Manuscript, variant of October, 2007; 1–14

Inertia elements versus Frobenius elements

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Introduction

Recall the generalized Chebotarev's Density Theorem, see SERRE [Se], which is one of the very fundamental facts in arithmetic geometry:

THEOREM (Generalized Dirichlet Density).

Let $f : Y \to X$ be a generically finite and Galois morphism of integral separated schemes of finite type over $\mathbb{Z}$. Let K and L denote the function fields of X, respectively Y, and let $G := \mathrm{Gal}(L|K)$ the group of (rational) automorphisms of Y over X. Then the following hold:

1) *There exists an open sub-scheme $U \subset X$ such that f is étale above U, and if $V = f^{-1}(U)$, then V is an open sub-scheme of Y, and $f : V \to V$ is an etale cover.*

2) *For every $\sigma \in G$, the set of all closed points $x \in U$ such that Frob_x is conjugated to σ has a Dirichlet density which equals $|\sigma^G|/|G|$.*

From this one gets a kind of "profinite variant" of the Chebotarev Density Theorem as follows: Let K be a finitely generated field (over its prime field). We consider normal model X of K, i.e., integral normal separated schemes of finite type over $\mathbb{Z}$ with function field equal to K. For for such models X consider sets $\Sigma \subseteq X$ of closed points which have Dirichlet density equal to 1, which we call `Frobenius sets`. We remark that the set of all Frobenius sets is inductive, i.e., for given Frobenius sets $\Sigma'_{X'} \subseteq X'$ and $\Sigma''_{X''} \subseteq X''$ there exists a Frobenius set $\Sigma \subseteq X$ such that $\Sigma_X \subseteq \Sigma'_{X'} \cap \Sigma''_{X''}$. Indeed, since X' and X'' are birationally equivalent, there exist affine open subsets $U' \subseteq X'$ and $U'' \subseteq X''$ which are isomorphic. Note that U' and U'' are normal models of K, and since their complements have dimensions strictly less than $\dim(X) =: d := \dim(X')$, it follows by SERRE [Se], that $\Sigma'_{U'} := \Sigma'_{X'} \cap U'$ and

* Ideas used in this work originate in part from my visit to the IAS Princeton during 2002. The present work I was partially supported by the NSF grants DMS 9729992 and DMS 0401056.

$\Sigma''_{U''} := \Sigma_{X''} \cap U''$ have Dirichlet density equal to 1. Hence $\Sigma'_{U'}$ and $\Sigma''_{U''}$ are Frobenius sets. Hence identifying U and U', say by setting $U' =: X := U''$, and denoting $\Sigma := \Sigma'_{U'} \cap \Sigma''_{U''} \subseteq X$, we get the desired result.

For every Galois extension $\tilde{K}|K$, and a given Frobenius set $\Sigma \subset X$, we define a `set of Frobenius liftings` $\mathfrak{Frob}_\Sigma$ in $\mathrm{Gal}(\tilde{K}|K)$ as follows: First let $\tilde{X} \to X$ be the normalization of X in the field extension $\tilde{K}|K$. For every point $x \in \Sigma$, let $(\mathcal{O}_x, \mathfrak{m}_x)$ be its local ring, and $\kappa_x = \mathcal{O}_x/\mathfrak{m}_x$ be the residue field at x, hence κ_x is a finite field. Let $\tilde{x}$ be a point of $\tilde{X}$ above x, and $T_{\tilde{x}|x} \subset Z_{\tilde{x}|x}$ the inertia, respectively decomposition, groups of $\tilde{x}|x$. Then one has a canonical exact sequence of profinite groups

$$1 \to T_{\tilde{x}|x} \to Z_{\tilde{x}|x} \to \mathrm{Gal}(\kappa_{\tilde{x}}|\kappa_x) \to 1.$$

We define a `Frobenius lifting` at x to be any fixed preimage $\sigma_x \in Z_{\tilde{x}|x}$ of the Frobenius element of $\mathrm{Gal}(\kappa_{\tilde{x}}|\kappa_x)$. In particular, if σ_x is a Frobenius lifting at x, then $\sigma_x T_{\tilde{x}|x}$ is the set of all the Frobenius liftings at x. Further, we define a `set of Frobenius liftings` of $\mathrm{Gal}(\tilde{K}|K)$ to be any subset $\mathfrak{Frob}_\Sigma \subset \mathrm{Gal}(\tilde{K}|K)$ which contains a Frobenius lifting σ_x for all closed points x of X. Then one has the following:

Theorem. *Let K be a finitely generated field, and $\tilde{K}|K$ an arbitrary Galois field extension. Then every set of Frobenius liftings $\mathfrak{Frob}_\Sigma \subset \mathrm{Gal}(\tilde{K}|K)$ is a dense subset in* $\mathrm{Gal}(\tilde{K}|K)$.

The aim of this note is twofold: First we show that the inertia and/or ramification elements of $\mathrm{Gal}(\tilde{K}|K)$ behave in a completely complementary way: They both build closed subsets in $\mathrm{Gal}(\tilde{K}|K)$. And second, the tame divisorial inertia elements are dense in the set of all the tame inertia elements. As an application, it turns out that this last assertion is the key technical point in a strategy for detecting the so called `decomposition graphs` of function fields $K|k$ with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite, or a global, field. Detecting the decomposition graphs is an essential technical step in tackling the Program (initiated by Bogomolov) for recovering function fields $K|k$ as above from their pro-ℓ *abelian-by-central* Galois theory, provided $\mathrm{td}(K|k) > 1$. See e.g. Pop [P] for more about this.

Thus let us give definitions and explain the matter in detail. Let K be an arbitrary field, and $\tilde{K}|K$ be some Galois field extension with Galois group $\mathrm{Gal}(\tilde{K}|K)$. Let v be a valuation of K. For $\tilde{v}$ a prolongation of v to $\tilde{K}$, we denote by $V_{\tilde{v}} \subseteq T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ are the ramification, respectively the inertia, respectively the decomposition, groups of $\tilde{v}$ in $\mathrm{Gal}(\tilde{K}|K)$. We denote by $\tilde{K}\tilde{v}$ and Kv are residue fields of $\tilde{v}$, respectively v. As in the case of local rings of (closed) points, the residue field extension $\tilde{K}\tilde{v}|Kv$ is a normal algebraic field extension (but in general not Galois). We set $G_{\tilde{v}} := \mathrm{Aut}(\tilde{K}\tilde{v}|Kv)$ and recall that one has an exact sequence of profinite groups of the form:

$$1 \to T_{\tilde{v}} \to Z_{\tilde{v}} \to G_{\tilde{v}} \to 1.$$

Further, $V_{\tilde{v}}$ is trivial if the residual characteristic $p := \mathrm{char}(Kv)$ is zero, respectively $V_{\tilde{v}}$ is the unique Sylow p-group of $T_{\tilde{v}}$ otherwise. An element $g \in T_{\tilde{v}}$ is called a v-`inertia element`, or an `inertia element` at v. And an element $g \in V_{\tilde{v}}$ is called a v-`ramification element`, or a `ramification element` at v. An inertia element element $g \in T_{\tilde{v}}$ is called a `tame inertia element`, if it satisfies the following equivalent conditions:

i) The order of g (as a super natural number) is prime to $p := \mathrm{char}(Kv)$.

ii) The closed subgroup generated by σ has trivial intersection with $V_{\tilde{v}}$.

To fix some notations, we denote by $\mathfrak{Ram}\big(\tilde{K}\big)$, and $\mathfrak{Inr}\big(\tilde{K}\big)$, and $\mathfrak{Inr.tm}\big(\tilde{K}\big)$, the sets of all the ramification, respectively inertia, respectively tame inertia, elements of $\mathrm{Gal}(\tilde{K}|K)$. The first fact we announce is the following:

Theorem A. *Let $\tilde{K}|K$ be an arbitrary Galois extension of fields. Then the following hold:*

1) *The sets $\mathfrak{Ram}\big(\tilde{K}\big)$, $\mathfrak{Inr}\big(\tilde{K}\big)$, and $\mathfrak{Inr.tm}\big(\tilde{K}\big)$, are closed in* $\mathrm{Gal}(\tilde{K}|K)$.

2) *More precisely, the following hold: Let $\Delta \subset \mathrm{Gal}(\tilde{K}|K)$ be a closed subgroup such that for every finite Galois sub-extension $K_i|K$ of $\tilde{K}|K$, there exists a valuation v_i on K_i such that: $\Delta|_{K_i} \subseteq V_{v_i}$, respectively $\Delta|_{K_i} \subseteq T_{v_i}$, respectively $\Delta|_{K_i} \subseteq T_{v_i} \backslash V_{v_i}$. Then there exists a valuation $\tilde{v}$ of $\tilde{K}$ such that $\Delta \subseteq V_{\tilde{v}}$, respectively $\Delta \subseteq T_{\tilde{v}}$, respectively $\Delta \subseteq T_{\tilde{v}} \backslash V_{\tilde{v}}$.*

The result above is a kind of "general non-sense" type result, and is proved as follows: Let $\mathrm{Val}\big(\tilde{K}\big)$ be the space of all the valuations of $\tilde{K}$ endowed with the `patch topology`, and $\mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big)$ the space of all the closed subgroups of $\mathrm{Gal}(\tilde{K}|K)$ endowed with the `étale topology`, see Section 2) for the definitions. Then the maps sending each $\tilde{v} \in \mathrm{Val}\big(\tilde{K}\big)$ to either $T_{\tilde{v}}$ or $V_{\tilde{v}}$ is continuous, and it turns out that the above Theorem is a reinterpretation of this fact. One should remark that the corresponding assertion is false, if one instead sends each $\tilde{v}$ to its decomposition group $Z_{\tilde{v}}$. This is in some sense the reason why the Chebotarev Density Theorem is possible in first place!

The next result is much more subtle, and does not follow by "general non-sense" type arguments: Let K be either a *finitely generated,* or a *function field* $K|k$ over some base field k. Recall that in the case K is finitely generated, a normal model for K is any separated integral normal scheme of finite type over $\mathbb{Z}$, whose function field equals K; and in the case $K|k$ is a function field, a normal model of K is any normal variety X over k with function field K. In both cases one defines a `(Zariski) prime divisor` of K to be any discrete valuation $\mathfrak{v}$ of K whose valuation ring is the local ring of (the generic point of) a Weil prime divisor $X_1 \subset X$ of some normal model X of K. Note that $K\mathfrak{v}$ is the function field of X_1 viewed as a scheme, and the following hold: If K is a finitely generated field, then $K\mathfrak{v}$ is a finitely generated field, whereas if $K|k$ is a function field, then $K\mathfrak{v}|k$ is a function field over k as well. And in particular, X_1 is a model for $K\mathfrak{v}$, and the normalization of X_1 in $K\mathfrak{v}$ is a normal model for $K\mathfrak{v}$. Coming back to inertia elements, in the above context we make the following definition: Let $\tilde{K}|K$ be an arbitrary Galois extension. We say that $g \in \mathrm{Gal}(\tilde{K}|K)$ is a `divisorial inertia element`, if g is an inertia element at some prime divisor $\mathfrak{v}$ of K as defined above. Finally, we denote by $\mathfrak{Inr.tm.div}\big(\tilde{K}\big)$ the set of all the divisorial tame inertia elements in $\mathrm{Gal}(\tilde{K}|K)$ in the case K is finitely generated; and in the case $K|k$ is a function field, we denote by $\mathfrak{Inr}\big(\tilde{K}|k\big)$, and $\mathfrak{Inr.tm}\big(\tilde{K}|k\big)$, the set of all the inertia, respectively tame inertia, elements at all the valuations of $\tilde{K}$ which are trivial on k; and by $\mathfrak{Inr.tm.div}\big(\tilde{K}|k\big)$ the set of of divisorial inertia elements in $\mathrm{Gal}(\tilde{K}|K)$. The second result we announce is the following:

Theorem B. *Let K be either a finitely generated field or a function field over some base field k. Then in the above notations the following hold:*

1) *If K is finitely generated, then the set of all the divisorial inertia elements $\mathfrak{Inr.tm.div}\big(\tilde{K}\big)$ is dense in $\mathfrak{Inr.tm}\big(\tilde{K}\big)$.*

2) *If $K|k$ is a function field, then $\mathfrak{Inr}\big(\tilde{K}|k\big)$, and $\mathfrak{Inr.tm}\big(\tilde{K}|k\big)$ are closed subsets of $\mathrm{Gal}(\tilde{K}|K)$, and $\mathfrak{Inr.tm.div}\big(\tilde{K}|k\big)$ is dense in $\mathfrak{Inr.tm}\big(\tilde{K}|k\big)$.*

1. Proof of Theorem A

First let us recall the basics concerning the `patch topology`. Let Ω be an arbitrary field, and let $\mathrm{Val}\big(\Omega\big)$ be the set of all the valuations rings, thus equivalence classes of valuations, or of places, of Ω. One defines the `Zariski topology` τ^{Zar} on $\mathrm{Val}\big(\Omega\big)$ as being the topology which has as a basis the sets of the form:

$$U_A := \{v \in \mathrm{Val}\big(\Omega\big) \mid v(a) \geq 0,\ a \in A\}, \quad \forall \text{ finite } A \subset \Omega.$$

It is easy to check that the trivial valuation (whose valuation ring is Ω itself) lies in all U_A, thus τ^{Zar} is not a Hausdorff topology. Further, τ^{Zar} is quasi-compact. The constructible, thus Hausdorff, topology generated by τ^{Zar} is called the `patch topology` on $\mathrm{Val}\big(\Omega\big)$, which we denote by τ^{pa}. A basis of this topology consists of all the sets of the form:

$$U_{A,B} := \{v \in \mathrm{Val}\big(\Omega\big) \mid v(a) \geq 0, v(b) > 0,\ a \in A,\, b \in B\}, \quad \forall \text{ finite } A, B \subset \Omega.$$

One of the basic facts concerning $\tau^{\rm pa}$ is that this topology is Hausdorff and compact, and that the basic open subsets $U_{A,B}$ are actually open and closed. Thus $\mathrm{Val}\big(\Omega\big)$ endowed with the patch topology is a profinite topological space.

The Zariski topology and the patch topology behave nicely under field extensions as follows: Let $\Omega'|\Omega$ be a field extension. Then the canonical restriction map

$$\mathrm{res} : \mathrm{Val}\big(\Omega'\big) \to \mathrm{Val}\big(\Omega\big), \quad v \mapsto v|_\Omega$$

is surjective (by the Chavalley's theorem on the prolongation of places), and continuous in both the Zariski topology and the patch topology. Moreover, if $(\Omega_i)_{i\in J}$ is an inductive family of fields, and $\Omega = \lim_i \Omega_i$, then $\mathrm{Val}\big(\Omega_i\big)$, $i \in I$, endowed with the (surjective) restrictions $\mathrm{res}_{ji} : \mathrm{Val}\big(\Omega_j\big) \to \mathrm{Val}\big(\Omega_i\big)$, $i \leq j$, is a projective system, and $\mathrm{Val}\big(\Omega\big)$ is in a canonical way the projective limit of this projective system.

Second, let G be a profinite group. Then the set of all the closed subgroups $\mathrm{Sbg}\big(G\big)$ of G carries in a canonical way the so called `étale topology` $\tau^{\rm et}$, which in some sense is similar to the Zariski topology on $\mathrm{Val}\big(\Omega\big)$. A basis of open subsets of $\tau^{\rm et}$ is given by:

$$U^{\rm et}_{G_1} := \{\Gamma \in \mathrm{Sbg}\big(G\big) \mid \Gamma \subseteq G_1\}, \quad \forall \text{ open } G_1 \subseteq G.$$

Clearly, $\tau^{\rm et}$ is quasi-compact and non-Hausdorff. The constructible topology on $\mathrm{Sbg}\big(G\big)$ is called the strict topology $\tau^{\rm st}$, and a basis of open subsets of this topology is given by

$$U^{\rm st}_{G_1,N} := \{\Gamma \in \mathrm{Sbg}\big(G\big) \mid \Gamma N = G_1\}, \quad \forall\; G_1, N \subseteq G \text{ open}, N \text{ normal}.$$

As above, it follows that $\tau^{\rm st}$ is Hausdorff and compact, and that $U^{\rm st}_{G_1,N}$ open and closed subsets of $\mathrm{Sbg}\big(G\big)$.

A special case of the above situation is when we consider a Galois extension of fields $\tilde K|K$, for which we fix notations as follows: Let $(K_i|K)_i$ is the family of all the finite Galois sub-extensions of $\tilde K|K$ inductively ordered by inclusion. For $K_i \subseteq K_j$, i.e., $i \leq j$, we denote:

1) $\mathrm{pr}_i : \mathrm{Gal}(\tilde K|K) \to \mathrm{Gal}(K_i|K)$ and $\mathrm{pr}_{ji} : \mathrm{Gal}(K_j|K) \to \mathrm{Gal}(K_i|K)$ the canonical surjective projections.

2) $\mathrm{res}_i : \mathrm{Val}\big(\tilde K\big) \to \mathrm{Val}\big(K_i\big)$ and $\mathrm{res}_{ji} : \mathrm{Val}\big(K_j\big) \to \mathrm{Val}\big(K_i\big)$ the canonical surjective restriction maps.

For $\tilde v \in \mathrm{Val}\big(\tilde K\big)$, let $V_{\tilde v} \subseteq T_{\tilde v} \subseteq Z_{\tilde v}$ be the ramification, respectively the inertia, respectively the decomposition, groups of $\tilde v$. Further, let $v_i := \tilde v|_{K_i}$ be the restriction of $\tilde v$ to K_i, and further let $V_{v_i} \subseteq T_{v_i} \subseteq Z_{v_i}$ be correspondingly defined. Then by Hilbert decomposition theory (for valuations), it follows that pr_i maps $V_{\tilde v} \subseteq T_{\tilde v} \subseteq Z_{\tilde v}$ onto $V_{v_i} \subseteq T_{v_i} \subseteq Z_{v_i}$. We conclude that there exist canonical maps

$$\psi^V, \psi^T, \psi^Z : \mathrm{Val}\big(\tilde{K}\big) \to \mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big)$$

defined by $\psi^V(\tilde{v}) := V_{\tilde{v}}$, $\psi^T(\tilde{v}) := T_{\tilde{v}}$, $\psi^Z(\tilde{v}) := Z_{\tilde{v}}$, and correspondingly for $K_i|K$, such that each pair of such maps fit into a commutative diagram

$$\begin{array}{ccc} \mathrm{Val}\big(\tilde{K}\big) & \xrightarrow{\psi^\bullet} & \mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big) \\ \big\downarrow \mathrm{res}_i & & \big\downarrow \mathrm{pr}_i \\ \mathrm{Val}\big(K_i\big) & \xrightarrow{\psi_i^\bullet} & \mathrm{Sbg}\big(\mathrm{Gal}(K_i|K)\big) \end{array}$$

where $\bullet$ is any of the letters V, T, Z respectively. After this preparation we can announce the following:

Theorem 1.1. *Let $\tilde{K}|K$ be a Galois field extension. Then in the above notations, the following hold:*

1) *The maps $\psi^V, \psi^T : \mathrm{Val}\big(\tilde{K}\big) \to \mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big)$ defined by $\psi^V(\tilde{v}) := V_{\tilde{v}}$ and $\psi^T(\tilde{v}) := T_{\tilde{v}}$ are continuous if we endow $\mathrm{Val}\big(\tilde{K}\big)$ with the patch topology τ^{pa} and $\mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big)$ with the étale topology τ^{et}.*

2) *Let $\mathcal{V} \subseteq \mathrm{Val}\big(\tilde{K}\big)$ be a τ^{pa}-closed subset. Then the sets $\mathfrak{Ram}_{\mathcal{V}}\big(\tilde{K}\big)$, $\mathfrak{Inr}_{\mathcal{V}}\big(\tilde{K}\big)$, and $\mathfrak{Inr.tm}_{\mathcal{V}}\big(\tilde{K}\big)$ of all the ramification, respectively inertia, respectively tame ramification, elements at valuations $v \in \mathcal{V}$ are closed in* $\mathrm{Gal}(\tilde{K}|K)$.

3) *More precisely, in the situation above, let $\Delta \subseteq \mathrm{Gal}(\tilde{K}|K)$ be a closed subgroup such that for every $K_i|K$, there exists a valuation $v_i \in \mathrm{res}_i(\mathcal{V})$ such that either* i) *$pr_i(\Delta) \subseteq V_{v_i}$, or* ii) *$pr_i(\Delta) \subseteq T_{v_i}$, or* iii) *$pr_i(\Delta) \subseteq T_{v_i} \backslash V_{v_i}$. Then there exists a valuation $\tilde{v} \in \mathcal{V}$ such that either* i) *$\Delta \subseteq V_{\tilde{v}}$, or* ii) *$\Delta \subseteq T_{\tilde{v}}$, or* iii) *$\Delta \subseteq T_{\tilde{v}} \backslash V_{\tilde{v}}$.*

Proof. To 1): By the discussion before the Theorem, without loss of generality we can suppose that $\tilde{K}|K$ is finite. If so, then $\mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big)$ consists of all the subgroups of $\mathrm{Gal}(\tilde{K}|K)$. Further, one checks immediately that the sets of the form $B_\Delta := \{\Gamma \in \mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big) \mid \Delta \subseteq \Gamma\}$, all $\Delta \subseteq \mathrm{Gal}(\tilde{K}|K)$, represent a basis for the τ^{et}-closed subsets in $\mathrm{Sbg}\big(\mathrm{Gal}(\tilde{K}|K)\big)$. (Indeed: First, the complement of B_Δ is the union of all the basic open subsets U_{G_1} with $\Delta \not\subseteq G_1$, hence an τ^{et} open set. Second, the basic closed set which is the complement of U_{G_1} is exactly the union of all the subsets B_Δ with Δ all the subgroups $\Delta \not\subseteq G_1$.) Our strategy to prove that ψ^T and ψ^V are continuous, is to show that the preimages of τ^{et}-closed subsets of the form B_Δ are τ^{pa}-closed.

First let us show that ψ^T is continuous. For $g \in \mathrm{Gal}(\tilde{K}|K)$ and $x \in \tilde{K}$ let us consider

$$\mathcal{U}_{g,x} := \{\tilde{v} \in \mathrm{Val}\big(\tilde{K}\big) \mid \tilde{v}(x) \geq 0,\ \tilde{v}(gx - x) \leq 0\}.$$

Then $\mathcal{U}_{g,x}$ is an $\tau^{\rm pa}$-open set, and if $\tilde v \in \mathcal{U}_{g,x}$, then $g \notin T_{\tilde v}$, by the definition of $T_{\tilde v}$. Therefore, $\mathcal{U}_g := \cup_x \mathcal{U}_{g,x}$ is $\tau^{\rm pa}$-open too, and its complement $\mathcal{V}_g$ in $\mathrm{Val}(\tilde K)$ is therefore closed. We now claim the following:

Claim. $\tilde v \in \mathcal{V}_g$ if and only if $g \in T_{\tilde v}$.

Indeed, if $\tilde v \in \mathcal{V}_g$, then for all $x \in K$ such that $\tilde v(x) \geq 0$, we must have $v(gx - x) > 0$. Equivalently, $g \in T_{\tilde v}$. The converse implication is obvious.

Now let $\Delta \subseteq \mathrm{Gal}(\tilde K|K)$ be a (closed) subgroup. Then $\mathcal{V}_\Delta := \cap_{g\in\Delta}\mathcal{V}_g$ is $\tau^{\rm pa}$-closed too, and by the Claim above one has: $v \in \mathcal{V}_\Delta$ if and only if $g \in T_{\tilde v}$ for all $g \in \Delta$; hence if and only if $\Delta \subset T_{\tilde v}$. Equivalently, $\mathcal{V}_\Delta$ is the preimage of B_Δ under ψ^T. We conclude that Ψ^T is continuous.

The continuity of ψ^V is proved in a similar way, but starting with $\tau^{\rm pa}$-open sets of the form

$$\mathcal{U}_{g,x} := \{\tilde v \in \mathrm{Val}(\tilde K) \mid \tilde v(x) \geq 0,\ \tilde v(gx - x) \leq \tilde v(x)\}.$$

and the resulting claim that if $\mathcal{V}_g$ is the complement of $U_{g,x}$, then $\tilde v \in \mathcal{V}_g$ if and only if $g \in V_{\tilde v}$, etc.

To 2) and 3): It is clear that the closeness of the sets $\mathfrak{Ram}_{\mathcal{V}}(\tilde K)$, $\mathfrak{Inr}_{\mathcal{V}}(\tilde K)$, and $\mathfrak{Inr.tm}_{\mathcal{V}}(\tilde K)$ immediately follows from assertion 3). Thus we are left with proving assertion 3). We make the proof only in the case i), as the case ii) is *mutatis mutandis* identical. Thus suppose that for every $K_i|K$ there exists $v_i \in \mathrm{res}_i(\mathcal{V})$ such that $\mathrm{pr}_i(\Delta) \subseteq T_{v_i}$. Hence in the notations from the proof of assertion 1), and taking into account the continuity of

$$\mathrm{pr}_i \circ \psi^T = \mathrm{res}_i \circ \psi_i^T : \mathcal{V} \to \mathrm{Sbg}\big(\mathrm{Gal}(K_i|K)\big)$$

it follows that the preimage $\mathcal{V}_i \subseteq \mathcal{V}$ of $\mathcal{V}_{\mathrm{pr}_i(\Delta)}$ under the continuous map above is closed and non-empty, by hypothesis. Thus $(\mathcal{V}_i)_i$ is a family of compact subsets of $\mathcal{V}$ which has the finite intersection property (as $\mathcal{V}_j \subseteq \mathcal{V}_i$ for $K_i \subseteq K_j$). Now take $\tilde v \in \cap_i \mathcal{V}_i$, and set $v_i := \tilde v|_{K_i}$. Then by general Hilbert decomposition theory (for valuations) we have $T_{v_i} = \mathrm{pr}_i(T_{\tilde v})$, hence from $\mathrm{pr}_i(\Delta) \subseteq T_{v_i}$ we get $\mathrm{pr}_i(\Delta) \subseteq \mathrm{pr}_i(T_{\tilde v})$. This being true for all K_i, we finally have $\Delta \subseteq T_{\tilde v}$, as claimed. □

2. Proof of Theorem B

First we remark that the assertion concerning the closeness of $\mathfrak{Inr}(K|k)$, and $\mathfrak{Inr.tm}(K|k)$, can be deduced from Theorem 1.1 above as follows: Let $\mathcal{V}$ be the set of all the valuations of $\tilde K$ which are trivial on k. Then $\tilde v \in \mathcal{V}$ if and only if $\forall\, x \in k^\times$ one has $v(x) = 0$. Hence $\mathcal{V} \subset \mathrm{Val}(\tilde K)$ is the intersection of the closed and open basic subsets $U_{\{x\}}$, $x \in k^\times$, thus $\tau^{\rm pa}$-closed, etc.

Now let us prove that $\mathfrak{Inr.tm.div}(K)$ is dense in $\mathfrak{Inr.tm}(K)$, respectively that $\mathfrak{Inr.tm.div}(K|k)$ is dense in $\mathfrak{Inr.tm}(K|k)$. The proofs are *mutatis mutandis* the same, therefore we will make the proofs at the same time.

We first remark that it sufficient to consider the case where $\tilde{K} = K^{\rm s}$ is the separable closure of K. Indeed, this follows immediately from the fact that for Galois field extensions $K \hookrightarrow L \hookrightarrow M$, and every valuation $v_M \in \mathrm{Val}(M)$ and its restriction v_L to L one has the following: Let $\mathrm{pr} : \mathrm{Gal}(M|K) \to \mathrm{Gal}(M|K)$ be the canonical surjective projection. Then pr maps T_{v_M} onto T_{v_M}, and V_{v_M} onto V_{v_M}, etc.

Therefore, without loss of generality, we will suppose that $\tilde{K} = K^{\rm s}$, hence $\mathrm{Gal}(\tilde{K}|K) = G_K$ is the absolute Galois group of K, and we will denote the valuation $\tilde{v}$ by v.

We introduce notations which will be used throughout the proof as follows:

• $\sigma \in G_K$ is a fixed tame inertia element, and $\Sigma \subset G_K$ the pro-cyclic closed subgroup of G_K generated by σ, and $|\Sigma| = |\sigma|$ denotes the order of Σ and of σ as a super natural numbers.

• $L|K$ is the fixed field of σ, hence of Σ, in $K^{\rm s}|K$. And for every finite Galois sub-extension $K_i|K$ of $K^{\rm s}|K$ we will denote by $L_i := L \cap K_i$ the fixed field of σ in K_i.

• $G_i := \mathrm{Gal}(K_i|L_i)$ is the cyclic group generated by $\sigma|_{K_i}$ in $\mathrm{Gal}(K_i|K)$. In particular, $\Sigma = G_L$ projects onto $G_i = \mathrm{Gal}(K_i|L_i)$, and so G_i is a finite quotient $\mathrm{pr}_i : \Sigma \to G_i$ of Σ.

Remarks 2.1. In the notations from above, we make the following more or less obvious remarks:

1) L and K_i are linearly disjoint over L_i, hence $[LK_i : L] =: n_i := [K_i : L_i]$, and the canonical projection below is an isomorphism:

$$\mathrm{Gal}(LK_i|L) \to \mathrm{Gal}(K_i|L_i) = \Sigma_i.$$

2) The assertion of Theorem B for σ is actually the following:

(∗) *For every finite Galois extension $K_i|K$, there exists some prime divisor $\mathfrak{v}_i$ of K_i such that $\mathrm{pr}_i(\sigma)$ is a tame inertia at $\mathfrak{v}_i$.*

3) Let $M|K$ be some finite field extension. Then $M^{\rm s} := MK^{\rm s}$ is a separable closure of M, and G_K contains G_M as an open subgroup in a canonical way. In particular, if the compositum ML inside $M^{\rm s}$ is purely inseparable over L, then the tame inertia element σ fixes ML point-wise, hence lies in G_M. Finally, let $pr : G_M \to \mathrm{Gal}(N|M) =: G$ the canonical projection.

In the above notations, by the functoriality of Hilbert decomposition for valuations, we immediately deduce that the assertion (∗) above for a given $K_i|K$ follows from the following:

(†) *There exists some finite field extension $M|K$, and a finite cyclic extension $N|M$ satisfying the following:*
  i) *$ML\,|\,L$ is purely inseparable, and $K_i \subset NL$.*
  ii) *$pr(\sigma)$ is a tame inertia element at some prime divisor $\mathfrak{v}$ of N.*

4) Thus in order to prove assertion $(*)$ for some finite Galois sub-extension $K_i|K$ of $K^{\rm s}|K$, we will show that assertion (†) above is satisfied for a properly chosen finite cyclic extension $N|M$.

5) Suppose that σ is a non-trivial tame inertia element at the valuation v as at the previous point e) above. Note that v is therefore non-trivial, and v is totally tamely ramified in $K^{\rm s}|L$. Hence v is also totally tamely ramified in the finite cyclic extensions $LK_i|L$ and $K_i|L_i$ too.

6) Since v is totally tamely ramified in $K^{\rm s}|L$, it follows that L contains the roots of unity $\mu_{|\Sigma|}$ of order $|\Sigma|$.

Step 1. *Getting started*

Let $K_i|K$ be a finite Galois sub-extension of $K^{\rm s}|K$, and let X be a proper model of K. By one of the main results of de Jong's alteration theory [dJ], Theorem 5.13, it follows that there exists an alteration $f : Y \to X$ of X such that the function field $N = \kappa(Y)$ of Y is a normal Galois extension of K, and the following are satisfied:

a) The group $\mathrm{Aut}(N|K)$ is contained in $\mathrm{Aut}(Y)$, and $\mathrm{Aut}(N|K)$ projects onto $\mathrm{Gal}(K_i|K)$ via the alteration $f : Y \to X$.

b) Y is regular.

Let $K'|K$ be the pure inseparable part of $N|K$. Then $K'^{\rm s} := K'K^{\rm s}$ is a separable closure of K', and one has a canonical identification $G_K = G_{K'}$, under which $L' := LK'$ is the fixed field of σ in $K'^{\rm s}$. Further, v has a unique prolongation w to $K'^{\rm s}$, and the groups $V_v \subseteq T_v$ are identified with $V_w \subseteq T_w$. Hence σ is a tame inertia element at w.

Let $M := N \cap L'$ be the fixed field of σ in N. Then $N|M$ is a cyclic extension, and $ML = K'L$ is purely inseparable over L, and $K_i \subset N$. Thus $N|M$ satisfies the condition i) from assertion (†) of Remark 2.1 above. Therefore, in order to prove assertion $(*)$ for $K_i|K$, it is sufficient to prove that $N|M$ satisfies condition ii) of loc.cit.

Recall that w is totally tamely ramified in the finite cyclic extension $N|M$. Hence for all $g \in \Sigma$ one has: $g\mathcal{O}_w = \mathcal{O}_w$ and $g\,c - c \in \mathfrak{m}_w$ for all $c \in \mathcal{O}_w$. Since Y is proper, it follows by the valuation criterion of properness, that there exists a unique local ring $(\mathcal{O}, \mathfrak{m})$ of Y such that $(\mathcal{O}_w, \mathfrak{m}_w)$ dominates $(\mathcal{O}, \mathfrak{m})$. From the uniqueness of $(\mathcal{O}, \mathfrak{m})$, and by the discussion above, we get the following:

Fact 2.2. *N contains regular local rings $(\mathcal{O},\mathfrak{m})$ such that $(\mathcal{O},\mathfrak{m})$ is dominated by $(\mathcal{O}_w,\mathfrak{m}_w)$ and $N=\mathrm{Quot}(\mathcal{O})$. And for every such ring $(\mathcal{O},\mathfrak{m})$ the following hold:*

1) *G acts faithfully on $\mathcal{O}$, in particular, every $g\in G$ maps $\mathfrak{m}$ isomorphically onto itself.*

2) *For all $g\in G$ and all $c\in\mathcal{O}$ one has $g\,c-c\in\mathfrak{m}$. This means that the action of G on $\mathcal{O}$ is totally ramified.*

3) *The residue field $\kappa:=\mathcal{O}/\mathfrak{m}$ is canonically embedable into the residue field $Nw:=\mathcal{O}_w/\mathfrak{m}_w$ of w on N. In particular, $\mathrm{char}(\kappa)\neq\ell$, and G acts trivially on κ via the canonical projection $\mathcal{O}\to\kappa$.*

We will conclude the proof of Theorem B by using the fact that for a properly chosen `cyclotomic alteration` followed by a sequence of `local modifications` of the local rings $(\mathcal{O},\mathfrak{m})$ from the Fact 2.3 above, one can reach a situation where *mutatis mutandis* the action of G on a properly chosen regular system of local parameters $(t_1,\dots,t_d)$ of $\mathcal{O}$, has very simple shape, namely:

- $g(t_k)=t_k$ for $k<d$.
- $g(t_k)=\zeta t_k$ for some primitive root of unity $\zeta\in\mu_{|G|}$.

If so, then the prime divisor $\mathfrak{v}$ of N defined by the t_d-adic valuation is totally tamely ramified in $N|M$, hence satisfies assertion (†).

Step 2. *Maximizing the decomposition groups*

Actually, the only alteration of the local rings as introduced in Fact 2.3 above, which is not a sequence of blowups, is the following `cyclotomic alteration`:

Lemma 2.3. *In the context and notations from Fact 2.3 above, denote $m:=|G|$, and consider the cyclotomic extension $N_1:=N[\mu_m]$. Then letting $\mathcal{O}\hookrightarrow\mathcal{O}^{\mathrm{n}}$ be the normalization of $\mathcal{O}$ in the finite field extension $N_1|N$, the following hold:*

1) *The field $M_1:=L'\cap N_1$ is actually $M_1=M[\mu_m]$. Hence the canonical restriction homomorphism $\mathrm{Gal}(N_1|M_1)\to\mathrm{Gal}(N|M)$ is an isomorphism, and $N_1|M_1$ satisfies condition* i) *form assertion* (†) *of Remark 2.1.*

2) *Let $(\mathcal{O}_1,\mathfrak{m}_1)$ be the unique localization of $\mathcal{O}^{\mathrm{n}}$ dominated by $(\mathcal{O}_w,\mathfrak{m}_w)$. Then $(\mathcal{O}_1,\mathfrak{m}_1)$ is a regular local ring, and $N_1|M_1$ endowed with $(\mathcal{O}_1,\mathfrak{m}_1)$ satisfy conditions* 1), 2), 3) *of Fact 2.3.*

Proof. The first assertion 1), follows from the fact that $\mu_m\subset\mu_{|\sigma|}$ are contained in L, by Remark 1.1, 7); hence we have $\mu_m\subset L'\cap N_1=:M_1$.

To 2): Since $\mathrm{char}(\kappa_x)\neq\ell$, it follows that the ring extension $\mathcal{O}_y\hookrightarrow\mathcal{O}^{\mathrm{n}}$ is an étale ring extension. Hence $\mathcal{O}^{\mathrm{n}}$ is a semi-local regular ring, as being an étale Galois cover of the local regular ring $\mathcal{O}_y$. Finally, the valuation ring of v dominates one of the localizations of $\mathcal{O}^{\mathrm{n}}$, etc. □

LEMMA 2.4. *In the context of Fact 2.3 above, set $m := |G|$, and suppose that $\mu_m \subset N$, hence $\mu_m \subset M = N \cap L$. Then for a properly chosen regular system of parameters $(t_1, \dots, t_d)$ of $\mathcal{O}$, the action of G on $(t_1, \dots, t_d)$ is given by a system of characters $\chi_k : G \to \mu_m$, $1 \le k \le d$, of G as follows:*

$$g t_k = \chi_k(g) t_k, \quad g \in G,\ 1 \le k \le d,$$

Proof. First, consider $V := \mathfrak{m}/\mathfrak{m}^2$ as κ-vector space. Since G acts on $\mathcal{O}$ and maps $\mathfrak{m}$ isomorphically onto itself, it follows that G acts on V too. On the other hand, since by condition 3) of Fact 2.3, $\mathrm{char}(\kappa)$ does not divide $|G|$, the action of G on V is semi-simple. Recall that G is cyclic of order $|G| = m$, and $\mu_m \subset \mathcal{O}$, hence $\mu_m \subset \kappa$. Therefore, if g_0 is a generator of G, then the minimal polynomial $P_{g_0}(X) = X^m - 1$ of g splits in linear factors over κ. This finally implies that the action of g_0 on V is diagonalizable, i.e., there exist characters

$$\chi_k : G \to \mu_m, \quad 1 \le k \le d$$

and a κ-basis $(\overline{u}_1, \dots, \overline{u}_d)$ of $V = \mathfrak{m}/\mathfrak{m}^2$ such that denoting by I_χ the diagonal matrix whose diagonal entries are the characters $\chi_1, \dots, \chi_d$, one has:

$$g\,(\overline{u}_1, \dots, \overline{u}_d) = (\overline{u}_1, \dots, \overline{u}_d) \cdot I_\chi(g), \quad g \in G.$$

Now let $\underline{u} := (u_1, \dots, u_d)$ be a preimage of $(\overline{u}_1, \dots, \overline{u}_d)$ in $\mathcal{O}$. Then $\underline{u}$ is a regular system of local parameters of $\mathcal{O}$, and by the discussion above we have: For every $g \in G$ there exists some $\underline{u}'_g = (u'_{g1}, \dots, u'_{gk})$ with $u'_{gk} \in \mathfrak{m}^2$ for all k such that:

$$g\,\underline{u} = \underline{u} \cdot I_\chi(g) + \underline{u}'_g, \quad g \in G.$$

We proceed by considering the I_χ-twisted G-action on the d-fold product $(N,+)^d := (N,+) \times \ldots \times (N,+)$ of the additive group of N, which is defined by $\widetilde{g}\,\underline{a} := g\,\underline{a}\,I_\chi(g^{-1})$, where $\underline{a} = (a_1, \dots, a_n) \in (N,+)^d$ is arbitrary, and $g(a_1, \dots, a_n) := (ga_1, \dots, ga_d)$ is the diagonal action of G on $(N,+)^d$. Then using the identity above, and setting $\underline{a}_g := \underline{u}' I_\chi(g^{-1})$ for all $g \in G$, we get:

$$\underline{a}_g = \widetilde{g}\,\underline{u} - \underline{u}, \quad g \in G.$$

In other words, $\{\underline{a}_g\}_g$ is a 1-cocycle of the twisted action of G with values in the additive subgroup $\mathfrak{m}^2 \times \ldots \times \mathfrak{m}^2$ of $(N,+)^d$. Since $m := |G|$ is invertible in $\mathcal{O}$, it follows that the cocycle $\underline{a}_g$ is trivial. Actually, setting $\underline{a} := -\frac{1}{|G|}\sum_g \underline{a}_g$, we get have $\underline{a} \in \mathfrak{m}^2 \times \ldots \times \mathfrak{m}^2$ and $\underline{a}_g = \widetilde{g}\,\underline{a} - \underline{a}$. Therefore, setting $\underline{t} := \underline{u} - \underline{a}$, we have $\widetilde{g}\,\underline{t} = \underline{t}$. Equivalently,

$$g\,\underline{t} = \underline{t} \cdot I_\chi(g), \quad g \in G,$$

and note that $\underline{t} = (t_1, \ldots, t_d)$ is a regular system of local parameters of $\mathcal{O}$, as $\underline{u} = (u_1, \ldots, u_d)$ was so, and $\underline{a} = (a_1, \ldots, a_d) \in \mathfrak{m}^2 \times \ldots \times \mathfrak{m}^2$. □

Step 3. *Maximazing an inertia group*

Let $(t_1, \ldots, t_d)$ be a system of regular local parameters of $\mathcal{O}$ as in the Lemma 2.6. We define a `local modification` of $\mathcal{O}$ to be a local regular ring in N obtained in two steps as follows:

1) First consider a blowup $\mathcal{Z} \to \operatorname{Spec}\mathcal{O}$ at any prime ideal of the form $\mathfrak{p}_{kl} = (t_k, t_l)$. Note that the zero set $V(\mathfrak{p}_{kl})$ is regular in $\operatorname{Spec}\mathcal{O}$, hence $\mathcal{Z}$ is regular, and the preimage of the closed point of $\mathcal{O}$ is the t_{kl}-projective line over κ, say with $t_{kl} := t_l/t_k$.

2) Second, replace $(\mathcal{O}, \mathfrak{m})$ by the local ring $(\mathcal{O}_{kl}, \mathfrak{m}_{kl})$ of $\mathcal{Z}$ defined by the zero $(t_{kl} = 0)$ of t_{kl}.

Remarks 2.5. Let $(\mathcal{O}', \mathfrak{m}') := (\mathcal{O}_{kl}, \mathfrak{m}_{kl})$ be a local modification of $(\mathcal{O}, \mathfrak{m})$ as above. Then the following hold:

1) The regular local ring $(\mathcal{O}_{kl}, \mathfrak{m}_{kl})$ is the localization of $\mathcal{O}[t_l/t_k]$ at its maximal ideal generated by $\mathfrak{m}$ and t_l/t_k, thus having $(t'_1, \ldots, t'_d)$ with $t'_j = t_j$ for $j \neq l$, and $t'_l := t_l/t_k$, as a local system of regular parameters.

2) The regular local ring $(\mathcal{O}', \mathfrak{m}')$ dominates $(\mathcal{O}, \mathfrak{m})$, and the resulting embedding of residue fields $\kappa \hookrightarrow \kappa'$ is a an isomorphism.

3) G acts on $(\mathcal{O}', \mathfrak{m}')$, as G maps both $\mathcal{O}[t_l/t_k]$, and its maximal ideal generated by $\mathfrak{m}$ and t_l/t_k, isomorphically onto themselves.

4) In particular, G acts trivially on κ', and therefore, G equals the inertia group of the action of G on $\mathcal{O}'$.

5) And the action of G on each t'_j is given by a character χ'_j of G such that $\chi'_j = \chi_j$ for $j \neq l$, and $\chi'_l = \chi_l \chi_k^{-1}$.

Lemma 2.6. *Let $N|M$ endowed with $(\mathcal{O}, \mathfrak{m})$ satisfy the conclusion of Lemma 2.4, There exists a character $\chi : G \to \mu_m$, and a finite sequence of local blowups as defined above such that the resulting regular local ring $(\mathcal{O}', \mathfrak{m}')$ has a regular system of local parameters $(t'_1, \ldots, t'_d)$ on which the action of every $g \in G$ is given by:*

$$g(t'_k) = t'_k \text{ for } k < d, \quad \text{and} \quad g(t'_d) = \chi(g)t'_d.$$

Proof. Let $\chi_1, \ldots, \chi_d$ be the characters of G defining the action of G on the given system of regular parameters $(t_1, \ldots, t_d)$ of $(\mathcal{O}, \mathfrak{m})$. We fix some primitive character $\chi_0 : G \to \mu_m$. Then for every $k = 1, \ldots, d$ there exists positive integers $e_k \leq m$ such that $\chi_k = \chi_0^{e_k}$. We make induction on the number $n > 0$ of non-trivial characters $\chi_k \neq 1$.

If $n = 1$, then after renumbering the parameters $(t_1, \dots, u_d)$, without loss of generality we can suppose that $\chi := \chi_d$ is the unique non-trivial one, etc.

Now suppose that there exist at least two non-trivial characters χ_k, χ_l. As above, after renumbering the parameters, we can suppose that $\chi_1 = \chi^{e_1}$ and $\chi_2 = \chi^{e_2}$ are non-trivial.

Claim. Let $e :=$ g.c.d.(e_1, e_2). There exists a sequence of local blowups of $\mathcal{O}$ such that the resulting $(\mathcal{O}', \mathfrak{m}')$ has a system of parameters $(t'_1, \dots, t'_d)$ satisfying the following condition:

i) $t'_k = t_k$, hence G acts by χ_k on t_k for $k > 2$.

ii) $g(t'_1) = t'_1$ for all $g \in G$.

ii) $g(t'_2) = \chi_0^e(g)\, t'_2$ for all $g \in G$.

Indeed, we make induction on $\tilde{e} := \max(e_1, e_2)$. Without loss of generality, after renumbering the parameters t_k, we can suppose that $e' = e_2 \geq e_1$.

Performing the local modification $(\mathcal{O}_{12}, \mathfrak{m}_{12})$, it follows by Remark 2.7, especially 5), that the resulting system of regular local parameters $(t'_1, \dots, t'_d)$ satisfies: $t'_k = t_k$ for $k \neq 2$, and $t'_2 = t_2/t_1$. Hence the action of G on t'_k is given by: $\chi'_k = \chi_k = \chi_0^{e_k}$, for $j \neq 2$; and $\chi'_2 = \chi_2\chi_1^{-1} = \chi_0^{e_2 - e_1}$. Hence we can choose the exponents e'_k such that $\chi'_k = \chi_0^{e'_k}$ and they satisfy:

i) $e'_k = e_k$, for $k \neq 2$.

ii) $e'_2 = e_2 - e_1 < e_2$, and $e'_2 \geq 0$ by the hypothesis $e_2 \geq e_1$.

Hence we have both: First $\tilde{e}' := \max(e'_1, e'_2) > \max(e_1, e_2) = \tilde{e}$, and second, g.c.d.$(e_1, e_2) =$ g.c.d.(e'_1, e'_2). Hence we can conclude the proof of the Claim by induction, and this finally concludes the proof of Lemma 2.8 too. □

Step 4. *Concluding the proof of Theorem B*

Let $(\mathcal{O}, \mathfrak{m})$ satisfying the conclusion of Lemma 2.8, and let $(t_1, \dots, t_d)$ be a system of local regular parameters of $\mathcal{O}$ such that G acts on t_k trivially for $k < d$, and acts on t_d via a character $\chi : G \to \mu_m$.

We first remark that since G acts faithfully on $N = \mathrm{Quot}(\mathcal{O})$, and its action on N factors through χ, it follows that the character χ must be a primitive character of G.

Second, we claim that G is contained in the inertia group of the principal ideal $\mathfrak{p} := t_d\mathcal{O}$. Indeed, the residue ring $\mathfrak{O} := \mathcal{O}/\mathfrak{p}$ is a regular local ring with maximal ideal $\mathfrak{n} = \mathfrak{m}/\mathfrak{p}$, having $(\bar{t}_1, \dots, \bar{t}_{d-1})$ as a regular system of local parameters, where $\bar{t}_k := t_k \pmod{\mathfrak{p}}$ for $k < d$, and residue field $\mathfrak{O}/\mathfrak{n} = \mathcal{O}/\mathfrak{m} = \kappa$. Since G acts trivially on t_k for $k < d$, it follows that G acts trivially on the local parameters $\bar{t}_k$ for $k = 1, \dots, d-1$. Since G acts trivially on the residue field $\kappa = \mathfrak{O}/\mathfrak{n}$ too, it follows that G acts trivially on $\mathfrak{O}$. Hence $\mathfrak{p}$ is totally ramified in $N|M$.

Therefore, the prime divisor $\mathfrak{v}$ defined by the t_d-adic valuation of N is totally tamely ramified in $N|M$.

This concludes the proof of Theorem B.

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