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ANABELIAN GEOMETRY

— A short survey —

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Lecture 1: $\widehat{\mathbf{GT}}$ versus I/OM

I) Motivation/Introduction

Non-tautological description of

$$\mathrm{Gal}_{\mathbb{Q}} = \mathrm{Aut}\big(\overline{\mathbb{Q}}\big)$$

The toy example: $\mathrm{Gal}_{\mathbb{R}}$ via π_1^{top}

- $\mathfrak{Var}_{\mathbb{R}}$ category of $\mathbb{R}$-varieties X.

- $\mathrm{Gal}_{\mathbb{R}} \cong \{\pm 1\}$ acts on $X^{\mathrm{an}} := X(\mathbb{C})$ and $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$

• Get a representation:

$$\rho_X : \mathrm{Gal}_{\mathbb{R}} \to \mathrm{Out}\big(\pi_1^{\mathrm{top}}(X)\big)$$

- Consider $\pi_1^{\mathrm{top}} : \mathfrak{Var}_{\mathbb{R}} \to \mathfrak{Groups}$ (outer hom's)...

• $(\rho_X)_X$ gives rise to repr $\imath_{\mathbb{R}} : \mathrm{Gal}_{\mathbb{R}} \to \mathrm{Aut}(\pi_1^{\mathrm{top}})$

FACT: *$\imath_{\mathbb{R}}$ is an isomorphism.*

Comment: Got non-tautological description of $\mathrm{Gal}_{\mathbb{R}}$...

- **What about other subfields of $\mathbb{C}$?**
  - $\Lambda \subseteq \mathbb{C}$ base field, e.g. $\mathbb{Q}$, $\mathbb{Q}(x_1, \ldots, x_n)$, $\mathbb{R}$, $\mathbb{Q}_p$
  - $\mathfrak{Var}_\Lambda$ geometrically integral Λ-varieties X.

Comment/Note: In general, Gal doesn't act on π_1^{top}

$*$ **How to do it:** Replace π_1^{top} by

- $\pi_1^{\mathrm{alg}} :=$ (profinite compl. of π_1^{top}) is a functor

  $\pi_1^{\mathrm{alg}} : \mathfrak{Var}_\Lambda \to \mathfrak{prof.Groups}$ (outer hom's)

  - There exists canonical exact sequence:

$$1 \to \pi_1^{\mathrm{alg}}(X, *) \to \pi_1^{\mathrm{et}}(X, *) \to \mathrm{Gal}_\Lambda \to 1$$

  - Represent $\rho_X : \mathrm{Gal}_\Lambda \to \mathrm{Out}\big(\pi_1^{\mathrm{alg}}(X)\big)$

- Finally $(\rho_X)_X$ gives rise to a morphism:

$$\imath_\Lambda : \mathrm{Gal}_\Lambda \to \mathrm{Aut}(\pi_1^{\mathrm{alg}})$$

II) Studying $\mathrm{Gal}_{\mathbb{Q}}$ via $\imath_{\mathbb{Q}}$

Question/Problem:

1) Find categories $\mathcal{V}$ for which $\mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{alg}})$ has "nice" topological/combinatorial description.

2) Find such categories $\mathcal{V}$ for which $\imath_{\mathcal{V}} : \mathrm{Gal}_K \to \mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{alg}})$ is an isomorphism.

- *This would give a new description of* Gal_{Λ}**!!!**

Example: Teichmüller modular tower

- $\mathcal{M}_{g,n}$ moduli space of curves (genus g, with n marked pts.)
- "Connecting" morphisms ("boundary" embeddings, gluings, etc.)
- $\mathcal{T} = \{\mathcal{M}_{g,n} \mid g, n\}$ category of varieties over $\Lambda = \mathbb{Q}$, the *Teichmüller modular tower.*
- Note: $\mathcal{M}_{0,4} \cong \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} = \mathbb{P}_{01\infty}$.

$$\mathcal{M}_{0,n} \cong (\mathcal{M}_{0,4})^{n-3} \setminus \{\text{fat diagonal}\}$$

FACTS (to Question/Problem 1):

- Harbater-Schneps: *Let* $\mathcal{V}_0 := \{\mathcal{M}_{0,4}, \mathcal{M}_{0,5}\}$*.*
  Then $\mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{V}_0}) = \widehat{GT}$ *is the famous Grothendieck–Teichmüller group.*

- [MANY]: *The variants* ${}^{I}GT$, ${}^{II}GT$, ${}^{IV}GT$, ${}^{\mathrm{new}}GT$ *are all of the form* ${}^{\mathcal{V}}\widehat{GT} := \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{V}})$...

- **Conclusion**: Question/Problem 1 has quite satisfactory answer(s) for $\mathcal{V} \subseteq \mathcal{T}$.

FACTS (to Question/Problem 2):

Injectivity: Λ *number field. Then*
$\imath_{\mathcal{V}} \colon \mathrm{Gal}_K \to \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{V}})$ *injective, provided:*

- Drinfel'd (using Belyi Thm): $\mathbb{P}_{01\infty} \in \mathcal{V}$
- Voevodsky: $C \in \mathcal{V}$, $C \subset E$ hyperbolic curve...
- Matsumoto: $C \in \mathcal{V}$, C affine hyperbolic curve
- Hoshi-Mochizuki: $C \in \mathcal{V}$, C hyperbolic curve

Surjectivity:

- **Ihara/Oda–Matsumoto Conj** (I/OM).
  $\imath_{\mathbb{Q}} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathfrak{Var}_{\mathbb{Q}}})$ *is isomorphism.*

FACTS:

- I/OM *has positive answer.*

- Actually much stronger assertions hold, e.g...

- Given $\Lambda \subset \mathbb{C}$ and $X \in \mathfrak{Var}_{\Lambda}$, $\dim(X) > 1$, set:

  - $\mathcal{V}_X = \{U_i \subset X\}_i \cup \{\mathbb{P}_{01\infty}\}$, with morphisms:
    inclusions $U_j \hookrightarrow U_i$, projections $U_i \to \mathbb{P}_{01\infty}$.

  - (P): $\imath_{\mathcal{V}_X} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{V}_X})$ *is isomorphism.*

Note: With $X = \mathbb{P}^2_{\mathbb{Q}}$, this gives (in principle) a pure topol./combin. construction of $\mathrm{Gal}_{\mathbb{Q}}$**!**

III) Variants

Replacing π_1^{alg} by variants

(compatible with the Galois action of Gal_Λ)

A) **The pro-ℓ variant**

$*$ Replace π_1^{alg} by its *pro-ℓ quotient*, hence:

- $\pi_1^{\mathrm{alg}}(X)$ by its pro-ℓ quotient $\pi_1^{\mathrm{alg},p}(X)$
- $\pi_{\mathcal{V}}$ by the corresponding $\pi_{\mathcal{V}}^{\ell}$
- $\mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{alg}})$ by $\mathrm{Aut}(\pi_{\mathcal{V}}^{\ell})$...

Questions:

1) Describe the image $\imath_{\mathcal{V}}^{\ell}(\mathrm{Gal}_\Lambda)$...
2) Consider the pro-ℓ I/OM...

- Lot of intensive research here concerning 1), e.g., by Matsumoto, Hain–M, Nakamura, etc.

Comment: This generalizes Serre's question/result about/on...

- The pro-ℓ I/OM is true in stronger form...

B) **The pro-ℓ abelian-by-central I/OM**

• Bogomolov's Program

Comment:

- The "Yoga" of Grothendieck's anabelian geometry...

<u>Bogomolov</u> (1990). Consider:

- $K|k$ function field, td.deg$(K|k) > 1$, $k = \overline{k}$.
- $\Pi^{c}_{K} \to \Pi_{K}$ *abelian-by-central/abelian* pro-ℓ quotients of Gal_K, $\ell \neq$ char.

Conjecture (Bogomolov's Program, 1990):
$K|k$ can be recovered from Π^{c}_{K} functorially.

Comments:

- tr.deg$(K|k) > 1$ is necessary, because...
- This goes far beyond Grothendieck's anabelian idea...

FACT (State of the Art):
Bogomolov's Progr OK for tr.deg $> \dim(k) + 1$.

Comments:

- $\dim(k) = 0$: tr.deg $> \dim(k) + 1$ is equiv to...
- The case tr.deg $> \dim(k) + 1$ follows by...

Back to: **Pro-ℓ abelian-by-central I/OM:**

$*$ Replace the profinite groups by the their
pro-ℓ abelian-by-central quotient, hence:

- $\pi_1^{\rm alg}(X)$ by its quotient $\Pi^{\rm c}(X) \to \Pi(X)$
- $\pi_{\mathcal{V}}$ by the corresponding $\Pi^{\rm c}_{\mathcal{V}} \to \Pi_{\mathcal{V}}$
- $\mathrm{Aut}(\pi_{\mathcal{V}}^{\rm alg})$ by $\mathrm{Aut}^{\rm c}(\Pi_{\mathcal{V}}) := \mathrm{im}\big(\mathrm{Aut}(\Pi^{\rm c}_{\mathcal{V}}) \to \mathrm{Aut}(\Pi_{\mathcal{V}})\big)$...

$\bullet$ Get the *pro-ℓ abelian-by-central* I/OM...

FACT (P): $\imath^{\rm c}_{\mathcal{V}_X} : \mathrm{Gal}_\Lambda \to \mathrm{Aut}^{\rm c}(\Pi_{\mathcal{V}_X})$ *is isom.*

Comments:

- Actually one proves the "birational variant" which is stronger.
- This is a special case of Bogomolov's Program.
- Bogomolov's Program would imply stronger assertions, like:

$*$ Let $\dim(X) > \dim(\Lambda) + 1$ be geom rigid...

- **Note:** If $\Lambda \subset \overline{\mathbb{Q}}$, then $\dim(X) > 2$ is enough.
- Consider $\mathcal{U}_X = \{U_i\}_i$ subcategory of $\mathcal{V}_X$.

FACTS (P): $\imath^{\rm c}_{\mathcal{U}_X} : \mathrm{Gal}_\Lambda \to \mathrm{Aut}^{\rm c}(\Pi_{\mathcal{U}_X})$ *and*
$\imath_{\mathcal{U}_X} : \mathrm{Gal}_\Lambda \to \mathrm{Aut}(\pi_{\mathcal{U}_X})$ *are isomorphisms.*

C) **The tempered $\widehat{\mathbf{GT}}$ and I/OM (André)**

$*$ Replace $\mathbb{C}$ by $\mathbb{C}_p$ and π_1^{alg} by the the corresp.
tempered (*alg.*) *fundam. group* π_1^{temp} hence:

- $\pi_{\mathcal{V}}$ by the corresponding $\pi_{\mathcal{V}}^{\mathrm{temp}}$
- $\mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{alg}})$ by $\mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{temp}})$...

- Get the tempered variant $\widehat{GT}^{\mathrm{temp}}$ of $\widehat{GT}$.
- Get the *tempered* I/OM for $\Lambda|\mathbb{Q}_p$ finite.

FACTS (André):

1) $\imath_\Lambda^{\mathrm{temp}} : \mathrm{Gal}_\Lambda \to \mathrm{Aut}(\pi^{\mathrm{temp}}_{\mathfrak{Var}_\Lambda})$ *is isom.*

2) $\imath_{\mathbb{Q}}(\mathrm{Gal}_{\mathbb{Q}}) \cap \widehat{GT}^{\mathrm{temp}} = \imath_{\mathbb{Q}_p}^{\mathrm{temp}}(\mathrm{Gal}_{\mathbb{Q}_p})$.

Comments:

- Actually $\pi_1^{\mathrm{temp}}(X) \subset \pi_1^{\mathrm{alg}}(X)$ for all X.
- There are "tempered variants" of other
aspects involving fundamental groups too.

IV) Final Question/Comments

- Arithmetical I/OM (question by David Burns)
- Pro-linear/pro-unipotent I/OM (question by Minhyong Kim)
- (Generalized) Drinfel'd upper half plane

Supplement 1: **On $\widehat{\mathbf{GT}}$**

- Exact sequence:

$$1 \to \widehat{F}_2 \to \pi_1(\mathbb{P}_{01\infty}) \to \mathrm{Gal}_{\mathbb{Q}} \to 1,$$

where $\mathbb{P}_{01\infty} := \mathbb{P}^1_{\mathbb{Q}} \backslash \{0, 1, \infty\}$ tripod...
and $\widehat{F}_2 = \langle \tau_0, \tau_1, \tau_\infty \,|\, \tau_0\tau_1\tau_\infty = 1 \rangle\widehat{\ } = \widehat{F}_{\tau_0, \tau_1} \ldots$

- $\widehat{GT} = \{(\lambda, f) \mid \lambda \in \widehat{\mathbb{Z}}^\times; \; f \in [\widehat{F}_2, \widehat{F}_2]$, rel. I, II, III$\}$
  the famous *Grothendieck–Teichmüller group.*
- $\exists$ can embedd $\mathrm{Gal}_{\mathbb{Q}} \hookrightarrow \mathrm{Aut}(\widehat{F}_2)$.

- Intensively studied by Drinfel'd, Ihara, Deligne, Schneps, Sch.–Lochak, Sch.–Nakamura, Sch.–Harbater, I–Matsumoto, Furusho, etc...
- Several variants ${}^{I}GT$, ${}^{II}GT$, ${}^{IV}GT$, etc. of $\widehat{GT}$.
- Actually: rel. I, II, III, are not independent (Schneps, Sch.–Lochak; Furusho: III suffices)
- Boggi–Lochak (to be thoroughly checked):
  Some variant ${}^{\mathrm{new}}GT$ *equals* $\mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{T}})$.

Supplement 2): **Belyi's Theorem**

- Recall RET: *There exists equiv of categories*

Topology&Geometry:	**Compl. algebraic curves:**
compact Riemann surfaces $\mathcal{X}$	projective smooth complex curves X

Function fields:

function fields $\mathcal{F}$ in one variable over $\mathbb{C}$

$$\mathcal{X} \quad \longleftrightarrow \quad \mathfrak{M}(\mathcal{C}) = \mathcal{F} = \mathbb{C}(X) \quad \longleftrightarrow \quad X$$

- **Basic Question** (Grothendieck):

  Which $\mathcal{X}$, hence which X, hence which $\mathcal{F}$, are defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$, hence number fields?

Theorem (Grothendieck/Belyi).

X is defined over $\overline{\mathbb{Q}} \Leftrightarrow \exists\ X \to \mathbb{P}_{01\infty}$ étale.

Proof: "$\Rightarrow$" by Belyi (nice and tricky!)

"$\Leftarrow$" by Grothendieck (étale fundam. groups)

Comments:

- This is the origin of Grothendieck's "Designs d'enfants".
- A cover $X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ as in Theorem is a *Belyi map*.
- Study the action of $\mathrm{Gal}_{\mathbb{Q}}$ on the space of "Designs"
  (many many people: Malle, Klüners–M., Schneps,
  Lochak–Sch., Zapponi, math-physicists, etc. etc. etc...)

Interesting open Question/Problem:

Higher dim extensions of Theorem above.

Two possible ways:

a) Describe all $X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ with at most n branch points.

b) Replace curves by higher dim varieties, e.g., surfaces (?!?).

- Several partial results to b), but...

Theorem (Ronkine 2004; unpublished).

The birat. class of a complex proj. surface X_0 of general type is defined over $\overline{\mathbb{Q}}$ iff $\exists$ smooth fibration $X_0 \twoheadrightarrow \mathbb{P}^1_{\mathbb{C}} \backslash \{0, 1, \infty\}$.

Supplement 3) **Bogomolov's Program**

Given: $\Pi^{\mathrm{c}}_K \to \Pi_K$. Reconstruct $K|k$ functorially.

<u>*Strategy of proof*</u> (P):

<u>Main Idea</u>: Consider $\mathcal{P}(K,+) := K^\times/k^\times$

the "projectivization" of the k-v.s. $(K,+)$.

- $(K,+,\cdot)$ can be recovered from

  $\mathcal{P}(K,+)$ endowed with its collineations,

  via Artin's *Fundam. Thm. Proj. Geometries*

NOW:

- Kummer Theory: $\widehat{K^\times} = \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell)$.

- And $\mathcal{P}(K,+) = K^\times/k^\times \hookrightarrow \widehat{K^\times}$.

<u>Hence to do list</u>: Given $\Pi^{\mathrm{c}}_K \twoheadrightarrow \Pi_K$,

1) Recover $K^\times/k^\times \hookrightarrow \widehat{K^\times}$.
2) Recover the collineations inside $K^\times/k^\times$.
3) Check compatibility with Galois Theory.

PLAN:

- Local Theory, i.e., recover:
    - primes of $K|k$; divisorial sets D_X of primes.
- Global Theory, i.e., recover:
    - Div(X), then $K^\times/k^\times$, then collineations; and finally check Galois compatibility.

THE GENERAL "NONSENSE"

Case $k = \overline{\mathbb{F}}_p$**:**

- $\mathrm{Pic}^0(X)$ is torsion group
- Valuations of k are trivial

Case $\mathrm{tr.deg}(K|k) > \dim k$**:**

- Specialization (Deuring, Roquette, Mumford)
- 1-motives techniques
- Reduce to the case $k = \overline{\mathbb{F}}_p$
- Recover the "nature" of k

Local Theory (few words):

- primes of $K|k$: DVR R_v with $k \subset R_v \subset K$

  such that $\mathrm{tr.deg}(Kv\,|\,k) = \mathrm{tr.deg}(K|k) - 1$.

- $D = \{v_i\}_i$ geometric, if $\exists$ normal model $X \to k$

  such that $D = D_X := \{v \mid \text{Weil prime div. of } X\}$.

- Quasi prime divisors

• Recovering the (quasi) primes:

- 1st Method: Use B.-Tsch. "commuting pairs"...
- 2nd Method: Use techniques developed by

  Ware, Koenigsmann, Míac et al, Topaz...

Comment: This is very very technical stuff...

• Recovering (quasi) decomposition graphs

• Recovering rational quotients

• $\cdots$

LECTURE 2: **Section conjectures**

I) Motivation/Introduction

Effective version of the Mordell Cojecture...

(Grothendieck: *Letter to Faltings,* June 1983)

• **Evidence**: Minhyong Kim's work...

Setting:

- k arbitrary base field; k^{i}, $k^{\mathrm{sep}} \subset \overline{k}$
- X_0 geom. integral k-variety, $d = \dim(X_0)$
- $X \subset X_0$ open k-subvariety

• Canonical exact sequence:

$$1 \to \pi_1^{\mathrm{alg}}(X, *) \to \pi_1^{\mathrm{et}}(X, *) \to \mathrm{Gal}_k \to 1$$

• For $x \in X_0$ regular $\exists\, v_x$ on $k(X)$ with:

$v_x(K^\times) = \mathbb{Z}^d$ and $\kappa(v_x) = \kappa(x)$.

- $\overline{v}_x | v_x$ prolong to $\overline{k(X)}$, get split exact seq

$$1 \to T_{\overline{v}_x} \to Z_{\overline{v}_x} \to \mathrm{Gal}_{\kappa(x)} \to 1.$$

Comments (about the splitting; tangential...)

Conclude: $\kappa(x) \subset k^{\rm ins} \Rightarrow \mathrm{Gal}_{\kappa(x)} = \mathrm{Gal}_k$ and

$pr_{k(X)} : \mathrm{Gal}_{k(X)} \to \mathrm{Gal}_k$ *has sections* $s_{\overline{v}_x}$.

- $\tilde{X} \to X$ pro-étale universal cover
- $\tilde{X}_0 \to X_0$ normaliz of X_0 in $k(X) \hookrightarrow k(\tilde{X})$
- $\mathrm{Gal}_{k(X)} \to \pi_1^{\rm et}(X)$ can projection
- $\tilde{x} \mapsto x$ centers of $\overline{v}_x$ on $\tilde{X}_0 \to X_0$
- Functoriality: $Z_{\overline{v}_x} \to Z_{\tilde{x}}$ and $T_{\overline{v}_x} \to T_{\tilde{x}}$.

Conclude: $\kappa(x) \subset k^{\rm ins} \Rightarrow \mathrm{Gal}_{\kappa(x)} = \mathrm{Gal}_k$ and

$pr_X : \pi_1(X) \to \mathrm{Gal}_k$ *has sections* $s_{\tilde{x}}$.

Cases:

a) $x \in X$: Then $T_{\tilde{x}} = \{1\}$ and $Z_{\tilde{x}} = \mathrm{Gal}_{\kappa(x)}$

- $x \in X$ with $\kappa(x) \subset k^{\rm ins}$ defines *conj class* of $s_{\tilde{x}}$.

b) $x \in X_0 \backslash X$: Then in general one has

$$T_{\tilde{x}} \neq \{1\} \text{ and } Z_{\tilde{x}} \neq \mathrm{Gal}_{\kappa(x)}.$$

- $x \in X_0 \backslash X$ with $\kappa(x) \subset k^{\rm ins}$ defines a *"bouquet" of sections* $\approx \mathrm{H}^1_{\rm cont}(\mathrm{Gal}_k, T_{\tilde{x}})$.

Special case: $X \subseteq X_0$ are smooth curves. Then:

- $x \in X_0 \backslash X$ iff $T_{\tilde{x}} \neq 1$.
- $\mathrm{char}(k) = 0 \Rightarrow T_{\tilde{x}} \cong \widehat{\mathbb{Z}}(1)$ as G_k-modules.
  Get $\mathrm{H}^1_{\mathrm{cont}}(\mathrm{Gal}_k, T_{\tilde{x}}) \cong \widehat{k^\times}$...

Curve SC (Section Conjecture / Grothendieck)

k fin gen infinite, $X \to k$ hyperbolic non-isotrivial. Then all sections of pr_X are of the form $s_{\tilde{x}}$.

Birational SC: *k, X as above.*

Then sections of $pr_{k(X)}$ are of the form $s_{\overline{v}_x}$.

- Variants...

- p**-adic Curve/Birat SC**: Replace k from Curve/Birat SC by a *finite extension* $k|\mathbb{Q}_p$.
- **Geom pro-$\mathfrak{C}$ Curve/Birat SC**: Replace π_1^{alg} from Curve/Birat SC by its *pro-$\mathfrak{C}$ completion...*
- Etc. e.g., [p**-adic**] **(pro-$\mathfrak{C}$) Curve/Birat SC**

- **Note:** ...Curve SC's $\Rightarrow$ ...Birat SC's [Comments]

II) Evidence/Facts SC:

A) **"No sections" results**...

- Stix, using the following local result:

• *$k|\mathbb{Q}_p$ finite, X hyperbolic. Then existence of sections of $pr_X \Rightarrow I(X)$ is a p-power.*

- Harari–Szamuely
  (using the *Manin-Brauer obstruction*)

- Hain: $k = \mathbb{Q}(\mathcal{M}_g)$, X_g gen curve of genus g.
  Curve SC true for X_g for $g \geq 5$...

• **Geom pro-p Curve SC**:

- Hoshi: *...does not hold over $k = \mathbb{Q}[\zeta_p]$ for $X_0^p + X_1^p + X_2^p \subset \mathbb{P}^2$ and p regular prime.*

• **Finally**: Curve/Birational SC are widely open.

• Conditional result on Birational SC: k # field.

- Esnault–Wittenberg: *X proj hyperbolic, with $\text{Ш}\big(\text{Jac}(X)\big)$ finite. If $pr : \text{Gal}_{k(X)}^{\text{geom.ab}} \to \text{Gal}_k$ has sections, then X has index 1.*

B) **Unconditional results**:

- Koenigsmann: *The p-adic Birat SC holds.*

Comment: Actually a much stronger fact holds:

- $k|\mathbb{Q}_p$ finite, and $K|k$ regular extension. Then sections of $\mathrm{Gal}_K \to \mathrm{Gal}_k$ originate from k-rational places of K.

  - In particular, if $K = k(X_0)$ with X_0 proj smooth curve, then the sections originate from $X_0(k)$...

Refinement: *Minimalistic p-adic Birational SC*

Context: $k|\mathbb{Q}_p$ finite, $\mu_p \subset k$, $K = k(X_0)$.

- $k''|k' \hookrightarrow K''|K'$ max $\mathbb{Z}/p$ metabelian ext's.
- $pr : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(k'|k)$ can projection.
- Liftable section s of pr is one coming from a section of $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(k''|k)$

- (P): *"Bouquets" of liftable sections $\leftrightarrow X_0(k)$.*

Comments: The less Galois theory, the better...

C) **Partial results**:

Tamagawa:

- k finite field, $X \subseteq X_0$ hyperbolic.
- $\tilde{X} \to X$ univ pro-étale cover, $\tilde{K} = k(\tilde{X})$.
- Given s section of pr_X, consider all

  $Y \to X$ finite sub-cover of $\tilde{X} \to X$

  such that $\mathrm{im}(s) \subset \pi_1^{\mathrm{et}}(Y)$.
- $|Y_0(k)| = \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\varphi_k)\big|\mathrm{H}^i\big(\overline{Y}_0, \mathbb{Z}_\ell(1)\big)$

• *A section s comes from a point $x \in X_0(k)$ iff $Y_0(k)$ non-empty for all $Y \to X$ as above.*

Nakamura:

- k number field, $X \subset X_0$ affine hyperbolic.

• *The points $x \in X_0 \backslash X$ are in bijection with conjugacy classes of max subgroups $\Delta \cong \widehat{\mathbb{Z}}$ of $\pi_1^{\mathrm{alg}}(X)$ of pure weight -2.*

For the next three results:

- $k|\mathbb{Q}_p$ fin, $X \subset X_0$ hyperbolic, $\tilde{X} \to X$ univ cover
- $pr_X : \pi_1^{\text{et}}(X) \to \text{Gal}_k$ can projection

Mochizuki: Suppose X_0 defined over $\overline{\mathbb{Q}}$.

- *Then* $X_0 \backslash X \ni x \leftrightarrow$ *maximal* $\Delta \cong \widehat{\mathbb{Z}} \subset \pi_1^{\text{alg}}(X)$ *on which* Gal_k *acts via the cycl character.*

Comment: ...the "absolute form" of the anab conj for curves.

Saidi: Defines "good sections" of pr_X...

- *The good sections* $\leftrightarrow X_0(k)$.

Comment: Proof relies on:

- Mochizuki's p-adic cuspidalization methods...
- Pop's methods developed for the "minimalistic" result...

P–Stix:

- *For every section* s *of* pr_X $\exists$ *valuation* $\tilde{w}$ *on* $k(\tilde{X})$ *such that* $\text{im}(s) \subset Z_{\tilde{w}}$.

Comments: ...relation to Mochizuki's "combinatorial SC

- Equivalently: *Every section comes from Berkovich points.*

D) **Section conjectures over $\mathbb{R}$**

Here: $k = \mathbb{R}$, $X \subseteq X_0$ hyperbolic, etc...

Wojtkowiak, Mochizuki, Wickelgren, etc...

$\pi_1^{\text{et}}(X) \to \text{Gal}_{\mathbb{R}}$ *has sections iff* $X_0(\mathbb{R}) \neq \emptyset$

• One cannot expect a Curve SC over $\mathbb{R}$...

Wickelgren:

- *The geometrically pro-2 curve version holds.*
- *The pro-2 birational SC holds.*

III) Final Comments

- Initial motivation of Grothendieck...

  ...Mordell Conjecture, now Faltings' Theorem...)
- No relation between Curve SC and

  (an effective) Mordell Conjecture yet...

Minhyong Kim:

Using pro-unipotent completions of π_1^{alg} designs an algorithm (of p-adic nature) which —under the conjectural properties of his *"Selmer varieties"*— produces the rational points of the curve in discussion; and more impressively, the effectiveness of the algorithm is guaranteed by the validity of the Curve SC. This — I would claim — sheds the right light on the relation and the rôle of the Curve SC to an *effective Mordell.*

IV) **Short list of open Problems**:

1) Prove/disprove:

$$\imath_{\mathcal{V}} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\pi_{\mathcal{V}}) \text{ is not onto for } \mathcal{V} \subseteq \mathcal{T}$$

2) Clarify/prove the pro-unipotent I/OM

3) Clarify the relation between the global and the p-adic Curve/Birational SC

4) Prove/disprove the global/p-adic (birational) section Conjecture.

5) Relation between the representations $\imath_{\mathcal{V}}$ and linear represent of $\mathrm{Gal}_{\mathbb{Q}}$, respectively $\mathrm{Gal}_{\mathbb{Q}_p}$