and $\mathcal{G}$ an étale quasi-compact set of closed subgroups, we denote $|\mathcal{G}| = \bigcup_{\Gamma} \Gamma$ ($\Gamma \in \mathcal{G}$) and remark that $|\mathcal{G}|$ endowed with $\mathcal{G}$ is a profinite group junk. For simplicity, we will denote this group junk by $\mathcal{G}$, because no confusion can appear. We say that G is the *(generalized) profinite free product on* $\mathcal{G}$ if for every finite group A and every morphism of profinite group junks $\varphi_{\mathcal{G}} : |\mathcal{G}| \longrightarrow A$ there exists exactly one group homomorphism $\varphi : G \longrightarrow A$ which extends $\varphi_{\mathcal{G}}$ to G.

We will say that G is $\mathcal{G}$-free if there exists an étale quasi-compact free set $\mathcal{H}$ of subgroups of G such that $|\mathcal{G}| \cap |\mathcal{H}| = \{1\}$ and G is the free product on $\mathcal{G} \cup \mathcal{H}$.

Let $(X, \mathcal{X})$ be a profinite group junk. Then there exists a unique profinite group $F_{X,\mathcal{X}}$ endowed with an étale quasi-compact set $\mathcal{F}$ of closed subgroups and a surjective morphism of profinite group junks $\theta : X \longrightarrow |\mathcal{F}|$ such that the following universality property is satisfied: Every profinite group junk homomorphism $\varphi : X \longrightarrow A$ into a finite group A factorizes through θ, ie there exists a morphism of profinite group junks $\psi_{\mathcal{F}} : |\mathcal{F}| \longrightarrow A$ such that $\varphi = \psi_{\mathcal{F}}\theta$ and secondly, there exists a unique prolongation of $\psi_{\mathcal{F}}$ to a morphism of profinite groups $\psi : F_{X,\mathcal{X}} \longrightarrow A$. In particular, $F_{X,\mathcal{X}}$ is the generalized profinite free product on $\mathcal{F}$.

To construct $\left(F_{X,\mathcal{X}}, \mathcal{F}, \theta\right)$ one can proceed as follows: Let $\mathfrak{F}$ be the discrete free group on $\mathcal{X}$ and $\jmath : X \longrightarrow \mathfrak{F}$ the structural mapping, thus θ maps every $\Gamma \in \mathcal{X}$ isomorphically onto the corresponding factor of $\mathfrak{F}$. Let $\mathcal{N}$ be the set of all normal subgroups N of finite index in $\mathfrak{F}$ for which the canonical mapping

$$\mathcal{X} \xrightarrow{\jmath} \mathfrak{F} \longrightarrow \mathfrak{F}/N$$

induces a homomorphism of profinite group junks $X \longrightarrow \mathfrak{F}/N$. An immediate verification shows that the set $\mathcal{N}$ is decreasingly filtered by inclusion. Hence one can consider the profinite completion of $\mathfrak{F}$ with respect to $\mathcal{N}$. We denote by $F_{X,\mathcal{X}}$ the profinite completion of $\mathfrak{F}$ with respect to $\mathcal{N}$ and let $\jmath_0 : \mathfrak{F} \longrightarrow F_{X,\mathcal{X}}$ be the structural morphism. We denote by $\mathcal{F}$ the image of $\jmath(\mathcal{X})$ by $\jmath_0$, hence that of $\mathcal{X}$ by $\jmath_0\jmath$. An easy verification shows that $\mathcal{F}$ is étale-quasi compact and that $\jmath_0\jmath$ induces a continuous surjective mapping $\theta : X \longrightarrow |\mathcal{F}|$, which turns to be a homomorphism of profinite group junks. Moreover, it is almost a tautology to verify that $F_{X,\mathcal{X}}$ endowed with the set of subgroups $\mathcal{F}$ and the morphism of profinite group junks $\theta : X \longrightarrow |\mathcal{F}|$ verifies the universality property we have given.

(2.2) Directly from the construction of the generalized profinite free product associated to a profinite group junk we get:

(1) The correspondence $(X, \mathcal{X}) \longrightarrow F_{X,\mathcal{X}}$ is covariant functorial.

(2) If $(X, \mathcal{X})$ is the disjoint union of two profinite group junks $(X_k, \mathcal{X}_k)$ $(k = 1, 2)$ then the canonical homomorphism $F_{X_1,\mathcal{X}_1} * F_{X_2,\mathcal{X}_2} \longrightarrow F_{X,\mathcal{X}}$ is an isomorphism.

(3) Let $(F_{X,\mathcal{X}}, \mathcal{F}, \theta)$ be the generalized profinite free product on some profinite group junk $(X, \mathcal{X})$ and let $F_{\mathcal{F}}$ denote the generalized profinite free product associated to the profinite group junk $\mathcal{F}$. Then the canonical homomorphism $F_{X,\mathcal{X}} \longrightarrow F_{\mathcal{F}}$ induced by $\theta : X \longrightarrow |\mathcal{F}|$ is an isomorphism.

We give two typical examples of generalized profinite free products.

1) The profinite free product of a profinite group on a boolean space

Let Γ be a profinite group and T a boolean space, ie a compact totally disconnected space. We define the profinite free product of Γ on the space T, notation $F_{\Gamma,T}$, in the following way: We consider the direct product $X = \Gamma \times T$ as a totally disconnected compact space and endow it with the set $\mathcal{X}$ of all isomorphic copies of Γ of the form $\Gamma_x = \Gamma \times \{x\}$ (all $x \in T$). The profinite free product of Γ on T is by definition the generalized profinite free product associated as above to the profinite group junk $(X, \mathcal{X})$. We will denote by $\mathcal{F}$ the image of $\mathcal{X}$ in $F_{\Gamma,T}$.

2) The profinite free envelope of a relatively projective group

It goes about the following: Let G be relatively projective with respect to an étale quasi-compact set $\mathcal{G}$ of subgroups. As mentioned above, $\mathcal{G}$ gives rise to a profinite group junk, which we will also denote by $\mathcal{G}$. We consider the generalized profinite free product associated to this profinite group junk and denote it by $F_{\mathcal{G}}$. As usually we denote by $\mathcal{F}$ the image of $\mathcal{G}$ in $F_{\mathcal{G}}$. Further, we consider the generalized profinite free product of $\hat{\mathbb{Z}}$ on the underlying topological space of the profinite group G and denote it by F_G. Finally we set $F_{G,\mathcal{G}} = F_G * F_{\mathcal{G}}$ and call it the free envelope of $(G, \mathcal{G})$. Using the separating theorem (1.7) from [P2] one verifies that the canonical mappings $\mathcal{G} \longrightarrow \mathcal{F}$ and $|\mathcal{G}| \longrightarrow |\mathcal{F}|$ are isomorphisms. Therefore there exists a canonical projection

$$F_{G,\mathcal{G}} \xrightarrow{\pi} G$$

which maps $\mathcal{F}$ and $|\mathcal{F}|$ isomorphically onto $\mathcal{G}$, respectively $|\mathcal{G}|$ and the generator 1_g of $\hat{\mathbb{Z}} \times \{g\}$ to g (all $g \in G$). In particular, if $\mathcal{G}_{\max}$ is a fundamental domain for $\mathcal{G}$ then $\mathcal{F}_{\max}$ is a fundamental domain for $\mathcal{F}$ and

$$(F_{G,\mathcal{G}}, \mathcal{F}_{\max}) \xrightarrow{\pi} (G, \mathcal{G}_{\max})$$

is a cover as defined in [P2], after theorem (1.7).

We now have the following generalization of Theorem 4 of Iwasawa [I].

(2.3) For a profinite group G endowed with an étale quasi-compact set $\mathcal{G}$ of closed subgroups we consider the following conditions:

i) Every $\mathcal{G}$-embedding problem for G has a proper, strong solution.

ii) $\mathcal{G}_{\max} = \mathcal{G}$ is a fundamental domain for $\mathcal{G}$.

(2.4) Theorem. *Let G_k $(k = 1, 2)$ be countably generated profinite groups which are endowed with étale quasi-compact sets $\mathcal{G}_k$ of closed subgroups such that the conditions* i), ii) *above are satisfied. Further suppose that there exists an isomorphism of profinite group junks $|\mathcal{G}_1| \cong |\mathcal{G}_2|$. Then there exists an isomorphism $G_1 \cong G_2$ which maps $\mathrm{con}(\mathcal{G}_1)$ isomorphically onto $\mathrm{con}(\mathcal{G}_2)$.*

Proof. The proof idea is the same as in Iwasawa's loc.cit., to which we refere. We explain the suplimentary difficulty one has here. In the context from **(2.3)** let us consider two profinite group junks homomorphism $\varphi, \varphi' : |\mathcal{G}| \longrightarrow C$ into some profinite group C. We say that φ, φ' are quasi-conjugated, notation $\varphi \sim \varphi'$, if for every $\Gamma \in \mathcal{G}$ there exists $c \in C$ such that $\varphi^c = \varphi'$ on Γ, ie $\varphi(g)^c = \varphi'(g)$ for all $g \in \Gamma$. Let now $\mathrm{EP} = (\gamma, \alpha)$ be some embedding problem for G and $\beta_g : |\mathcal{G}| \longrightarrow B$ a morphism of profinite group junks. We say that β_g is compatible with EP if $\alpha\beta_g = \gamma$ on $|\mathcal{G}|$. We say that β_g is quasi-compatible with EP if $\alpha\beta_g \sim \gamma$ as profinite group junks from $|\mathcal{G}|$ into A. If so, then β_g gives rise in a canonical way to a $\mathcal{G}$-embedding problem $\mathrm{EP}_g = (\gamma, \alpha, \mathcal{B})$ for G by setting $\mathcal{B} = \beta_g(\mathcal{G})$. We will call such a $\mathcal{G}$-embedding problem a quasi-cover for $(G, \mathcal{G})$ and denote it by $(\gamma, \alpha, \beta_g)$. The suplimentary difficulty we have spoken about, consists in showing that in the situation from **(2.3)** the following holds: Every finite quasi-cover $(\gamma, \alpha, \beta_g)$ for $(G, \mathcal{G})$ has proper solutions β such that $\beta \sim \beta_g$ when viewed as homomorphisms of profinite group junks from $|\mathcal{G}|$ into B. This assertion is the content of the lemma below. After proving the lemma the induction procedure from loc.cit. does work without any changes. Therefore we will omit the straightforward verifications.

Lemma. *Let G together with $\mathcal{G}$ satisfying the conditions* i), ii) *above. Then every finite quasi-cover $\mathrm{EP}_g = (\gamma, \alpha, \beta_g)$ for $(G, \mathcal{G})$ has proper solutions β such that the induced profinite group junk homomorphism $\beta : |\mathcal{G}| \longrightarrow B$ is quasi-conjugated to β_g.*

Proof. We first show that there exists another profinite group junk homomorphism $\beta'_g : |\mathcal{G}| \longrightarrow B$ which is compatible with EP and quasi-conjugated to β_g. Indeed, for every $\Gamma \in \mathcal{G}$ we take $a \in A$ such that $\alpha\beta_g = \gamma^a$ on Γ. Then Γ has an étale open quasi-compact neighbourhood $\mathcal{G}_\Gamma$ such that $\alpha\beta_g = \gamma^a$ on $|\mathcal{G}_\Gamma|$. Clearly, the family $(\mathcal{G}_\Gamma)_\Gamma$ is an open covering of $\mathcal{G}$, hence we can extract from this a finite subcovering, say $(\mathcal{G}_k)_k$. Moreover, replacing this subcovering by a finer one we can suppose that $\beta_g(\Gamma') = 1$ if Γ' is element of two different members of the covering. We now define the profinite group junk homomorphism β'_g we are looking for as follows: For every $\mathcal{G}_k$ there exist some $a_k \in A$ such that $\alpha\beta_g = \gamma^{a_k}$ on $|\mathcal{G}_k|$. We take a preimage b_k

of a_k in B and define β'_g to be the unique mapping from $|\mathcal{G}|$ into B such that ${\beta'_g}^{b_k} = \beta_g$ on $|\mathcal{G}_k|$ (all k). One easily verifies that β'_g exists and has the properties we asked for. Hence, without loss of generality we can suppose that $\alpha\beta_g = \gamma$ on $|\mathcal{G}|$.

We next remark that it is sufficient to prove the lemma in the case where α is injective on every $\beta_g(\Gamma)$. Indeed, there exists a finite quotient $A' = G/D$ of G such that $D \subseteq \ker(\gamma)$ and β_g factorizes through the canonical projection $\gamma' : G \longrightarrow A'$. Using the same construction as in the proof of Theorem A we find an embedding problem $\mathrm{EP}' = (\gamma', \alpha')$ for G which dominates EP and such that the deduced morphism of profinite group junks β'_g is compatible with EP' and moreover, α' is injective on the image of β'_g. Hence α' is injective on every $\beta'_g(\Gamma)$. Finally, if β' is a proper, strong solution of $(\gamma', \alpha', \beta'_g)$ such that β' is quasi-conjugated to β'_g when viewed as profinite group junks from $|\mathcal{G}|$ into B', then clearly, the induced solution β for EP will have the properties we asked for. Hence, without loss of generality, we can suppose that α is injective on every $\beta_g(\Gamma)$.

Hence it remains to prove the lemma in the case where $\alpha\beta_g = \gamma$ on $|\mathcal{G}|$ and α is injective on $\beta_g(\Gamma)$ (all $\Gamma \in \mathcal{G}$). To prove the lemma in this particular case we shall use the profinite free envelope $F_{G,\mathcal{G}} \xrightarrow{\pi} G$ of $(G, \mathcal{G})$. We first remark that we can suppose that B is a quotient $\phi : F \longrightarrow B$ of $F := F_{G,\mathcal{G}}$ and so, $\beta_g \pi = \phi$ on $|\mathcal{F}|$ and if $\Phi \in \mathcal{F}$ corresponds to $\Gamma \in \mathcal{G}$ then also $\phi(\Phi) = \beta_g(\Gamma)$.

We proceed as follows: For every open normal subgroup C of F we set $D = \pi(C)$ and consider the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\pi} & G \\ \downarrow{\scriptstyle \phi_C} & & \downarrow{\scriptstyle \gamma_C} \\ F/C & \xrightarrow{\alpha_C} & G/D \end{array}$$

where the morphisms are the canonical projections. Clearly, denoting by $\mathcal{G}_C$ the image of $\mathcal{G}$ in G/D and defining $\mathcal{F}_C$ correspondingly, it follows that $\alpha_C(\mathcal{F}_C) = \mathcal{G}_C$, hence $\alpha_C(\mathrm{con}(\mathcal{F}_C)) = \mathrm{con}(\mathcal{G}_C)$. Moreover, the family of all mappings

$$\alpha_C : \mathrm{con}(\mathcal{F}_C) \longrightarrow \mathrm{con}(\mathcal{G}_C)$$

is a projective system of finite sets which has as limit $\pi : \mathrm{con}(\mathcal{F}) \longrightarrow \mathrm{con}(\mathcal{G})$. On the other hand,

$$(F_{G,\mathcal{G}}, \mathcal{F}) \xrightarrow{\pi} (G, \mathcal{G})$$

is a cover as defined in [P2], after theorem 1.8, as we already have remarked above. Hence it follows by loc.cit. lemma (1.11) that for every $\Delta \in \mathrm{con}(\mathcal{F})$ the preimage of $\pi(\Delta)$ in $\mathrm{con}(\mathcal{F})$ is exactly $\Delta^{\ker(\pi)}$. Next, for every C and $\Delta \in \mathrm{con}(\mathcal{F})$ we let $\mathcal{F}_{\Delta/C}$ be the preimage of $\alpha_C(\Delta/C)$ in $\mathrm{con}(\mathcal{F}_C)$. Then the sets

$$\mathcal{X}_C = \{\mathcal{F}_{\Delta/C} \mid \Delta/C \in \mathrm{con}(\mathcal{F}_C)\}$$

form a projective system with respect to the canonical projections $F/C' \longrightarrow F/C$. Their projective limit is the set

$$\mathcal{X} = \{ \mathcal{F}_\Delta \mid \Delta \in \mathcal{F} \},$$

where $\mathcal{F}_\Delta$ is the preimage of $\pi(\Delta)$ in $\mathrm{con}(\mathcal{F})$. Hence, $\mathcal{X}$ consists of all subsets of $\mathcal{F}$ of the form $\Delta^{\ker(\pi)}$ ($\Delta \in \mathrm{con}(\mathcal{F})$), as we have already remarked above. By a standard limit argument, for every C there exists C' containing C such that the following conditions are satisfied: for all C'' which contain C', the image of $\mathcal{X}$ and that of $\mathcal{X}_{C''}$ in the set of subsets of $\mathcal{F}_C$ are equal. In particular, for such a C' the image of every $\mathcal{F}_{\Delta/C'}$ in $\mathcal{F}_C$ consists exactly of the orbit of Δ/C under the conjugation with elements from $\ker(\pi)/C$. We now set $C = \ker(\phi)$ and for such a C' let β' be a proper, strong solution of the quasi-cover $(\gamma_{C'}, \alpha_{C'}, \beta'_g)$ defined by $\beta'_g := \phi_{C'}\pi^{-1}$. Let β be the compositum of β' with the canonical projection $F/C' \longrightarrow F/C = B$. Then β is a proper, strong solution of $(\gamma, \alpha, \beta_g)$. We are going to show that β satisfies the other requirement of the lemma. Indeen, take $\Gamma \in \mathcal{G}$ arbitrary and let $\Phi \in \mathcal{F}$ be its preimage. Then

$$\alpha_{C'}(\Phi/C') = \Gamma/D' = \alpha_{C'}\beta'(\Gamma),$$

hence $\beta'(\Gamma)$ lies in $\mathcal{F}_{\Phi/C'}$. Thus $\beta(\Gamma)^b = \Phi/C = \beta_g(\Gamma)$ for some $b \in \ker(\pi)/C$. We claim that actually $\beta^b = \beta_g$ on Γ. Indeed, as b lies in $\ker(\pi)/C$, hence in $\ker(\alpha)$, it follows that $\alpha\beta^b = \alpha\beta$. Hence we have $\alpha\beta_g = \gamma = \alpha\beta^b$ on Γ. Thus $\beta^b = \beta_g$ on Γ, as α is injective on $\beta_g(\Gamma)$. The proof of the lemma is finished.

(2.5) Corollary. *Let G be a countably generated profinite group endowed with an étale quasi-compact set of subgroups $\mathcal{G}$ such that the conditions from* **(2.3)** *above are satisfied. Then G is isomorphic to the profinite free product*

$$G \cong F_g * F_\omega,$$

of the generalized profinite free product on $\mathcal{G}$ with an isomorphic copy of the profinite ω-free group F_ω. Moreover, denoting the image of $\mathcal{G}$ in F_g by $\mathcal{F}$, the above isomorphism maps $\mathrm{con}(\mathcal{G})$ isomorphically onto $\mathrm{con}(\mathcal{F})$.

In particular, denoting by G_g the normal closed subgroup of G which is generated by $\mathcal{G}$, one has:

1) *G/G_g is ω-free.*
2) *G_g is the profinite free product on some étale quasi-compact subset of $\mathrm{con}(\mathcal{G})$.*

Proof. It is clear that $F_g * F_\omega$ is a countably generated profinite group. Moreover, if $\mathcal{F}$ is the image of $\mathcal{G}$ in F_g, then $F_g * F_\omega$ is $\mathcal{F}$-projective and satisfies the conditions i), ii) from above. Furthermore, $\mathcal{F} = \mathcal{F}_{\max}$ is a fundamental domain for $\mathcal{F}$. As $\mathcal{G}$ and $\mathcal{F}$ are isomorphic as profinite group junks, we can apply theorem **(2.4)**. The remaining assertions 1) and 2) immediately follow from the structure of $F_g * F_\omega$. □

We now come to the proof of Theorem B. We first remark that every $\mathcal{G}_\kappa$-embedding problem $\mathrm{EP}_{\mathcal{G}_\kappa}$ for G_κ has proper solutions. Indeed, by [P2], theorem (3.3), it follows that $\mathrm{EP}_{\mathcal{G}_\kappa}$ has solutions, as G_κ is $\mathcal{G}_\kappa$-projective. Using the procedure from the proof of Theorem A it follows that $\mathrm{EP}_{\mathcal{G}_\kappa}$ is dominated by a split embedding problem EP for G_κ. By the main theorem there exist proper solutions for EP, and therefore also for $\mathrm{EP}_{\mathcal{G}_\kappa}$. Next, by [P2], Ch 2, §1, A), we can suppose without loss of generality that $\mathcal{L}_\kappa$ is closed in the subfield topology. Hence applying [P3], theorem (4) together with the proposition preceding it, it follows that G_κ is strongly $\mathcal{G}_\kappa$-projective. Moreover, if we in lemma (6) from loc.cit. consider only proper solutions β_C of the emebdding problems EP_C in discussion there (these proper solutions exist by the remark above), it follows that all β_C are actually proper, strong solutions. Combining this result with lemma (5) from loc.cit. it follows that every $\mathcal{G}_\kappa$-embedding problem has proper, strong solutions. Finally, as κ is countably generated it follows that its absolute Galois group G_κ is countably generated. In particular, $\mathcal{G}_\kappa$ has a fundamental domain, say $\mathcal{G}$. Hence the profinite group G endowed with $\mathcal{G}$ satisfies the conditions from **(2.3)**. To conclude we apply **(2.5)**.

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