RECOVERING FIELDS FROM THEIR DECOMPOSITION GRAPHS

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1. Introduction

Recall that the birational anabelian conjecture originating in ideas presented in Grothendieck's *Esquisse d'un Programme* [G1] and *Letter to Faltings* [G2], asserts roughly the following: First, there should exist a group theoretical recipe by which one can recognize the absolute Galois groups G_K of finitely generated infinite fields K among all the profinite groups. Second, if $G = G_K$ is such an absolute Galois group, then the group theoretical recipe recovers the field K from G_K in a functorial way. Third, the recipe should be invariant under open homomorphisms of absolute Galois groups. In particular, the category of finitely generated infinite fields (up to Frobenius twist) should be equivalent to the category of their absolute Galois groups and open outer homomorphisms between these groups. A first instance of this situation is the celebrated Neukirch–Uchida Theorem, which says that global fields are characterized by their absolute Galois groups. I will not go into further details about the results concerning Grothendieck's (birational) anabelian geometry, but the interested reader can find more about this in Szamuely's Bourbaki Séminaire talk [Sz], and Faltings' Séminare Bourbaki talk [Fa], and newer results by Stix [St], Mochizuki [Mz], Saidi–Tamagawa [S–T], and Koenigsmann [Ko], Minhyong Kim [Ki] concerning the (birational) section conjecture.

The idea behind Grothendieck's anabelian geometry is that the *arithmetical Galois action* on rich geometric fundamental groups (like the geometric absolute Galois group) makes objects very rigid, so that there is no room left for non-geometric morphisms between such rich fundamental groups endowed with arithmetical Galois action.

On the other hand, Bogomolov [Bo] advanced at the beginning of the 1990's the idea that one should have anabelian type results in a total absence of an arithmetical action as follows: Let ℓ be a fixed rational prime number. Consider function fields $K|k$ over algebraically closed fields k of characteristic $\neq \ell$. For each such a function field $K|k$, let $G''_K := \mathrm{Gal}(K''|K)$ be the Galois group of a maximal pro-ℓ abelian-by-central Galois extension $K''|K$. Note that if $G^{(i+1)} := [G^{(i)}, G_K](G^{(i)})^{\ell^\infty}$, where $i \geq 0$ and $G^{(0)} = G_K$, are the central ℓ^∞ terms of the absolute Galois group G_K of K, then we have $G''_K = G^{(0)}/G^{(2)}$. Further, $G'_K = G^{(0)}/G^{(1)}$ is the Galois group of the maximal pro-ℓ abelian sub-extension $K'|K$ of $K''|K$; and denoting by $G^{(\infty)}$ the intersection of all the $G^{(i)}$, it follows that $G_K(\ell) := G_K/G^{(\infty)}$ is the maximal pro-ℓ quotient of G_K. Now the program initiated by Bogomolov in loc.cit. has as ultimate goal to recover function fields $K|k$ with $\mathrm{tr.deg}(K|k) > 1$ as above from G''_K in a functorial way. (Note

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that BOGOMOLOV denotes G''_K by PGal^c_K.) If successful, this program would go far beyond Grothendieck's birational anabelian conjectures, as k being algebraically closed implies that there is no arithmetical action in the game. The program initiated by BOGOMOLOV is not completed yet, and the present paper is a contribution towards trying to settle that program. See the Historical Note below for more about this.

In [P1] a sketch of a strategy to recover $K|k$ was presented, which has as starting point the following idea: Let $\widehat{K} := \varprojlim_n K^\times/n$, $n = \ell^e$, be the ℓ-adic completion of the multiplicative group $K^\times$ of $K|k$. Since the cyclotomic character of K is trivial, one can identify the ℓ-adic Tate module $\mathbb{T}_{\ell,K} := \varprojlim_e \mu_{\ell^e,K}$ of K with $\mathbb{Z}_\ell$, and let $\imath_K : \mathbb{T}_{\ell,K} \to \mathbb{Z}_\ell$ be such an identification. Then via Kummer Theory, one has isomorphisms of ℓ-adically complete groups:

$$\widehat{K} = \mathrm{Hom}_{\mathrm{cont}}(G'_K, \mathbb{T}_{\ell,K}) \xrightarrow{\imath_K} \mathrm{Hom}_{\mathrm{cont}}(G'_K, \mathbb{Z}_\ell),$$

i.e., $\widehat{K}$ can be recovered from G'_K, hence from G''_K via the canonical projection $G''_K \to G'_K$. On the other hand, since $k^\times$ is divisible, $\widehat{K}$ equals the ℓ-adic completion of the free abelian group $K^\times/\,k^\times$. Now the idea of recovering $K|k$ is as follows:

a) First, give a recipe to recover the image $\jmath_K(K^\times) = K^\times/k^\times$ of the ℓ-adic completion functor $\jmath_K : K^\times \to K^\times/k^\times \subset \widehat{K}$ inside the "known" $\widehat{K} = \mathrm{Hom}_{\mathrm{cont}}(G'_K, \mathbb{Z}_\ell)$.

b) Second, interpreting $K^\times/k^\times =: \mathcal{P}(K)$ as the projectivization of the (infinite) dimensional k-vector space $(K, +)$, give a recipe to recover the projective lines $\mathfrak{l}_{x,y} := (kx+ky)^\times/k^\times$ inside $\mathcal{P}(K)$.

c) Third, apply the *Fundamental Theorem of Projective Geometries,* see e.g., ARTIN [Ar], and deduce that $K|k$ can be recovered from $\mathcal{P}(K)$ endowed with all the lines $\mathfrak{l}_{x,y}$.

d) Finally, check that these recipes are invariant under isomorphisms of profinite groups $G'_K \cong G'_L$ which are induced by ones between G''_K and G''_L, hence such isomorphisms originate actually from geometry.

The strategy from [P1] to tackle the above problems a), b), c), d), above is in principle similar to the strategies (initiated by Neukirch and Uchida) to tackle Grothendieck's anabelian conjectures. It has two main parts as follows, the terms being as introduced later:

Part I. *Local theory*: It has as input Galois theoretic information, like G''_K, or like $G_K(\ell)$ as considered in [P3]. It should be a recipe which in a first approximation recovers the decomposition groups of prime divisors of $K|k$ together with their inertia subgroups. The final output of the local theory should be the (geometric) decomposition graphs $\mathcal{G}_{\mathcal{D}_K}$ of G'_K together with their rational quotients $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$. This recipe should be invariant under isomorphisms $G'_K \cong G'_L$ which are induced by isomorphisms between G''_K and G''_L.

Part II. *Global theory*: Its input is a geometric decomposition graph $\mathcal{G}_{\mathcal{D}_K}$ for $K|k$ together with its set of rational quotients $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$. It should be a group theoretical recipe to recover $\mathcal{P}(K)$ and its projective lines, thus finally to recover the function field $K|k$. And this recipe should be invariant under isomorphisms of decomposition graphs $\mathcal{G}_{\mathcal{D}_K} \cong \mathcal{G}_{\mathcal{D}_L}$ which are compatible with rational quotients.

The present manuscript concerns Part II of the above strategy, and the final answer to that is Theorem 4.1, which we reproduce below in a slightly modified form as the Main Result. Before that let us define the geometric decomposition graphs $\mathcal{G}_{\mathcal{D}_K}$ of $K|k$ and their sets of rational quotients $\mathfrak{A} = \{\Phi_{\kappa_x}\}_{\kappa_x}$.

- *The valuation theoretical side*

Let K be an arbitrary field. The space of all the equivalence classes of valuations Val_K of K is in a canonical way a partially ordered set by: $w \leq v$ iff $\mathcal{O}_v \subseteq \mathcal{O}_w$ iff $\mathfrak{m}_w \subseteq \mathfrak{m}_v$, and if so, then $\mathfrak{m}_w \subset \mathcal{O}_v$ is a prime ideal, and $\mathcal{O}_w$ is the localization $\mathcal{O}_w = (\mathcal{O}_v)_{\mathfrak{m}_w}$. The unique minimal element of Val_K is the trivial valuation v_0 which has $\mathcal{O}_{v_0} = K$ as valuation ring. Further, the minimal *non-trivial* elements of Val_K are exactly the rank 1 valuation rings of K (which then correspond to the equivalence classes of non-archimedean absolute values of K). Note that if $w \leq v$, then $\mathcal{O}_v/\mathfrak{m}_w$ is a valuation ring in the residue field Kw of w. We denote the corresponding valuation of Kw by v/w, and call it the quotient of v by w. Conversely, given $w \in \mathrm{Val}_K$ and a valuation w_1 of Kw, the preimage $\mathcal{O}$ of $\mathcal{O}_{w_1}$ under $\mathcal{O}_w \to Kw$ is a valuation ring of a valuation $v \geq w$ such that $v/w = w_1$. We denote $w_1 \circ w := v$, and call it the composition of w_1 and w.

Val_K has in a canonical way the structure of a (half-oriented) graph with origin $K = Kv_0$ as follows:

- The vertices are the residue fields Kv indexed by $v \in \mathrm{Val}_K$.

- The set of edges from Kw to Kv is non-empty if and only if $w \leq v$ and $\mathrm{rank}(v/w) \leq 1$. If so, then v/w is the unique edge from Kw to Kv. We say that v/w is a non-trivial oriented edge if $\mathrm{rank}(v/w) = 1$, respectively we call v/w a trivial non-oriented edge if $v = w$, i.e., v/w is the trivial valuation of Kw.

We will call the graph defined above the valuation graph for K. There are two functorial constructions one should mention here:

1) *Embeddings.* Let $\imath : L \hookrightarrow K$ be a field embedding, and $\varphi_\imath : \mathrm{Val}_K \to \mathrm{Val}_L$, $v \mapsto v_L := v|_L$ be the canonical restriction map. Then $\varphi_\imath$ is surjective and compatible with the ordering of valuations. And if $w \leq v$ in Val_K, then $w_L \leq v_L$ in Val_L, and $\mathrm{rank}(v_L/w_L) \leq \mathrm{rank}(v/w)$. Hence if the edge v/w from Kw to Kv exists, then the edge v_L/w_L from Lw_L to Lv_L exists too. Therefore, $\varphi_\imath$ defines a canonical projection from the valuation graph of K onto the valuation graph of L, under which Kv is mapped to Lv_L, and the edge v/w from Kw to Kv (if it exists) is mapped to the edge v_L/w_L from Lw_L to Lv_L. Note that if v/w is a non-trivial oriented edge such that $v_L = w_L$, then v/w is mapped to the trivial non-oriented edge of $Lv_L = Lw_L$.

2) *Restrictions.* Let Kw be the residue field of w, and $\mathrm{Val}_w = \{v \in \mathrm{Val}_K \mid v \geq w\}$ be the set of all refinements of v. Then $\mathrm{Val}_w \to \mathrm{Val}_{Kw}$, $v \mapsto v/w$, is a canonical bijection which respects the ordering, thus defines an isomorphism of the subgraph Val_w of the valuation graph for K onto the valuation graph Val_{Kw} for Kw.

Geometric prime divisor graphs

Now let us come back to the case where K is a function field $K|k$ over k as above. Recall that a (Zariski) prime divisor of a function field $K|k$ is a discrete valuation v of K whose valuation ring is the local ring $\mathcal{O}_{X,x_1}$ of the generic point x_1 of some Weil prime divisor of some normal model $X \to k$ of $K|k$. If so, then the residue field Kv of v is the function field $Kv = \kappa(x_1)$, thus $\mathrm{tr.deg}(Kv|k) = \mathrm{tr.deg}(K|k) - 1$. A set of Zariski prime divisors D of $K|k$ is called a geometric set, if there exists a quasi-projective normal model $X \to k$ of $K|k$ such that $D = D_X$ is the set of the valuations v_{x_1} defined by the generic points x_1 of all the Weil prime divisors of X. Further, for a k-valuation $\tilde{v}$ of K the following are equivalent:

i) There exists a chain $w_1 < \cdots < w_r := \tilde{v}$ and $\mathrm{tr.deg}(K\tilde{v}|k) = \mathrm{tr.deg}(K|k) - r$.

ii) $\tilde{v}$ is the composition $\tilde{v} = v_r \circ \cdots \circ v_1$, where v_1 is a prime divisor of $K_0 := K$, and setting $K_i = K_{i-1}v_i$ for $0 < i \leq r$, one has: v_i is a prime divisor of K_{i-1}.

We will say that a valuation $\tilde{v}$ of K is a **prime r-divisor** of $K|k$, if $\tilde{v}$ satisfies the above equivalent conditions. In particular, one has $r \leq \mathrm{tr.deg}(K|k)$. If the rank r is not essential for the context, we will simply say that $\tilde{v}$ is a generalized prime divisor of $K|k$.

We define the **total prime divisor graph** $\mathcal{D}_K^{\mathrm{tot}}$ of K to be the sub-graph of Val_K whose vertices are index by <u>all</u> the generalized prime divisors $\tilde{v}$ of $K|k$. Equivalently, for each vertex $\tilde{v}$ one has: The set $D_{\tilde{v}}$ of all the non-trivial edges starting at $K\tilde{v}$ equals the set of all the prime divisors of $K\tilde{v}$. Note that all the maximal branches of non-trivial edges of $\mathcal{D}_K^{\mathrm{tot}}$ start at the origin vertex $K = K_0$ and have length equal to $\mathrm{tr.deg}(K|k)$.

A **geometric prime divisor graph** for $K|k$ is any connected subgraph $\mathcal{D}_K$ of $\mathcal{D}_K^{\mathrm{tot}}$ which satisfies: First, for each vertex $K\tilde{v}$ of $\mathcal{D}_K$, the set $D_{\tilde{v}}$ of all the non-trivial edges of $\mathcal{D}_K$ starting at $K\tilde{v}$ is a geometric set of prime divisors of $K\tilde{v}$. Second, all maximal branches of non-trivial edges of $\mathcal{D}_K$ originate at $K = K_0$ and have length equal to $\mathrm{tr.deg}(K|k)$. Equivalently, $\mathcal{D}_K$ is a half-oriented connected graph having $K = K_0$ as origin and satisfying:

I) The vertices of $\mathcal{D}_K$ are distinct function fields $K_i|k$ over k.

II) For every vertex K_i, the trivial valuation of K_i is the only edge from K_i to itself. And the set of non-trivial edges starting at K_i is a geometric set of prime divisors of K_i, and if $v_{i'}$ is a nontrivial edge from K_i to $K_{i'}$, then $K_{i'} = K_i v_{i'}$, hence $\mathrm{tr.deg}(K_{i'}|k) = \mathrm{tr.deg}(K_i|k) - 1$.

III) All the maximal branches of non-trivial edges of $\mathcal{D}_K$ have length equal to $\mathrm{tr.deg}(K|k)$.

The functorial behavior of geometric prime divisor graphs is as follows:

1) *Embeddings.* Let $L|l \hookrightarrow K|k$ be an embedding of function fields which maps l isomorphically onto k. Then the canonical restriction map $\mathrm{Val}_K \to \mathrm{Val}_L$ gives rise to a morphism of the total prime divisor graphs $\varphi_\imath : \mathcal{D}_K^{\mathrm{tot}} \to \mathcal{D}_L^{\mathrm{tot}}$, which moreover is surjective. The relation between *geometric prime divisor graphs* $\mathcal{D}_K$ and $\mathcal{D}_L$ is a little bit more subtle. One shows the following, see Proposition 3.13: Given geometric prime divisor graphs $\mathcal{D}_K$ and $\mathcal{D}_L$, there exist geometric prime divisor graphs $\mathcal{D}_K^0$ and $\mathcal{D}_L^0$ containing $\mathcal{D}_K$, respectively $\mathcal{D}_L$, such that $\varphi_\imath$ defines a surjective morphism of geometric prime divisor graphs:

$$\varphi_\imath : \mathcal{D}_K^0 \to \mathcal{D}_L^0.$$

2) *Restrictions.* Given a generalized divisor $\tilde{w}$ of $K|k$, let $\mathcal{D}_{\tilde{w}}^{\mathrm{tot}}$ be the set of all the generalized prime divisors $\tilde{v}$ of $K|k$ with $\tilde{w} \leq \tilde{v}$. Then the restriction map

$$\mathcal{D}_{\tilde{w}}^{\mathrm{tot}} \to \mathcal{D}_{K\tilde{w}}^{\mathrm{tot}}, \quad \tilde{v} \mapsto \tilde{v}/\tilde{w}$$

is an isomorphism of valuation graphs. Moreover, if $\tilde{v}$ is a vertex of some geometric prime divisor graph $\mathcal{D}_K$ for $K|k$, then one has: The maximal subgraph $\mathcal{D}_{K\tilde{v}}$ of $\mathcal{D}_K$ whose initial vertex is $K\tilde{v}$ is isomorphic to a geometric graph of prime divisors of $K\tilde{w}$.

- *The Galois decomposition theoretical side*

Let ℓ be a fixed prime number as above. For every field K, let $K'|K$ be a maximal pro-ℓ abelian extension, and $G_K' = \mathrm{Gal}(K'|K)$ denote its Galois group. For $v \in \mathrm{Val}_K$, and prolongations v' of v to K', we have: The inertia/decomposition groups $T_{v'} \subseteq Z_{v'}$ of the several prolongations $v'|v$ are conjugated under G_K'. Hence these groups are equal, as G_K' is commutative. We will denote them by $T_v \subseteq Z_v$, and call them the inertia/decomposition groups at v. Recall that $G_{Kv}' = Z_v/T_v$ canonically.

Via Hilbert decomposition theory and Galois correspondence, we attach to Val_K a graph $\mathcal{G}_{\mathrm{Val}_K}$ of pro-ℓ abelian groups with an origin vertex as follows: The vertices of $\mathcal{G}_{\mathrm{Val}_K}$ are indexed by the (distinct) residue fields Kv with $v \in \mathrm{Val}_K$, and we endow each Kv with the abelian pro-ℓ group G_{Kv}'. Concerning edges, if v/w is the unique edge from some Kw to some Kv (hence, either $w = v$ and v/w is the trivial valuation on $Kv = Kw$, or $w < v$ and $\mathrm{rank}(v/w) = 1$ on Kw), we endow the edge v/w with the pair of groups $T_{v/w} \subseteq Z_{v/w}$. Note that in the case v/w is the trivial valuation, we have merely by definition: $T_{v/w} = 1$ and $Z_{v/w} = G_{Kv}'$.

We will call $\mathcal{G}_{\mathrm{Val}_K}$ the valuation decomposition graph of K, or of G_K'.

Finally, remark that the above functorial constructions concerning embeddings and restrictions, give rise functorially to corresponding functorial constructions on the Galois side as follows:

1) *Embeddings.* Let $\imath : L \hookrightarrow K$ be an embedding of fields, and consider a prolongation $\imath' : L' \hookrightarrow K'$ of $\imath$. Then $\imath'$ gives rise to a projection $\Phi_\imath : G_K' \to G_L'$, which in turn gives rise canonically to a morphism of valuation decomposition graphs, which we denote by $\Phi_\imath$ again:

$$\Phi_\imath : \mathcal{G}_{\mathrm{Val}_K} \to \mathcal{G}_{\mathrm{Val}_L}.$$

Note that $\Phi_\imath$ maps the profinite group G_{Kv}' at the vertex Kv into the profinite group G_{Lv_L}' at the corresponding vertex Lv_L. And concerning edges, $\Phi_\imath$ maps $T_{v/w} \subseteq Z_{v/w}$ into the pair $T_{v_L/w_L} \subseteq Z_{v_L/w_L}$ of the corresponding inertia/decomposition subgroups of v_L/w_L in G_{Lw_L}'.

2) *Restrictions.* For $v \in \mathrm{Val}_w$, one has $Z_v \subseteq Z_w$ and $T_w \subseteq T_v$. And under the canonical projection $Z_w \to G_{Kw}'$, every $T_v \subseteq Z_v$ is mapped onto $T_{v/w} \subseteq Z_{v/w}$ in G_{Kw}', etc.

Geometric decomposition graphs

Next suppose that K is a function field $K|k$ as considered above. Then we have the following, see e.g., [P4], Introduction, for a discussion of these facts: For every prime divisor v of $K|k$ one has $T_v \cong \mathbb{T}_{\ell,K}$, and for every prime r-divisor $\tilde{v}$ one has $T_{\tilde{v}} \cong \mathbb{T}_{\ell,K}^r$. Further, for generalized prime divisors $\tilde{v}_1$ and $\tilde{v}_2$ one has: $Z_{\tilde{v}_1} \cap Z_{\tilde{v}_2} \neq 1$ if and only if $\tilde{v}_1, \tilde{v}_2$ are not independent as valuations, i.e., $\mathcal{O} := \mathcal{O}_{\tilde{v}_1}\,\mathcal{O}_{\tilde{v}_2} \neq K$ is a proper valuation ring of K. Moreover, if $\tilde{v}$ is the valuation defined by $\mathcal{O}$, then $\tilde{v}$ is the unique generalized prime divisor of $K|k$ with $T_v = T_{\tilde{v}_1} \cap T_{\tilde{v}_2}$, and also the unique generalized prime divisor of $K|k$ maximal with the property $Z_{\tilde{v}_1}, Z_{\tilde{v}_2} \subseteq Z_{\tilde{v}}$.

In the above context, let $\mathcal{D}_K$ be a geometric graph of prime divisors of $K|k$. We attach to $\mathcal{D}_K \subset \mathrm{Val}_K$ the corresponding subgraph $\mathcal{G}_{\mathcal{D}_K} \subset \mathcal{G}_{\mathrm{Val}_K}$ of $\mathcal{G}_{\mathrm{Val}_K}$, and call $\mathcal{G}_{\mathcal{D}_K}$ a geometric decomposition graph for $K|k$ (or G_K').

Next recall that the isomorphy type of (the maximal abelian pro-ℓ quotient of) the fundamental group $\pi_1'(X)$ of complete regular models $X \to k$, if such models exist, does depend on

$K|k$ only, and not on $X \to k$. Moreover, one can recover $\pi_1'(X)$ as being $\pi_1'(X) = G_K'/T_K$, where T_K is the subgroup of G_K generated by all the inertia groups T_v with v prime divisor of $K|k$. This justifies introducing the following terminology: We set $\pi_{1,K}' := G_K'/T_K$ and call it the **birational fundamental group** for $K|k$. As discussed in Fact 2.15, there always exist quasi projective normal models $X \to k$ for $K|k$ such that $\pi_{1,K}'$ classifies the connected normal covers of X which are unramified above D_X, i.e., unramified in co-dimension 1. We call such models $X \to k$ and the corresponding geometric sets $D := D_X$ **complete regular like**. And we will say that a geometric decomposition graph $\mathcal{G}_{\mathcal{D}_K}$ is **complete regular like**, if for all the vertices $\tilde{v}$ of $\mathcal{G}_{\mathcal{D}_K}$ one has: If $\operatorname{tr.deg}(K\tilde{v}|k) > 1$, then the set $D_{\tilde{v}}$ of non-trivial edges of $\mathcal{G}_{\mathcal{D}_K}$ originating from $K\tilde{v}$ is complete regular like, and if $\operatorname{tr.deg}(K\tilde{v}|k) = 1$, then $D_{\tilde{v}}$ consists of all the prime divisors of $K\tilde{v}|k$ (and in particular, it is complete regular like).

By the functorial properties of embeddings we get the following: Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields, and $\mathcal{D}_K$ and $\mathcal{D}_L$ be geometric prime divisor graphs of $K|k$, respectively $L|l$, such that $\imath$ gives rise to a morphism $\varphi_\imath : \mathcal{D}_L \to \mathcal{D}_K$. Then $\imath$ gives rise canonically to a morphism of decomposition graphs:

$$\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_L} .$$

Rational quotients

Let $K|k$ be a function field as above satisfying $\operatorname{tr.deg}(K|k) > 1$. For every non-constant function $t \in K$, let K_t be the relative algebraic closure of $k(t)$ in K. Since $\operatorname{tr.deg}(K_t|k) = 1$, it follows that K_t has a unique complete normal model $X_t \to k$, which turns out to be a projective smooth curve. Therefore, the set of prime divisors of $K_t|k$ is actually in bijection with the (local rings at the) closed points of X_t, thus with the set of Weil prime divisors of X_t. Therefore, the total prime divisor graph $\mathcal{D}_{K_t}^{\text{tot}}$ for K_t is actually the *geometric prime divisor graph,* whose non-trivial edges are in bijection with D_{X_t}. We will denote $\mathcal{D}_{K_t}^{\text{tot}}$ simply by $\mathcal{D}_{K_t}$.

Let $\imath_t : K_t \to K$ be the canonical embedding, and $\Phi_{K_t} : G_K' \to G_{K_t}'$ the (surjective) canonical projection. Then by the functoriality of embeddings we get: For every geometric decomposition graph $\mathcal{G}_{\mathcal{D}_K}$ for $K|k$, the projection Φ_{K_t} gives rise to a morphism of geometric decomposition graphs $\Phi_{K_t} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{K_t}$.

In the above context, we say that $\Phi_{K_t} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{K_t}$ is a **rational quotient** of $\mathcal{G}_{\mathcal{D}_K}$, if $K_t = k(t)$, i.e., $k(t)$ is relatively algebraically closed in K. We will call t a "general element" of K, and usually denote general elements of K by x, and K_x by κ_x, in order to distinguish them from the "usual" K_t. A "birational Bertini" type argument shows that there are "many" general elements in K, see LANG [L2], Ch.VIII, and Fact 3.17: For any given algebraically independent functions $x, t \in K$, not both inseparable, $t_{a',a} := t/(a'x+a)$ is a general element of K for almost all $a', a \in k$. A set of general elements $\Sigma \subset K$ is a **Bertini set** if Σ contains almost all elements $t_{a',a}$ for all x, t as above. We denote by $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$ the set of all the rational quotients of $K|k$, and consider subsets $\mathfrak{A} \subset \mathfrak{A}_K$ containing all the $\Phi_{k(t_i)} \in \mathfrak{A}$, $t_i \in \Sigma$, and Σ some Bertini set of general elements. For short, we call such subsets $\mathfrak{A} \subset \mathfrak{A}_K$, **Bertini type sets** of rational quotients.

The relation between rational projections and morphisms of geometric decomposition graphs is as follows: Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields with $\imath(l) = k$, and $K|\imath(L)$ a separable field extension. Then there exists a Bertini type set $\mathfrak{B} = \{\Phi_{\kappa_y}\}_{\kappa_y}$ for $\mathcal{G}_{\mathcal{D}_L}$

such that $\kappa_x := \imath(\kappa_y)$ is relatively algebraically closed in K for all κ_y. Hence for all $\Phi_{\kappa_y} \in \mathfrak{B}$ and the corresponding $\Phi_{\kappa_x} \in \mathfrak{A}_K$, $\kappa_x := \imath(\kappa_y)$, we get: The isomorphism $\Phi_{\kappa_x\kappa_y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ defined by $\imath_{\kappa_x\kappa_y} := \imath|_{\kappa_y}$ satisfies the condition:

$$\Phi_{\kappa_y} \circ \Phi_\imath = \Phi_{\kappa_x\kappa_y} \circ \Phi_{\kappa_x} .$$

Because of this property, we will say that $\Phi_\imath$ is compatible with the rational quotients.

- *Abstract decomposition graphs*

It is one of our main tasks in the present manuscript to define and study abstract decomposition graphs which resemble the geometric decomposition graphs $\mathcal{G}_{\mathcal{D}_K}$. And to define proper morphisms of such abstract decomposition graphs, in particular their rational quotients. The abstract decomposition graphs, which endowed with families of rational quotients resemble the complete regular like geometric decomposition graphs as introduced above will be called geometric like abstract decomposition graphs. The main result of the global theory we mentioned above as Part II is the following, see Theorem 4.1, and Definitions 2.16, 3.2, 3.9, 3.15, for the precise definitions of all the terms:

Main Theorem.
Let $K|k$ be a function field with $\mathrm{tr.deg}(K|k) > 1$, *and let $\mathcal{G}_{\mathcal{D}_K}$ be a complete regular like geometric decomposition graph for $K|k$. We endow $\mathcal{G}_{\mathcal{D}_K}$ with a Bertini type set $\mathfrak{A}$ of rational quotients, and view it as a* geometric like abstract decomposition graph.

1) *Let $\mathcal{H}$ endowed with a family of rational quotients $\mathfrak{B}$ be a geometric like abstract decomposition graph. Then up to multiplication by ℓ-adic units, and composition with automorphisms $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_K}$ defined by embedding of function fields $\imath : K|l \to K|k$ such that $K|\imath(K)$ is purely inseparable, there exists* at most one isomorphism *$\Phi : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}$ of abstract decomposition graphs which is compatible with the rational quotients $\mathfrak{A}$ and $\mathfrak{B}$.*

2) *Let $L|l$ be a further function field with* $\mathrm{tr.deg}(L|l) > 1$, *and let $\mathcal{H}_{\mathcal{D}_L}$ be a complete regular like abstract decomposition graph for $L|l$. We endow $\mathcal{H}_{\mathcal{D}_L}$ with a Bertini type set $\mathfrak{B}$ of rational quotients, and view it as a* geometric like abstract decomposition graph. *Let*

$$\Phi : G_K^{'} \to G_L^{'}$$

be an open group homomorphism which defines a proper morphism $\Phi : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}_{\mathcal{D}_L}$ of abstract decomposition graphs compatible with the rational quotients $\mathfrak{B}$ and $\mathfrak{A}$. Then there exist an ℓ-adic unit ϵ and an embedding of function fields

$$\imath : L|l \to K|k$$

such that $\Phi = \epsilon \cdot \Phi_\imath$, where $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}_{\mathcal{D}_L}$ is the canonical morphism defined by $\imath$ as indicated above.

Further, $\imath(l) = k$, and $\imath$ is unique up to Frobenius twists.

As mentioned previously, the Main Theorem above settles the global theory, hence reduces the problem of recovering function fields $K|l$ from $G_K^{''}$ to recovering: First, the *complete regular like geometric decomposition graphs* $\mathcal{G}_{\mathcal{D}_K}$ for $K|k$. Second, the sets of *rational quotients* $\mathfrak{A}_K = \{\kappa_x\}_{\kappa_x}$ of such decomposition graphs $\mathcal{G}_{\mathcal{D}_K}$. So far, the manuscripts [P4], together with ideas from [P3] can be used to completely answer the two questions above in the case k is an algebraic closure of a finite field (and a manuscript about this is in preparation). In a similar but more technical way, one can recover from $G_K^{''}$ the complete regular like geometric decomposition graphs for $K|k$ and their rational quotients in the case k is an algebraic closure of a global field and $K|k$ is "very general", i.e., the birational fundamental group $\pi_{1,K}$ is finite, see Definition 2.16, for the definition of $\pi_{1,K}$. Thus after building the corresponding local theory, one gets the following "target result":

> *There exists a group theoretical recipe which does the following: First it recognizes the Galois groups of the form* G''_K *among all the profinite groups, where* $K|k$ *is a function field with* $\mathrm{tr.deg}(K|k) > 1$ *over fields* k *which are algebraic closures of either finite fields or global fields with* $\mathrm{char}(k) \neq \ell$. *Second, this recipe gives the isomorphy type of* $K|k$ *and is invariant under isomorphisms, i.e., if* $L|l$ *is a further function field over an algebraically closed field* l, *and* $G''_K \cong G''_L$ *is a given isomorphism, then its abelianization* $G'_K \cong G'_L$ *is induced by an isomorphism of function fields* $L^{\mathrm{i}}|l \cong K^{\mathrm{i}}|k$.

Historical note.

The first attempt to give a recipe to recover $K|k$ from G''_K was made by Bogomolov in his fundamental paper [Bo]. Although the loc.cit. is too sketchy in order to be sure what the author precisely proposes, a thorough inspection shows that loc.cit. provides a fundamental tool for recovering inertia elements of valuations v of K (which nevertheless may be non-trivial on k). This is Bogomolov's theory of *liftable commuting pairs,* see Bogomolov–Tschinkel [B–T1] for detailed proofs. Nevertheless, there are serious technical issues and difficulties when one wants to develop a global theory along the lines suggested in [Bo].

A sketch of a viable global theory —at least in the case *k is an algebraic closure of a finite field* can be found in the notes of my MSRI Talk from the Fall of 1999, see [P1]. That was followed by [P2], where several technical details from [P1] were worked out.

The present manuscript is actually an elaborated version of [P2], with the main result here, which is Theorem 4.1, being a generalization of that of loc.cit.: Here we prove namely the *Hom-form* of the (*Isom-form* of the) main result from [P2]. I should though mention that the manuscript [P2] considers the mixed "arithmetic + geometric situation", which might be applied to the case when k is not necessarily algebraically closed.

Finally, I would also like to mention the manuscripts:

- Bogomolov–Tschinkel [B–T2], which considers the case $K = k(X)$ with $X \to k$ a (projective, smooth) surface over an algebraic closure of a finite field k. (In its initial variant, Bogomolov–Tschinkel considered only the case when $\pi_1(X)$ is finite.)
- [P3] gives a recipe to recover $K|k$ from $G_K(\ell)$ under the hypothesis k an algebraic closure of a finite field. The full $G_K(\ell)$ and not only G''_K was used, because the "right" local theory, now developed in [P4], was not available back then.

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2. PRO-ℓ ABSTRACT DECOMPOSITION GRAPHS

In this section we develop an abelian pro-ℓ prime divisor decomposition theory for "abstract function fields", which is similar in some sense to the abstract class field theory.

A) *Axioms and definitions*

Throughout this chapter let ℓ be a fixed prime number.

Definition 2.1. A level $\delta \geq 0$ (pro-ℓ) abstract decomposition graph is a connected half oriented graph $\mathcal{G}$ whose vertices are endowed with pro-ℓ abelian groups, and whose edges are endowed with pairs of pro-ℓ abelian groups, satisfying the following axioms:

Axiom I): The vertices of $\mathcal{G}$ are pro-ℓ abelian free groups G_i, and $\mathcal{G}$ has an origin, which we denote by $G = G_0$.

Axiom II):

i) For every vertex G_i there exists a unique non-oriented edge v_{i0} from G_i to itself, and v_{i0} is endowed with the pair of pro-ℓ groups $\{1\} \subseteq G_i$. For every other vertex $G_{i'} \neq G_i$ there exists at most one edge $v_{i'}$ from G_i to $G_{i'}$. If $v_{i'}$ exists, we say that $v_{i'}$ is the oriented edge from G_i to $G_{i'}$, and $v_{i'}$ is endowed with a pair $T_{v_{i'}} \subseteq Z_{v_{i'}}$ of subgroups of G_i such that $T_{v_{i'}} \cong \mathbb{Z}_\ell$, and $G_{i'} = Z_{v_{i'}}/T_{v_{i'}}$.

The edges of $\mathcal{G}$ are also called valuations of $\mathcal{G}$, in particular the edges originating from G_i are called valuations of G_i. And the edge v_{i0} is called the trivial valuation of G_i, whereas $v_{i'}$ are called non-trivial valuations of G_i.

The groups $T_{v_{i'}} \subseteq Z_{v_{i'}}$ are called the inertia, respectively decomposition, groups of $v_{i'}$; and $G_{i'}$ is called the residue group of $v_{i'}$. For the sake of uniformity, we will also say that $\{1\} =: T_{i0} \subseteq Z_{i0} := G_i$ are the inertia, respectively decomposition, groups of the trivial valuation v_{i0}; and note that $G_i = Z_{i0}/T_{i0}$ is the residue group of v_{i0}.

ii) For distinct non-trivial edges $v_{i'} \neq v_{i''}$ originating at G_i, one has $Z_{v_{i'}} \cap Z_{v_{i''}} = \{1\}$ in G_i, hence $T_{v_{i'}} \cap T_{v_{i''}} = \{1\}$ too.

For every co-finite subset $\mathfrak{U}_i$ of the set of non-trivial edges $v_{i'}$ originating at G_i, let $T_{\mathfrak{U}_i}$ be the closed subgroup of G_i generated by all the $T_{v_{i'}}$, $v_{i'} \in \mathfrak{U}_i$. A system $(\mathfrak{U}_{i,\alpha})_\alpha$ of such co-finite subsets is called co-final, if every finite set of valuations $v_{i'}$ as above is contained in the complement of $\mathfrak{U}_{i,\alpha}$ for some α.

iii) There exist co-final systems $(\mathfrak{U}_{i,\alpha})_\alpha$ such that $T_{v_{i'}} \cap T_{\mathfrak{U}_{i,\alpha}} = \{1\}$ for all α and all $v_{i'} \notin \mathfrak{U}_{i,\alpha}$.

Axiom III): All maximal branches of non-trivial edges of $\mathcal{G}$ have the length δ.

Definition/Remark 2.2. Let $\mathcal{G}$ be an abstract decomposition graph of level $\delta_{\mathcal{G}} \geq 0$. We will say that $\mathcal{G}$ is a level $\delta_{\mathcal{G}}$ abstract decomposition graph on $G = G_0$. A valuation $v_{i'}$ of G_i will be called an i'-edge of $\mathcal{G}$. If no confusion is possible, we will denote the 1-edges of $\mathcal{G}$ simply by v; thus the corresponding pro-ℓ groups involved are denoted $T_v \subseteq Z_v$ and $G_v := Z_v/T_v$.

1) Consider any δ such that $0 \leq \delta \leq \delta_{\mathcal{G}}$. By induction on δ it is easy to see that $\mathcal{G}$ has a unique maximal connected abstract decomposition sub-graph containing the origin G_0 of $\mathcal{G}$ and having all branches of length δ.

2) Let $\tilde{v} = (v_\delta, \dots, v_1)$ be a path of non-trivial valuations starting at G_0 and having length $\delta_{\tilde{v}} := \delta > 0$. Hence v_1 is a non-trivial valuation of G_0, and inductively, if G_i is the residue group of v_i, then v_{i+1} is a non-trivial valuation of G_i for all $i < \delta$. In particular, G_δ is the residue group of v_δ. Then there exists a unique maximal connected subgraph $\mathcal{G}_{\tilde{v}}$ of $\mathcal{G}$ having G_δ as origin. Clearly, $\mathcal{G}_{\tilde{v}}$ is in a natural way an abstract decomposition graph of level $\delta_{\mathcal{G}} - \delta$ on G_δ.

We say that $\mathcal{G}_{\tilde{v}}$ is a δ-residual abstract decomposition graph of $\mathcal{G}$. In particular, the unique 0-residual abstract decomposition graph of $\mathcal{G}$ is $\mathcal{G}$ itself.

3) For every branch $\tilde{v}$ of length $\delta_{\tilde{v}} = \delta$ as above, we will say that $G_{\tilde{v}} := G_\delta$ is a δ-residual group of $\mathcal{G}$; or precisely, that $G_{\tilde{v}}$ is the *$\tilde{v}$-residual group* of $\mathcal{G}$. One can further elaborate here as follows: For $\delta > 1$ we define inductively the following: Set $\tilde{w} = (v_{\delta-1}, \dots, v_1)$, and suppose $T_{\tilde{w}} \subseteq Z_{\tilde{w}}$ are defined, hence in particular $Z_{\tilde{w}}/T_{\tilde{w}} =: G_{\tilde{w}} = G_{\delta-1}$ is the residue group of $\tilde{w}$. We then define the inertia/decomposition groups $T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ of $\tilde{v}$ in G as being the pre-images of $T_{v_\delta} \subseteq Z_{v_\delta}$ via $Z_{\tilde{w}} \to Z_{\tilde{w}}/T_{\tilde{w}} = G_{v_{\delta-1}}$.

Note that we have $Z_{\tilde{v}}/T_{\tilde{v}} =: G_{\tilde{v}} = G_\delta$, and $T_{\tilde{v}} \cong \mathbb{Z}_\ell^\delta$.

We call $\tilde{v}$ a generalized valuation of G, or a multi-index of lenth $\delta_{\tilde{v}}$ of $\mathcal{G}$. And we will say that $\delta_{\tilde{v}}$ is the rank of $\tilde{v}$, or that $\tilde{v}$ is a generalized r-valuation if $r = \delta_{\tilde{v}}$.

Given generalized valuations $\tilde{v} = (v_r, \dots, v_1)$, $\tilde{w} = (w_s, \dots, w_1)$, we say that $\tilde{w} \le \tilde{v}$ if $s \le r$, and $v_i = w_i$ for all $i \le s$. Directly from the definitions one gets: If $\tilde{w} \le \tilde{v}$, then $Z_{\tilde{v}} \subseteq Z_{\tilde{w}}$, and $T_{\tilde{w}} \subseteq T_{\tilde{v}}$. On the other hand, by Axiom II, ii), it immediately follows that the converse of (any of) these assertions is also true. We will say that $\tilde{v}$ and $\tilde{w}$ are dependent, if there exists some $t > 0$ such that $v_i = w_i$ for $i \le t$. For dependent generalized valuations $\tilde{v}$ and $\tilde{w}$ as above the following are equivalent:

a) t is maximal such that $v_i = w_i$ for $i \le t$.

b) $T_{\tilde{v}} \cap T_{\tilde{w}} \cong \mathbb{Z}_\ell^t$.

c) t is maximal such that $Z_{\tilde{v}}, Z_{\tilde{w}}$ are both contained in the decomposition group of some generalized t-valuation of G.

4) In order to have a uniform notation, we view $\tilde{v} = v_0$ to be the trivial multi-index, or the trivial path, of $\mathcal{G}$ as the unique one having length equal to 0. We further set $Z_{v_0} := G$ and $T_{v_0} = \{1\}$. In particular, one has $G_{v_0} = Z_{v_0}/T_{v_0} = G$, which is compatible with the other notations/conventions. Further, $v_0 \le \tilde{v}$ for all multi-indices $\tilde{v}$.

Definition/Remark 2.3. Let $\mathcal{G}$ be a level $\delta_{\mathcal{G}}$ abstract decomposition graph on $G = G_0$. In the notations from above we consider the following:

1) Denote $\widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big)$. Since G is a pro-ℓ free abelian group, $\widehat{\mathcal{L}}_{\mathcal{G}}$ is a free ℓ-adically complete $\mathbb{Z}_\ell$-module (in ℓ-adic duality with G).

From now on suppose that $\delta_{\mathcal{G}} > 0$. Recall that $T_v \subset Z_v$ and $G_v = Z_v/T_v$ denote respectively the inertia, the decomposition, and the residue, groups at the 1-edges v of $\mathcal{G}$, i.e., at the valuations v of G.

2) Denote by $T \subseteq G$ the closed subgroup generated by all the inertia groups T_v (all v as above). We set $\pi_{1,\mathcal{G}} := G/T$ and call it the abstract fundamental group of $\mathcal{G}$. One has a canonical exact sequence

$$1 \to T \to G \to \pi_{1,\mathcal{G}} := G/T \to 1\,.$$

Taking continuous $\mathbb{Z}_\ell$-Homs, we get an exact sequence of the form

$$0 \to \widehat{U}_{\mathcal{G}} := \mathrm{Hom}\big(\pi_{1,\mathcal{G}}, \mathbb{Z}_\ell\big) \xrightarrow{\mathrm{can}} \widehat{\mathcal{L}}_{\mathcal{G}} := \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big) \xrightarrow{\jmath^{\mathcal{G}}} \widehat{\mathcal{L}}_T := \mathrm{Hom}\big(T, \mathbb{Z}_\ell\big).$$

We will call $\widehat{U}_{\mathcal{G}} = \mathrm{Hom}\big(\pi_{1,\mathcal{G}}, \mathbb{Z}_\ell\big)$ the unramified part of $\widehat{\mathcal{L}}_{\mathcal{G}}$. And if no confusion is possible, we will identify $\widehat{U}_{\mathcal{G}}$ with its image in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

3) Now let us have a closer look at the structure of $\widehat{\mathcal{L}}_{\mathcal{G}}$. For an arbitrary 1-edge v as above, the inclusions $T_v \hookrightarrow Z_v \hookrightarrow G$ give rise to restriction homomorphisms as follows:

$$\jmath^v : \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\mathrm{res}_{Z_v}} \widehat{\mathcal{L}}_{Z_v} := \mathrm{Hom}\big(Z_v, \mathbb{Z}_\ell\big) \xrightarrow{\mathrm{res}_v} \widehat{\mathcal{L}}_{T_v} := \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big).$$

We set $\widehat{U}^1_v = \ker(\mathrm{res}_{Z_v})$ and $\widehat{U}_v = \ker(\jmath^v)$ and call them the principal v-units, respectively the v-units, in $\widehat{\mathcal{L}}_{\mathcal{G}}$. And remark that the unramified part of $\widehat{\mathcal{L}}_{\mathcal{G}}$ is exactly $\widehat{U}_{\mathcal{G}} = \cap_v \ker(\jmath^v)$.

We further denote $\widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}} = \{\, x \in \widehat{\mathcal{L}}_{\mathcal{G}} \mid \jmath^v(x) = 0 \text{ for almost all } v \,\}$. We remark that by Axiom II, iii), $\widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$ is dense in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Indeed, consider a co-final system $(\mathfrak{U}_\alpha)_\alpha$ of subsets of the set of 1-edges v. As at loc.cit., denoting $G_\alpha = G/T_{\mathfrak{U}_\alpha}$ and $T_\alpha = T/T_{\mathfrak{U}_\alpha}$, we have a canonical exact sequence

$$1 \to T_\alpha \to G_\alpha \to \pi_{1,\mathcal{G}} \to 1\,,$$

and T_α is generated by the images $T_{v,\alpha}$ of T_v (all $v \notin \mathfrak{U}_\alpha$) in G_α. Clearly, the image of the inflation map $\mathrm{inf}_\alpha : \mathrm{Hom}\big(G_\alpha, \mathbb{Z}_\ell\big) \to \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big)$ is exactly

$$\Delta_\alpha := \{x \in \widehat{\mathcal{L}}_{\mathcal{G}} \mid \jmath^v(x) = 0 \text{ for all } v \in \mathfrak{U}_\alpha\,\} = \cap_{v \in \mathfrak{U}_\alpha} \ker(\jmath^v)\,.$$

Finally, taking inductive limits over the co-final system $(\mathfrak{U}_\alpha)_\alpha$, the density assertion follows.

A closed $\mathbb{Z}_\ell$-submodule $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ is said to be a finite co-rank submodule, if $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$ and $\Delta / \widehat{U}_{\mathcal{G}}$ is a finite $\mathbb{Z}_\ell$-module; or equivalently, Δ is contained in $\ker(\jmath^v)$ for almost all v. Clearly, the sum of two finite co-rank submodules of $\widehat{\mathcal{L}}_{\mathcal{G}}$ is again of finite co-rank. Thus the set of such submodules is inductive. And one has:

$$\widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}} = \cup_\Delta \,(\text{all finite co-rank } \Delta) = \cup_\alpha \Delta_\alpha\,.$$

4) By the discussion above, the family $(\jmath^v)_v$ gives rise canonically to a continuous homomorphism $\widehat{\oplus_v}\, \jmath^v$ of ℓ-adically complete $\mathbb{Z}_\ell$-modules

$$\widehat{\oplus_v}\, \jmath^v : \widehat{\mathcal{L}}_{\mathcal{G}} \to \widehat{\mathcal{L}}_T \hookrightarrow \widehat{\oplus_v}\, \widehat{\mathcal{L}}_{T_v} = \widehat{\oplus_v}\, \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big)\,.$$

We identify $\widehat{\mathcal{L}}_T$ with its image inside $\widehat{\oplus_v}\, \widehat{\mathcal{L}}_{T_v}$. Therefore, $\jmath^{\mathcal{G}} = \widehat{\oplus_v}\, \jmath^v$ on $\widehat{\mathcal{L}}_{\mathcal{G}}$.

We denote $\widehat{\mathrm{Div}}_{\mathcal{G}} := \widehat{\oplus_v}\, \widehat{\mathcal{L}}_{T_v}$ and call it the ℓ-adic abstract divisor group of $\mathcal{G}$.

Finally, we set $\widehat{\mathfrak{Cl}}_{\mathcal{G}} = \mathrm{coker}(\jmath^{\mathcal{G}})$, and call it the ℓ-adic abstract divisor class group of $\mathcal{G}$. And remark that we have a canonical exact sequence

$$0 \to \widehat{U}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\mathcal{G}}} \widehat{\mathrm{Div}}_{\mathcal{G}} \xrightarrow{\mathrm{can}} \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0\,.$$

5) We say that $\mathcal{G}$ is complete curve like if the following holds: There exist generators τ_v of T_v such that $\prod_v \tau_v = 1$, and this is the only pro-relation satisfied by the system of elements $\mathfrak{T} = (\tau_v)_v$. We call such a system $\mathfrak{T}$ a distinguished system of inertia generators.

We remark the following: Let $\mathfrak{T} = (\tau'_v)_v$ be another distinguished system of inertia generators. Then $\tau'_v = \tau_v^{\epsilon_v}$ for some ℓ-adic units $\epsilon_v \in \mathbb{Z}_\ell$, as both τ_v and τ'_v are generators of T_v.

Hence we have $1 = \prod_v \tau'_v = \prod_v \tau_v^{\epsilon_v}$. By the uniqueness of the relation $\prod_v \tau_v = 1$, it follows that $\epsilon_v = \epsilon$ for some fixed ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell$.

Next consider some δ with $0 < \delta \leq \delta_{\mathcal{G}}$. We say that $\mathcal{G}$ is level δ complete curve like if all the $(\delta-1)$-residual abstract decomposition graphs $\mathcal{G}_{\tilde{v}}$ are residually complete curve like. In particular, "level 1 complete curve like" is the same as "complete curve like".

6) For every 1-vertex v consider the exact sequence $1 \to T_v \to Z_v \to G_v \to 1$ given by Axiom II, i). Let $\inf_v : \mathrm{Hom}\big(G_v, \mathbb{Z}_\ell\big) \to \mathrm{Hom}\big(Z_v, \mathbb{Z}_\ell\big)$ be the resulting inflation homomorphism. Since $T_v = \ker(Z_v \to G_v)$, it follows that $\mathrm{res}_{Z_v}(\widehat{U}_v)$ is the image of the inflation map infl_v. Therefore there exists a canonical exact sequence:

$$0 \to \widehat{U}_v^1 \longrightarrow \widehat{U}_v \xrightarrow{\jmath_v} \mathrm{Hom}\big(G_v, \mathbb{Z}_\ell\big) = \widehat{\mathcal{L}}_{\mathcal{G}_v} \to 0\,,$$

and we call $\jmath_v$ the v-reduction homomorphism.

7) In particular, if $\delta_{\mathcal{G}} > 1$, then $\delta_{\mathcal{G}_v} > 0$ for every 1-vertex v. Hence we have the corresponding exact sequence for the residual abstract decomposition graph $\mathcal{G}_v$:

$$0 \to \widehat{U}_{\mathcal{G}_v} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}_v} \xrightarrow{\jmath^{\mathcal{G}_v}} \widehat{\mathrm{Div}}_{\mathcal{G}_v}.$$

We will say that $\mathcal{G}$ is ample, if $\delta_{\mathcal{G}} > 0$, and the following conditions are satisfied:

i) $\jmath^\Sigma : \widehat{\mathcal{L}}_{\mathcal{G}} \longrightarrow \oplus_{v\in\Sigma} \mathcal{L}_{T_v}$ is surjective for every finite set Σ, where $\jmath^\Sigma := \oplus_{v\in\Sigma}\jmath^v$.

ii) If $\delta_{\mathcal{G}} > 1$, then the following hold:

   a) $\jmath_v(\widehat{U}_{\mathcal{G}}) \subseteq \widehat{U}_{\mathcal{G}_v}$ and $\widehat{U}_{\mathcal{G}_v} + \jmath_v(\widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}} \cap \widehat{U}_v) = \widehat{\mathcal{L}}_{\mathcal{G}_v,\mathrm{fin}}$ for every v.

   b) For every finite co-rank submodule $\Delta \subseteq \widehat{\mathcal{L}}_{\mathcal{G}}$, there exists v such that $\Delta \subseteq \widehat{U}_v$, and $\ker(\Delta \xrightarrow{\jmath_v} \widehat{\mathcal{L}}_{\mathcal{G}_v} \xrightarrow{\jmath^{\mathcal{G}_v}} \widehat{\mathrm{Div}}_{\mathcal{G}_v}) \subseteq \widehat{U}_{\mathcal{G}}$.

Note that the condition ii) above is empty in the case $\delta_{\mathcal{G}} = 1$. Thus if $\delta_{\mathcal{G}} = 1$, then condition i) is necessary and sufficient for $\mathcal{G}$ to be ample.

Next consider $0 < \delta \leq \delta_{\mathcal{G}}$. We say that $\mathcal{G}$ is ample up to level δ, if all the residual abstract decomposition graphs $\mathcal{G}_{\tilde{v}}$ for $\tilde{v}$ such that $0 \leq \delta_{\tilde{v}} < \delta$ are ample. In particular "ample up to level 1" is the same as "ample".

B) *Abstract $\mathbb{Z}_{(\ell)}$ divisor groups*

Definition 2.4. 1) Let M be the ℓ-adic completion of a free $\mathbb{Z}$-module. We say that a $\mathbb{Z}_{(\ell)}$-submodule $\mathcal{M}_{(\ell)} \subseteq M$ of M is a $\mathbb{Z}_{(\ell)}$-lattice in M, (for short, a lattice) if $\mathcal{M}_{(\ell)}$ is a free $\mathbb{Z}_{(\ell)}$-module, and it is ℓ-adically dense in M, and it satisfies the following equivalent conditions:

a) $M/\ell = \mathcal{M}_{(\ell)}/\ell$

b) $\mathcal{M}_{(\ell)}$ has a $\mathbb{Z}_{(\ell)}$-basis $\mathfrak{B}$ which is ℓ-adically independent in M.

c) Every $\mathbb{Z}_{(\ell)}$-basis of $\mathcal{M}_{(\ell)}$ is ℓ-adically independent in M.

2) Let $N, \mathcal{M}_{(\ell)} \subseteq M$ be $\mathbb{Z}_{(\ell)}$-submodules of M such that $N, M/N$ are ℓ-adically complete and torsion free. We call $\mathcal{M}_{(\ell)}$ an N-lattice in M, if $(\mathcal{M}_{(\ell)} + N)/N$ is a lattice in M/N.

3) Finally, in the context above, a true lattice in M is a free Abelian subgroup $\mathcal{M}$ of M such that $\mathcal{M}_{(\ell)} := \mathcal{M} \otimes \mathbb{Z}_{(\ell)}$ is a lattice in M in the above sense. And we will say that a $\mathbb{Z}$-submodule $\mathcal{M} \subseteq M$ is a true N-lattice in M, if $(\mathcal{M} + N)/N$ is a true lattice in M/N.

4) Next we introduce the ℓ-adic equivalence: Let M be an arbitrary $\mathbb{Z}_\ell$-module. We say that subsets M_1, M_2 of M are ℓ-adically equivalent, if there exists an ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell$ such that $M_2 = \epsilon \cdot M_1$ inside M. Further, given systems $S_1 = (x_i)_i$ and $S_2 = (y_i)_i$ of elements of M, we will say that S_1 and S_2 are ℓ-adically equivalent, if there exists an ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell$ such that $x_i = \epsilon\, y_i$ (all i).

5) We define correspondingly the ℓ-adic N-equivalence of N-lattices, etc.

Construction 2.5. Let $\mathcal{G}$ be an abstract decomposition graph on G which is *level δ complete curve like* and *ample up to level δ* for some given $\delta > 0$. Recall the last exact sequence from point 4) from Definition/Remark 2.3:

$$0 \to \widehat{U}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\mathcal{G}}} \widehat{\mathrm{Div}}_{\mathcal{G}} \xrightarrow{\mathrm{can}} \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0\,.$$

The aim of this subsection is to describe the ℓ-adic equivalence class of a lattice $\mathrm{Div}_{\mathcal{G}}$ in $\widehat{\mathrm{Div}}_{\mathcal{G}}$ —in the case it exists, which will be called an abstract divisor group of $\mathcal{G}$. By construction, this will be equivalent to giving the equivalence class of a $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$, which will turn out to be the pre-image of $\widehat{\mathrm{Div}}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

The case $\delta = 1$, i.e., $\mathcal{G}$ complete curve like and ample.

In the notations from Definition/Remark 2.3, 5) above, let $\mathfrak{T} = (\tau_v)_v$ be a distinguished system of inertia generators. Further let $\mathcal{F}_{\mathfrak{T}}$ be the Abelian pro-ℓ free group on the system $\mathfrak{T}$ (written multiplicatively). Then one has a canonical exact sequence of pro-ℓ groups

$$1 \to \tau^{\mathbb{Z}_\ell} \to \mathcal{F}_{\mathfrak{T}} \to T \to 1\,,$$

where $\tau = \prod_v \tau_v$ in $\mathcal{F}_{\mathfrak{T}}$ is the pro-ℓ product of the generators τ_v (all v). Remarking that $\mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) \cong \widehat{\mathrm{Div}}_{\mathcal{G}}$ in a canonical way, and taking ℓ-adically continuous Hom's, we get an exact sequence

$$0 \to \widehat{\mathcal{L}}_T = \mathrm{Hom}\big(T, \mathbb{Z}_\ell\big) \to \widehat{\mathrm{Div}}_{\mathcal{G}} = \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) \to \mathbb{Z}_\ell = \mathrm{Hom}\big(\tau^{\mathbb{Z}_\ell}, \mathbb{Z}_\ell\big) \to 0,$$

where the last homomorphism maps each φ to its "trace": $\varphi \mapsto \big(\tau \mapsto \sum_v \varphi(\tau_v)\big)$. Thus $\widehat{\mathcal{L}}_T$ consists of all the homomorphisms $\varphi \in \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big)$ with trivial trace.

Consider the system $\mathfrak{B} = (\varphi_v)_v$ of all the functionals $\varphi_v \in \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) = \widehat{\mathrm{Div}}_{\mathcal{G}}$ defined by $\varphi_v(\tau_w) = 1$ if $v = w$, and $\varphi_v(\tau_w) = 0$ for all $v \neq w$. We denote by

$$\mathrm{Div}_{\mathfrak{T}} = \, < \mathfrak{B} >_{(\ell)} \subset \widehat{\mathrm{Div}}_{\mathcal{G}}$$

the $\mathbb{Z}_{(\ell)}$-submodule of $\mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) = \widehat{\mathrm{Div}}_{\mathcal{G}}$ generated by $\mathfrak{B}$. Then $\mathrm{Div}_{\mathfrak{T}}$ is a lattice in $\widehat{\mathrm{Div}}_{\mathcal{G}}$, and $\mathfrak{B}$ is an *ℓ-adic basis* of $\widehat{\mathrm{Div}}_{\mathcal{G}}$. We next set

$$\mathrm{Div}^0_{\mathfrak{T}} := \{\, \textstyle\sum_v a_v\, \varphi_v \in \mathrm{Div}_{\mathfrak{T}} \mid \sum_v a_v = 0 \,\} = \mathrm{Div}_{\mathfrak{T}} \,\cap\, \widehat{\mathcal{L}}_T.$$

Clearly, $\mathrm{Div}^0_{\mathfrak{T}}$ is a lattice in $\widehat{\mathcal{L}}_T$. And moreover, the system $(e_v = \varphi_{v'} - \varphi_v)_{v' \neq v}$ is an ℓ-adic $\mathbb{Z}_{(\ell)}$-basis of $\mathrm{Div}^0_{\mathfrak{T}}$ for every fixed v.

The dependence of $\mathrm{Div}_{\mathfrak{T}}$ on $\mathfrak{T} = (\tau_v)_v$ is as follows. Let $\mathfrak{T}' = (\tau'_v)_v = \mathfrak{T}^\epsilon$ with $\epsilon \in \mathbb{Z}_\ell^\times$ be another distinguished system of inertia generators. If $\mathfrak{B}' = (\varphi'_v)_v$ is the dual basis to $\mathfrak{T}'$, then $\epsilon \cdot \mathfrak{B}' = \mathfrak{B}$. Thus $\mathfrak{B}$ and $\mathfrak{B}'$ are ℓ-adically equivalent, and we have: $\mathrm{Div}_{\mathfrak{T}} = \epsilon \cdot \mathrm{Div}_{\mathfrak{T}'}$ and $\mathrm{Div}^0_{\mathfrak{T}} = \epsilon \cdot \mathrm{Div}^0_{\mathfrak{T}'}$.

Therefore, all the subgroups of $\widehat{\mathrm{Div}_{\mathcal{G}}}$ and $\widehat{\mathcal{L}}_T$ of the form $\mathrm{Div}_{\mathfrak{T}}$, respectively $\mathrm{Div}^0_{\mathfrak{T}}$, are ℓ-adically equivalent (for all distinguished $\mathfrak{T}$). Hence the ℓ-adic equivalence classes of $\mathrm{Div}_{\mathfrak{T}}$ and $\mathrm{Div}^0_{\mathfrak{T}}$ do not depend on $\mathfrak{T}$, but only on $\mathcal{G}$.

Fact 2.6. In the above context, denote by $\mathcal{L}_{\mathfrak{T}}$ the pre-image of $\mathrm{Div}^0_{\mathfrak{T}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Further consider all the finite co-rank submodules Δ of $\widehat{\mathcal{L}}_{\mathcal{G}}$. Then the following hold:

(i) $\mathcal{L}_{\mathfrak{T}}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$, and $\mathcal{L}_{\mathfrak{T}} \subset \widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$.

(ii) $\Delta \cap \mathcal{L}_{\mathfrak{T}}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in Δ (all Δ as above).

Moreover, $\jmath^v(\mathcal{L}_{\mathfrak{T}}) = \mathbb{Z}_{(\ell)}\varphi_v$ (all v).

Proof. Clear. □

Definition 2.7. In the context of Fact 2.6 above, we define objects as follows:

1) A lattice of the form $\mathrm{Div}_{\mathfrak{T}} \subset \widehat{\mathrm{Div}_{\mathcal{G}}}$ will be called an abstract divisor group of $\mathcal{G}$. We will further say that $\mathrm{Div}^0_{\mathfrak{T}}$ is the abstract divisor group of degree 0 in $\mathrm{Div}_{\mathfrak{T}}$.

2) The $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{\mathfrak{T}}$ is called a divisorial $\widehat{U}_{\mathcal{G}}$-lattice for $\mathcal{G}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$. And we will say that $\mathcal{L}_{\mathfrak{T}}$ and $\mathrm{Div}_{\mathfrak{T}}$ correspond to each other, and that $\mathfrak{T}$ defines them.

The case: $\delta > 1$.

We begin by mimicking the construction from the case $\delta = 1$, and then conclude the construction by induction on δ. Thus let $\mathfrak{T} = (\tau_v)_v$ be any system of generators for the inertia groups T_v (all 1-edges v). Further let $\mathcal{F}_{\mathfrak{T}}$ be the Abelian pro-ℓ free group on the system $\mathfrak{T}$ (written multiplicatively). Then T is a quotient $\mathcal{F}_{\mathfrak{T}} \to T \to 1$ in a canonical way. Remarking that $\mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) \cong \widehat{\mathrm{Div}_{\mathcal{G}}}$ in a canonical way, by taking ℓ-adic Hom's we get an exact sequence

$$0 \to \mathrm{Hom}\big(T, \mathbb{Z}_\ell\big) \to \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) = \widehat{\mathrm{Div}_{\mathcal{G}}}.$$

Next let $\mathfrak{B} = (\varphi_v)_v$ be the system of all the functionals $\varphi_v \in \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big)$ defined by $\varphi_v(\tau_w) = 1$ if $v = w$, and $\varphi_v(\tau_w) = 0$ for all $v \neq w$. We denote by

$$\mathrm{Div}_{\mathfrak{T}} = <\mathfrak{B}>_{(\ell)} \subset \mathrm{Hom}\big(\mathcal{F}, \mathbb{Z}_\ell\big)$$

the $\mathbb{Z}_{(\ell)}$-submodule of $\mathrm{Hom}\big(\mathcal{F}, \mathbb{Z}_\ell\big)$ generated by $\mathfrak{B}$. Then $\mathfrak{B}$ *is an ℓ-adic basis* of $\mathrm{Hom}\big(\mathcal{F}, \mathbb{Z}_\ell\big)$, i.e., $\mathrm{Div}_{\mathfrak{T}}$ is ℓ-adically dense in $\widehat{\mathrm{Div}_{\mathcal{G}}} = \mathrm{Hom}\big(\mathcal{F}, \mathbb{Z}_\ell\big)$, and there are no non-trivial ℓ-adic relations between the elements of $\mathfrak{B}$. We will call $\mathfrak{B} = (\varphi_v)_v$ the "dual basis" to $\mathfrak{T}$, and remark that $\mathrm{Div}_{\mathfrak{T}}$ is a lattice in $\mathrm{Hom}\big(T, \mathbb{Z}_\ell\big)$.

Finally, let $\mathfrak{T}' = (\tau'_v)_v$ be another system of inertia generators, and suppose that $\mathfrak{T}' = \mathfrak{T}^\epsilon$ for some $\epsilon \in \mathbb{Z}_\ell$. If $\mathfrak{B}' = (\varphi'_v)_v$ is the dual basis to $\mathfrak{T}'$, then $\epsilon\,\varphi'_v = \varphi_v$ inside $\mathrm{Hom}\big(T, \mathbb{Z}_\ell\big)$. Thus $\epsilon\cdot\mathfrak{B}' = \mathfrak{B}$. In other words, $\mathfrak{B}$ and $\mathfrak{B}'$ are ℓ-adically equivalent, and we have: $\mathrm{Div}_{\mathfrak{T}} = \epsilon\cdot\mathrm{Div}_{\mathfrak{T}'}$.

Fact 2.8. In the notations from above, let a $\widehat{U}_{\mathcal{G}_v}$-lattice $\mathcal{L}_{\mathcal{G}_v} \subset \widehat{\mathcal{L}}_{\mathcal{G}_v}$ with $\widehat{U}_{\mathcal{G}_v} \subset \mathcal{L}_{\mathcal{G}_v}$ be given for every valuation v of $\mathcal{G}$. Then the following hold:

1) Up to ℓ-adic equivalence, there exists at most one $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ such that first, $\widehat{U}_{\mathcal{G}} \subset \mathcal{L}_{\mathcal{G}} \subset \widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$, and second, for every finite co-rank submodule $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$ and the corresponding $\Delta_v := \jmath_v(\Delta \cap \widehat{U}_v) \subset \widehat{\mathcal{L}}_{\mathcal{G}_v,\mathrm{fin}}$ the following hold:

   i) $\mathcal{L}_\Delta := \Delta \cap \mathcal{L}_{\mathcal{G}}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in Δ.

ii) $\jmath_v(\mathcal{L}_\Delta \cap \widehat{U}_v)$ is a $\widehat{U}_{\mathcal{G}_v}$-lattice in Δ_v, which is ℓ-adically $\widehat{U}_{\mathcal{G}_v}$-equivalent to $\mathcal{L}_{\mathcal{G}_v} \cap \Delta_v$.

Moreover, if the $\widehat{U}_\mathcal{G}$-lattice $\mathcal{L}_\mathcal{G}$ exists, then its ℓ-adic equivalence class depends only on the ℓ-adic equivalence classes of the $\widehat{U}_{\mathcal{G}_v}$-lattices $\mathcal{L}_{\mathcal{G}_v}$ (all v).

2) In the above context, suppose that $\mathcal{G}$ is ample, and that the $\widehat{U}_\mathcal{G}$-lattice $\mathcal{L}_\mathcal{G}$ satisfying i),ii), exists. Then $\widehat{U}_{\mathcal{G}_v} + \jmath_v(\mathcal{L}_\mathcal{G} \cap \widehat{U}_v)$ is a $\widehat{U}_{\mathcal{G}_v}$-lattice which is ℓ-adically $\widehat{U}_{\mathcal{G}_v}$-equivalent to $\mathcal{L}_{\mathcal{G}_v}$ (all v).

Proof. To 1): Let $\mathcal{L}_\mathcal{G}, \mathcal{L}'_\mathcal{G}$ be $\widehat{U}_\mathcal{G}$-lattices in $\widehat{\mathcal{L}}_\mathcal{G}$ satisfying the conditions from 1) above. Let $\Delta \in \widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$ be chosen such that $\Delta \neq (0)$ and $\Delta \cap \widehat{U}_\mathcal{G} = (0)$. Then by the ampleness of $\mathcal{G}$, it follows that there exists v such that $\Delta \subseteq \widehat{U}_v$, and $\jmath_v$ maps Δ injectively into $\widehat{\mathcal{L}}_{\mathcal{G}_v}$, and $\jmath_v(\Delta) \cap \widehat{U}_{\mathcal{G}_v} = (0)$.

For Δ as above, set $\mathcal{L}'_\Delta = \Delta \cap \mathcal{L}'_\mathcal{G}$. Then by hypothesis i), it follows that $\mathcal{L}_\Delta$ and $\mathcal{L}'_\Delta$ are both $\widehat{U}_\mathcal{G}$-lattices in Δ. And since $\Delta = \Delta \cap \widehat{U}_v$, by hypothesis ii) it follows that $\jmath_v(\mathcal{L}_\Delta)$ and $\jmath_v(\mathcal{L}'_\Delta)$ are both lattices in $\Delta_v = \jmath_v(\Delta)$, which are $\widehat{U}_{\mathcal{G}_v}$-equivalent to the $\widehat{U}_{\mathcal{G}_v}$-lattice $\mathcal{L}_{\mathcal{G}_v} \cap \Delta_v$. Therefore, there exists $\epsilon = \epsilon_{\Delta,v} \in \mathbb{Z}_\ell^\times$ such that $\jmath_v(\mathcal{L}'_\Delta) = \epsilon \cdot \jmath_v(\mathcal{L}_\Delta)$. Now since $\jmath_v$ is an isomorphism on Δ, the above equality implies that $\mathcal{L}'_\Delta = \epsilon \cdot \mathcal{L}_\Delta$. Equivalently, $\mathcal{L}'_\Delta$ and $\mathcal{L}_\Delta$ are equivalent lattices in Δ. On the other hand, since

$$(*) \qquad\qquad \mathcal{L}_\mathcal{G} = \cup_\Delta \mathcal{L}_\Delta \quad \text{and} \quad \mathcal{L}'_\mathcal{G} = \cup_\Delta \mathcal{L}'_\Delta\,,$$

one immediately gets that $\mathcal{L}'_\mathcal{G} = \epsilon \cdot \mathcal{L}'_\mathcal{G}$, as claimed.

To 2): First, since $\mathcal{L}_\mathcal{G} = \cup_\Delta \mathcal{L}_\Delta$ as at $(*)$ above, it follows from the hypothesis i), ii), that $\widehat{U}_{\mathcal{G}_v} + \jmath_v(\mathcal{L}_\mathcal{G} \cap \widehat{U}_v)$ is ℓ-adically equivalent to some $\widehat{U}_{\mathcal{G}_v}$-sublattice of $\mathcal{L}_{\mathcal{G}_v}$, as this is the case for all the $\widehat{U}_{\mathcal{G}_v} + \jmath_v(\mathcal{L}_\Delta \cap \widehat{U}_v)$. After replacing $\mathcal{L}_{\mathcal{G}_v}$ by some properly chosen ℓ-adic multiple, say $\epsilon \cdot \mathcal{L}_{\mathcal{G}_v}$ with $\epsilon \in \mathbb{Z}_\ell^\times$, without loss of generality, we can suppose that $\jmath_v(\mathcal{L}_\mathcal{G} \cap \widehat{U}_v) \subseteq \mathcal{L}_{\mathcal{G}_v}$, thus $\widehat{U}_{\mathcal{G}_v} + \jmath_v(\mathcal{L}_\mathcal{G} \cap \widehat{U}_v) \subseteq \mathcal{L}_{\mathcal{G}_v}$. For the converse inclusion, let $\Gamma \subseteq \widehat{\mathcal{L}}_{\mathcal{G}_v}$ be a finite co-rank submodule. Then by the ampleness of $\mathcal{G}$, see Definition/Remark 2.3, 7), ii), there exists a finite co-rank submodule $\Delta \subseteq \widehat{\mathcal{L}}_\mathcal{G}$ such that $\Gamma \subseteq \widehat{U}_{\mathcal{G}_v} + \jmath_v(\Delta \cap \widehat{U}_v)$. But then by the properties i), ii), we get: $\Gamma \cap \mathcal{L}_{\mathcal{G}_v} \subseteq \widehat{U}_{\mathcal{G}_v} + \jmath_v(\mathcal{L}_\Delta \cap \widehat{U}_v) \subseteq \widehat{U}_{\mathcal{G}_v} + \jmath_v(\mathcal{L}_\mathcal{G} \cap \widehat{U}_v)$. Since Γ was arbitrary, and $\mathcal{L}_{\mathcal{G}_v} = \widehat{U}_{\mathcal{G}_v} + \cup_\Gamma(\Gamma \cap \mathcal{L}_{\mathcal{G}_v})$, the converse inclusion follows. ☐

Let $\mathcal{G}$ be an abstract decomposition graph which is both level δ complete curve like and ample up to level δ for some $\delta > 1$. In particular, all residual abstract decomposition graphs $\mathcal{G}_{\tilde{v}}$ to non-trivial indices $\tilde{v}$ of length $\delta_{\tilde{v}} < \delta$ are both *level* $(\delta - \delta_{\tilde{v}})$ *complete curve like and ample up to level* $(\delta - \delta_{\tilde{v}})$; and if $\delta_{\tilde{v}} = \delta - 1$, then $\mathcal{G}_{\tilde{v}}$ is complete curve like and ample. Hence if $\delta_{\tilde{v}} = \delta - 1$, then $\mathcal{G}_{\tilde{v}}$ has an abstract divisor group $\mathrm{Div}_{\mathcal{G}_{\tilde{v}}}$ as defined/introduced in Definition 2.7. In the above context, let us fix notations as follows:

Definition 2.9. In the above context, we define an abstract divisor group of $\mathcal{G}$ (if it exists) to be the lattice defined by any particular system $\mathfrak{T}$ of inertia generators as above

$$\mathrm{Div}_\mathcal{G} := \mathrm{Div}_\mathfrak{T} \subset \widehat{\mathrm{Div}}_\mathcal{G}\,,$$

which together with its preimage $\mathcal{L}_\mathcal{G}$ in $\widehat{\mathcal{L}}_\mathcal{G}$ satisfies inductively on δ the following:

i) Abstract divisor groups $\mathrm{Div}_{\mathcal{G}_v}$ exist for all residual abstract decomposition graphs $\mathcal{G}_v$. Let $\mathcal{L}_{\mathcal{G}_v}$ be the preimage of $\mathrm{Div}_{\mathcal{G}_v}$ in $\widehat{\mathcal{L}}_{\mathcal{G}_v}$ (all v).

ii) $\mathcal{L}_{\mathcal{G}}$ satisfies conditions i), ii) from Fact 2.8 for all finite co-rank submodules $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ with respect to the pre-images $\mathcal{L}_{\mathcal{G}_v}$ defined at i) above.

- Note that $\mathcal{L}_{\mathcal{G}} \subset \widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$ by its very definition: If $x \in \mathcal{L}_{\mathcal{G}}$, then $\jmath^v(x) = 0$ for almost all v.

Remark 2.10. Let $\mathcal{G}$ be an abstract decomposition graph which is level δ complete curve like and ample up to level δ for some $\delta > 0$. Suppose that an abstract divisor group $\mathrm{Div}_{\mathcal{G}} := \mathrm{Div}_{\mathfrak{X}}$ for $\mathcal{G}$ exists, and let $\mathcal{L}_{\mathcal{G}}$ be its pre-image in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Then one has:

1) The canonical homomorphism $\jmath^v : \widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big) \xrightarrow{\mathrm{res}_v} \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big) = \mathbb{Z}_\ell \varphi_v$ gives rise by restriction to a surjective homomorphism

$$\jmath^v : \mathcal{L}_{\mathcal{G}} \to \mathbb{Z}_{(\ell)}\, \varphi_v\,.$$

Indeed, by condition i) of the ampleness, see Definition/Remark 2.3, 7), it follows that $\jmath^v(\widehat{\mathcal{L}}_{\mathcal{G}}) = \mathbb{Z}_\ell\, \varphi_v$. Further, since $\mathcal{L}_{\mathcal{G}}$ is ℓ-adically dense in $\widehat{\mathcal{L}}_{\mathcal{G}}$, it follows that $\jmath^v(\mathcal{L}_{\mathcal{G}})$ is dense in $\mathbb{Z}_\ell\, \varphi_v$. Thus the assertion.

2) Moreover, the $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{\mathcal{G}}$ endowed with all the valuation homomorphisms $\jmath^v$ determines $\mathrm{Div}_{\mathcal{G}}$, as being the additive subgroup

$$\mathrm{Div}_{\mathcal{G}} = \textstyle\sum_v \mathbb{Z}_{(\ell)}\varphi_v = \sum_v \jmath^v(\mathcal{L}_{\mathcal{G}}) \subset \widehat{\mathrm{Div}}_{\mathcal{G}}$$

generated by the $\jmath^v(\mathcal{L}_{\mathcal{G}})$ for all the v. Therefore, giving an abstract divisor group $\mathrm{Div}_{\mathcal{G}}$, is *equivalent* to giving a $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ such that inductively we have:

i) $\mathcal{L}_{\mathcal{G}}$ satisfies the conditions i), ii) from Fact 2.8 with respect to the pre-images $\mathcal{L}_{\mathcal{G}_v}$ of some abstract divisor groups $\mathrm{Div}_{\mathcal{G}_v}$ (all v).

ii) $\jmath^v(\mathcal{L}_{\mathcal{G}}) \cong \mathbb{Z}_{(\ell)}$ (all v), and $\mathcal{L}_{\mathcal{G}}$ is the pre-image of $\oplus_v \jmath^v(\mathcal{L}_{\mathcal{G}})$ via $\jmath^{\mathcal{G}}$.

3) Finally, for an abstract divisor group $\mathrm{Div}_{\mathcal{G}}$ for $\mathcal{G}$, and its pre-image $\mathcal{L}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$, we set $\mathfrak{Cl}_{\mathcal{L}_{\mathcal{G}}} = \mathrm{Div}_{\mathcal{G}}/\jmath^{\mathcal{G}}(\mathcal{L}_{\mathcal{G}})$, and call it the abstract ideal class group of $\mathcal{L}_{\mathcal{G}}$. Thus one has a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & \widehat{U}_{\mathcal{G}} \arrow[r, hook] \arrow[d] & \mathcal{L}_{\mathcal{G}} \arrow[r, "\jmath^{\mathcal{G}}"] \arrow[d] & \mathrm{Div}_{\mathcal{G}} \arrow[r, "\mathrm{can}"] \arrow[d] & \mathfrak{Cl}_{\mathcal{G}} \arrow[r] \arrow[d, "\jmath_{\mathfrak{Cl}}"] & 0 \\
0 \arrow[r] & \widehat{U}_{\mathcal{G}} \arrow[r, hook] & \widehat{\mathcal{L}}_{\mathcal{G}} \arrow[r, "\jmath^{\mathcal{G}}"] & \widehat{\mathrm{Div}}_{\mathcal{G}} \arrow[r, "\mathrm{can}"] & \widehat{\mathfrak{Cl}}_{\mathcal{G}} \arrow[r] & 0
\end{tikzcd} \qquad (*)$$

where the first three vertical morphisms are the canonical inclusions, and the last one is the ℓ-adic completion homomorphism.

Proposition 2.11. *Let $\mathcal{G}$ be an abstract decomposition graph which is level δ complete curve like and ample up to level $\delta > 0$. Then any two abstract divisor groups $\mathrm{Div}_{\mathcal{G}}$ and $\mathrm{Div}'_{\mathcal{G}}$ for $\mathcal{G}$ are ℓ-adically equivalent as lattices in $\widehat{\mathrm{Div}}_{\mathcal{G}}$. Equivalently, their pre-images $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}'_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ are ℓ-adically equivalent $\widehat{U}_{\mathcal{G}}$-lattices in $\widehat{\mathcal{L}}_{\mathcal{G}}$.*

Proof. We prove this assertion by induction on δ. For $\delta = 1$, the uniqueness is already shown, see Fact 2.6, and Definition in case $\delta = 1$. Now suppose that $\delta > 1$. Let $\mathrm{Div}_{\mathcal{G}_v}$ and $\mathrm{Div}'_{\mathcal{G}_v}$ be abstract divisor groups for $\mathcal{G}$ used for the definition of $\mathrm{Div}_{\mathcal{G}}$, respectively $\mathrm{Div}'_{\mathcal{G}}$ (all v). By the induction hypothesis, $\mathrm{Div}_{\mathcal{G}_v}$ and $\mathrm{Div}'_{\mathcal{G}_v}$ are ℓ-adically equivalent. Thus their pre-images $\mathcal{L}_v$ and $\mathcal{L}'_v$ in $\widehat{\mathcal{L}}_{\mathcal{G},v}$ are ℓ-adically equivalent $\widehat{U}_{\mathcal{G},v}$-lattices. Therefore, by Fact 2.8, the lattices $\mathcal{L}_{\mathcal{G}}$

and $\mathcal{L}'_{\mathcal{G}}$ (which are the pre-images of $\mathrm{Div}_{\mathcal{G}}$ respectively $\mathrm{Div}'_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$) are ℓ-adically equivalent. Finally, use Remark 2.10, 2), above to conclude. $\square$

Definition 2.12. Let $\mathcal{G}$ be an abstract decomposition graph which is level δ complete curve like and ample up to level δ. We will say that $\mathcal{G}$ is a divisorial abstract decomposition graph, if it has abstract divisor groups $\mathrm{Div}_{\mathcal{G}} = \mathrm{Div}_{\mathfrak{T}}$ as introduced above. If this is the case, we will denote by $\mathcal{L}_{\mathcal{G}}$ the pre-image of $\mathrm{Div}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$, and call it a divisorial $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

C) *Example: abstract decomposition graphs arising from algebraic Geometry*

Let $K|k$ be a function field over an algebraically closed field k with $\mathrm{char}(k) \neq \ell$. Generalizing the geometric prime divisor graphs from the Introduction, we define a level δ geometric prime divisor graph for $K|k$ as being a (half) oriented graph $\mathcal{D}_K$ defined as follows:

I) The vertices of $\mathcal{D}_K$ are distinct function fields $K_i|k$ over k. And $\mathcal{D}_K$ has an origin which is $K_0 := K$.

II) For every vertex K_i, the trivial valuation v_{i0} of K_i is the only edge from K_i to itself, and we view this edge as a non-oriented one, or a trivial edge. Further, the set of all the oriented edges starting at K_i is a geometric set D_i of prime divisors $v_{i'}$ of K_i. We call these edges non-trivial, and if $v_{i'} \in D_i$ is such a nontrivial edge from K_i to $K_{i'}$, then $K_{i'} = K_i v_{i'}$. In particular, $\mathrm{tr.deg}(K_{i'}|k) = \mathrm{tr.deg}(K_i|k) - 1$.

III) All the maximal branches of non-trivial edges of $\mathcal{D}_K$ have length equal to δ, hence $\delta \leq \mathrm{tr.deg}(K|k)$.

As indicated in the Introduction, using Hilbert decomposition theory for valuations, we attach to $\mathcal{D}_K$ the corresponding subgraph $\mathcal{G}_{\mathcal{D}_K} \subset \mathcal{G}_{\mathrm{Val}_K}$. Hence by definition one has:

I) The vertices of $\mathcal{G}_{\mathcal{D}_K}$ are in bijection with the vertices of $\mathcal{D}_K$, via the Galois correspondence, i.e., the vertices of $\mathcal{G}_{\mathcal{D}_K}$ are the pro-ℓ groups G'_{K_i} with K_i vertex of $\mathcal{D}_K$. In particular, $G'_{K_0} := G'_K$ is the origin of $\mathcal{G}_{\mathcal{D}_K}$.

II) The trivial edge v_{i0} from K_i to itself is endowed with $\{1\} =: T_{v_{i0}} \subset Z_{v_{i0}} := G'_{K_i}$, i.e., with $\{1\} \subset G'_{K_i}$. The set of non-trivial edges originating from G'_{K_i} equals the set D_i of non-trivial edges $v_{i'}$ originating from K_i, and $v_{i'} \in D_i$ is endowed with the inertia/decomposition groups $T_{v_{i'}} \subseteq Z_{v_{i'}}$. In particular, if $v_{i'}$ is an edge from K_i to $K_{i'} = K_i v_{i'}$, then $G'_{K_{i'}} = Z_{v_{i'}}/T_{v_{i'}}$.

III) All the maximal branches originating from G'_{K_0} and consisting of non-trivial edges of $\mathcal{G}_{\mathcal{D}_K}$ have length δ.

Proposition 2.13. *In the above notations, $\mathcal{G}_{\mathcal{D}_K}$ is a level δ abstract decomposition graph.*

Proof. Indeed, all the axioms of an abstract decomposition graph are more or less well known facts concerning Hilbert decomposition theory for valuations. For instance, if $v_{i'}$ is a valuation of K_i, then all the prolongations $v'_{i'}$ of $v_{i'}$ to K'_i are conjugated under G'_{K_i}, and therefore, their inertia, respectively decomposition, groups are equal; say equal to $T_{v_{i'}} \subseteq Z_{v_{i'}}$. Further, $T_{v_{i'}} \cong \mathbb{Z}_\ell$, and the residue field $K'_i v'_{i'}$ equals $(K_i v_{i'})'$, thus $G_{i'} = (K_{i'} v_{i'})' = Z_{v_{i'}}/T_{v_{i'}}$, etc. Moreover, for Zariski prime divisors $v_{i'} \neq w_{i'}$ of $K_i|k$ one has the following, see e.g. Pop [P4], Introduction, and especially Proposition 2.5, 2): The decomposition groups $Z_{v_{i'}}$ and $Z_{w_{i'}}$ have trivial intersection. And finally, if $X_i \to k$ is a normal quasi-projective variety such that $D_i = D_{X_i}$ is the set of Weil prime divisors of X_i, then every open subgroup of G'_{K_i}

contains almost all inertia groups $T_{v_{i'}}$. Indeed, in every finite extension of K_i only finitely Weil prime divisors of X_i are ramified, etc. $\square$

Remarks 2.14. Let $\mathcal{G} := \mathcal{G}_{\mathcal{D}_K}$ be a level δ abstract decomposition graph as introduced above. Then by Kummer Theory we have:

1) $\widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{Hom}\big(G'_K, \mathbb{Z}_\ell\big) = \widehat{K}$.

In order to compute $\widehat{U}_{\mathcal{G}}$, $\widehat{\mathrm{Div}}_{\mathcal{G}}$, and $\widehat{\mathfrak{Cl}}_{\mathcal{G}}$, we do the following: First let $X \to k$ be a quasi-projective normal model of $K|k$ such that its set of Weil prime divisors D_X equals the set of non-trivial 1-edges v of $\mathcal{G}$. Let $\mathcal{H}(X)$ denote the group of principal divisors of X, and consider the canonical exact sequence $0 \to \mathcal{H}(X) \longrightarrow \mathrm{Div}(X) \xrightarrow{pr} \mathfrak{Cl}\,(X) \to 0$. Then passing to ℓ-adic completions, we get an exact sequence of ℓ-adic complete groups of the form:

$$0 \to \mathbb{T}_{\ell,X} \longrightarrow \widehat{\mathcal{H}}(X) \longrightarrow \widehat{\mathrm{Div}}(X) \longrightarrow \widehat{\mathfrak{Cl}}\,(X) \to 0\,,$$

where $\mathbb{T}_{\ell,X} = \varprojlim_n {}_n\mathfrak{Cl}\,(X)$, $n = \ell^e$, is the ℓ-adic Tate module of $\mathfrak{Cl}\,(X)$.

2) Note that for every prime divisor v of $K|k$ one has a commutative diagram of the form:

$$(*)_v \qquad \begin{tikzcd}
\widehat{K} \arrow[r, "v"] \arrow[d] & \widehat{vK} \arrow[d, "\theta^v"] \\
\mathrm{Hom}\big(G'_K, \mathbb{Z}_\ell\big) \arrow[r, "\jmath^v"] & \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big)
\end{tikzcd}$$

This diagram gives rise to a canonical inertia generator $\tau_v \in T_v$, which we call the **arithmetical inertia generator**, and which is defined as follows: Let γ_v be the unique positive generator of $vK = \mathbb{Z}v$. Then $\tau_v \in T_v$ is the unique generator of T_v such

$$\theta^v(\gamma_v)\big(\tau_v\big) = 1\,.$$

The diagrams $(*)_v$ with $v \in D_X$ give rise canonically to a commutative diagram:

$$\begin{tikzcd}
\widehat{K} \arrow[r] \arrow[d] & \widehat{\mathrm{Div}}(X) = \widehat{\oplus}_v\, vK \arrow[r] \arrow[d, "\widehat{\oplus}\theta^v"] & \widehat{\mathfrak{Cl}}\,(X) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}\big(G'_K, \mathbb{Z}_\ell\big) \arrow[r, "\jmath^{\mathcal{G}}"] & \widehat{\oplus}_v\, \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big) \arrow[r, "\mathrm{can}"] & \widehat{\mathcal{P}}_X \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are isomorphisms, and $\widehat{\mathcal{P}}_X$ is simply the quotient of the middle group by the first one.

3) From this we deduce: $\widehat{\mathrm{Div}}_{\mathcal{G}} = \widehat{\mathrm{Div}}(X)$ and $\widehat{\mathfrak{Cl}}_{\mathcal{G}} = \widehat{\mathcal{P}}_X = \widehat{\mathfrak{Cl}}(X)$.

Concerning $\widehat{U}_{\mathcal{G}}$, the situation is a little bit more complicated. Let me recall that we defined $\widehat{U}_{\mathcal{G}} = \mathrm{Hom}\big(\pi'_{1,\mathcal{G}}, \mathbb{Z}_\ell\big)$, where $\pi'_{1,\mathcal{G}} := G'_K/T$ and T is the group generated by all the inertia groups T_v with $v \in D_X$. In geometrical terms, this means the following: $\pi'_{1,\mathcal{G}}$ classifies all the generically finite integral ℓ-abelian Galois covers $Y \to X$ which are not ramified at any $v \in D_X$. In particular, we have a canonical projection

$$\pi'_{1,\mathcal{G}} \to \pi'_1(X) := \pi_1^{\ell,\mathrm{ab}}(X)$$

which is not injective is in general. Nevertheless, if X is regular, then by the *purity of the branch locus,* it follows that $\pi'_{1,\mathcal{G}} = \pi'_1(X)$.

4) Let D^1_K be the set of all prime divisors of $K|k$. For $D \subseteq D^1_K$ a subset, let $T_D \subseteq G'_K$ be the subgroup generated by all the T_v with $v \in D$, and set

$$\pi'_{1,D} := G'_K / T_D \,.$$

In particular, $\pi'_{1,K} := \pi'_{1,D^1_K}$ is a birational invariant of K, which we will call the abelian pro-ℓ birational fundamental group for K. We note the following:

i) For every $D \subseteq D^1_K$ there is a canonical (obvious) surjective projection $\pi'_{1,D} \to \pi'_{1,K}$.

ii) If $X \to k$ is a quasi-projective normal model of $K|k$, then there is a canonical surjective projection $\pi'_{1,D_X} \to \pi'_1(X)$ which is not necessarily an isomorphism.

iii) Nevertheless, if in the context of ii) above X is regular, then $\pi'_{1,D_X} \to \pi'_1(X)$ is an isomorphism by the purity of the branch locus.

Now let us show the following:

Fact 2.15. In the above context, the following hold:

a) If $X \to k$ is complete normal model of $K|k$, then there exists a canonical surjective projection $\pi'_{1,K} \to \pi'_1(X)$.

b) For every given geometric subset $D_{X'} \subset D^1_K$, there exist geometric sets $D_X \subset D^1_K$ such that $D_{X'} \subseteq D_X$ and the canonical projection $\pi'_{1,D_X} \to \pi'_{1,K}$ is an isomorphism.

c) If X is a affine normal curve, and $\pi'_{1,D_X} \cong \pi'_{1,K}$, then either $X \cong \mathbb{A}^1_k$, or $X \cong E\backslash\{\text{pt}\}$ with E an elliptic curve.

In particular, $\pi'_{1,K}$ as well as all the π_{1,D_X} are always finitely generated.

Proof. First, one gets a) easily by the fact that a base change of an étale cover is étale. Now since $X \to k$ is a complete variety, for every $v \in D^1_K$, there exists a dominant canonical k-morphism $\operatorname{Spec} \mathcal{O}_v \to X$. If $X' \to X$ is some étale cover classified by $\pi'_1(X)$, then X' is integral, and $X' \times_X \operatorname{Spec} \mathcal{O}_v$ is étale over $\mathcal{O}_v$. Equivalently, v is unramified in the field extension $k(X) \hookrightarrow k(X')$, hence the image of the inertia group T_v in $\pi'_1(X)$ is trivial.

For the assertion b), first consider a small enough affine open subset $X_0 \subset X'$ such that X_0 is regular. Then $X_0 \to k$ is a quasi-projective regular model for $K|k$, hence $\pi'_{1,D_{X_0}} = \pi'_1(X_0)$, by the purity of the branch locus. Therefore, $\pi'_{1,D_{X_0}}$ is finitely generated, hence a finite $\mathbb{Z}_\ell$-module as it is an abelian pro-ℓ group. Since $\pi'_{1,D_{X_0}} \to \pi'_{1,K}$ is surjective, $\pi'_{1,K}$ is a finite $\mathbb{Z}_\ell$-module too. But then

$$\Delta = \ker(\pi'_{1,D_{X_0}} \to \pi'_{1,K})$$

is also a finite $\mathbb{Z}_\ell$-module. Finally, by the definition of $\pi'_{1,K}$, it follows that for every $g \in \Delta$ there exists some $v \in D^1_K$ such that $g \in T_v$. Since Δ is finitely generated, there exists a finite set $\Sigma \subset D^1_K$ such that the images of T_v (all $v \in \Sigma$) in $\pi'_{1,D_{X_0}}$ generate Δ. In order to conclude, consider any quasi-projective normal model $X \to k$ such that $D_{X'}, \Sigma \subseteq D_X$ (hence in particular, $D_{X_0} \subseteq D_X$ too).

Finally, assertion c) follows immediately from the structure theorem for (the Abelian pro-ℓ quotient of the) fundamental groups of a normal curve. $\square$

Definition 2.16. 1) We will say that a normal quasi-projective model $X \to k$ for $K|k$ is regular complete like, if $\pi'_{1,D_X} = \pi'_{1,K}$. And we will say that a geometric set D_X of prime divisors is regular complete like, if $X \to k$ is so.

2) Let $\mathcal{D}_K$ be a level δ geometric graph of prime divisors for $K|k$. For each vertex $\tilde{v}$ of $\mathcal{D}_K$, let $X_{\tilde{v}} \to k$ be a quasi-projective model of $K\tilde{v} \,|\, k$ such that $D_{X_{\tilde{v}}}$ is the set of non-trivial edges originating from $K\tilde{v}$. We say that $\mathcal{D}_K$ is regular complete like, for all $\tilde{v}$ one has: $X_{\tilde{v}} \to k$ is regular complete like if $\dim(X_i) > 1$, and $X_i \to k$ is a complete normal curve if $\dim(X_i) = 1$.

Proposition 2.17. *In the above notations and context one has:*

1) *$\mathcal{G}_{\mathcal{D}_K}$ is level δ complete curve like if (and only if) $\delta = \operatorname{tr.deg}(K|k)$, <u>and</u> each δ-residual variety $X_{\tilde{v}}$ is a complete normal curve.*

2) *Let $\mathcal{G}_{\mathcal{D}_K}$ be complete regular like. Then $\mathcal{G}_{\mathcal{D}_K}$ is ample up to level δ.*

3) *Suppose that $\delta = \operatorname{tr.deg}(K|k)$, and let $\mathcal{G}_{\mathcal{D}_K}$ be complete regular like. Then $\mathcal{G}_{\mathcal{D}_K}$ is a divisorial abstract decomposition graph, and $\operatorname{Div}(X)_{(\ell)} := \operatorname{Div}(X) \otimes \mathbb{Z}_{(\ell)} \subset \widehat{\operatorname{Div}}(X)$ is an abstract divisor group of $\mathcal{G}_{\mathcal{D}_K}$, which we call the canonical abstract divisor group of $\mathcal{G}_{\mathcal{D}_K}$.*

Finally, $\widehat{U}_{\mathcal{G}_{\mathcal{D}_K}}$ is the ℓ-adic dual of $\pi'_{1,K}$, and the preimage $\mathcal{L}_X$ of $\operatorname{Div}(X)_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}} = \widehat{K}$ will be called the canonical divisorial $\widehat{U}_{\mathcal{G}_{\mathcal{D}_K}}$-lattice of $\mathcal{G}_{\mathcal{D}_K}$.

Proof. To simplify notations set $\mathcal{G} := \mathcal{G}_{\mathcal{D}_K}$.

To (1): Clear by the structure of G'_L in the case $L|k$ is a function field of a curve. It is nevertheless more/quite difficult to prove the "only if" part of (1), which we will not directly use, thus omit the proof here.

To (2): We make induction on $d = \operatorname{tr.deg}(K|k)$. In the notations from Definition/Remarks 2.3, 3), let $\Delta \subset \widehat{K}$ be a finite co-rank submodule such that $\Delta \cap \widehat{U}_{\mathcal{G}} = 0$. Then there exists an open affine subset $X' \subset X$ such that for all $v \in \mathcal{D}_{X'}$ one has: $\Delta \subseteq \widehat{U}_v$. Hence the canonical projection $\pi_1(X') \to \pi_1(X)$ gives rise to an embedding $\widehat{U}(X) \hookrightarrow \widehat{U}(X')$, and $\Delta \subset \widehat{U}(X')$. Using de Jongs's alterations, and the inclusion/norm maps for divisors and divisor class groups, w.l.o.g. we can suppose that X is a smooth k-variety. Finally, let $X_1, \dots, X_n$ be the finitely many Weil prime divisors in $X \backslash X'$, and $\Sigma = \{x_1, \dots, x_n\} \subset D^1_K$ be their generic points. Then the $T_{v_{x_m}}$ with $x_m \in \Sigma$ generate $T_{X',X} := \ker\big(\pi_1(X') \to \pi_1(X)\big)$.

Now let $\imath : X \to \mathbb{P}^N_k$ be some k-embedding. For a general hyper-plane $H \subset \mathbb{P}^N_k$, we set $Y = H \cap X$, $Y' = H \cap X'$, $Y_m = X_m \cap H$ $(1 \le m \le n)$. Then by a Bertini type argument, each Y_m is a prime divisor of Y. Let $T := \{y_1, \dots, y_n\}$ be the set of the generic points of the Y_m $(1 \le m \le n)$. By Bertini we have:

i) The canonical projections $\pi_1(Y) \to \pi_1(X)$ and $\pi_1(Y') \to \pi_1(X')$ are surjective.

ii) Set $T_{Y',Y} := \ker\big(\pi_1(Y') \to \pi_1(Y)\big)$. Then $T_{Y',Y}$ is generated by the inertia groups at all the y_m $(1 \le m \le n)$.

iii) Finally, $T_{Y',Y}$ is mapped surjectively onto $T_{X',X}$ under the projection $\pi_1(Y') \to \pi_1(X')$ from i) above.

This is now exactly the translation of the fact that for the Zariski prime divisor v of K defined by the Weil prime divisor $Y = H \cap X$ of X, the assertion from Definition/Remarks 2.3, 3), holds for Δ at v.

To (3): It follows immediately from (1) and (2) above. $\square$

3. FROM ABSTRACT DECOMPOSITION GRAPHS TO FIELDS

A) *Morphisms of abstract decomposition graphs*

Let $\mathcal{G}$ and $\mathcal{H}$ be given abstract decomposition graphs of levels $\delta_{\mathcal{G}}$ and $\delta_{\mathcal{H}}$, based on $G = G_0$, respectively $H = H_0$. We denote as usually by $T_v \subset Z_v$ and $G_v = Z_v/T_v$ the 1-edges, respectively the 1-vertices of $\mathcal{G}$, and correspondingly by $T_w \subset Z_w$ and $G_w = Z_w/T_w$ the ones for $\mathcal{H}$. Further, $\mathcal{G}_v$ and $\mathcal{H}_w$ are the corresponding 1-residual abstract decomposition graphs, which have then level $\delta_{\mathcal{G}} - 1$, respectively $\delta_{\mathcal{H}} - 1$. We also recall that v_0 and w_0 are the trivial valuations of G, respectively H, and that their inertia groups are trivial by definition.

Definition/Remark 3.1. 1) In the context above, let $\Phi : G_0 \to H_0$ be a (continuous) group homomorphism. Let $\tilde{v}$ and $\tilde{w}$ be multi-indices for $\mathcal{G}$ and $\mathcal{H}$. We define inductively on the length of $\tilde{v}$ the fact that $\tilde{w}$ corresponds to $\tilde{v}$ via Φ as follows, see Definition/Remark 2.2, especially loc.cit. 3 and 4), to recall notations:

i) The trivial multi-index $\tilde{w} = w_0$ corresponds to $\tilde{v}$ if and only if $\Phi(T_{\tilde{v}}) = 1$ and $\Phi(Z_{\tilde{v}})$ is open in H_0. And the only $\tilde{w}$ which corresponds to the trivial multi-index $\tilde{v} = v_0$ is the trivial multi-index $\tilde{w} = w_0$.

ii) Suppose that $\tilde{w} = (w_s, \dots, w_1)$ and $\tilde{v} = (v_r, \dots, v_1)$ are both non-trivial, and let us set $\tilde{v} = (\tilde{v}_1, v_1)$ and $\tilde{w} = (\tilde{w}_1, w_1)$ with $\tilde{v}_1$ and $\tilde{w}_1$ the corresponding multi-indices for the residual abstract decomposition graphs $\mathcal{G}_{v_1}$, respectively $\mathcal{H}_{w_1}$. (Note that $\tilde{v}_1$ and/or $\tilde{w}_1$ might be trivial.) Then we say that $\tilde{w}$ corresponds to $\tilde{v}$ if and only if one of the following hold:

- a) If $\Phi(T_{v_1}) = 1$, then under the induced homomorphism $\Phi_{v_1} : G_{v_1} = Z_{v_1}/T_{v_1} \to H_0$, inductively one has: $\tilde{w}$ corresponds to $\tilde{v}_1$.
- b) If $\Phi(T_{v_1}) \neq 1$, then $\Phi(T_{v_1}) \subseteq T_{w_1}$ and $\Phi(Z_{v_1}) \subseteq T_{w_1}$ are open subgroups, and under the induced homomorphism $\Phi_{v_1} : G_{v_1} = Z_{v_1}/T_{v_1} \to Z_{w_1}/T_{w_1} = H_{w_1}$ inductively one has: $\tilde{w}_1$ corresponds to $\tilde{v}_1$.

2) We remark that if $\tilde{w}$ corresponds to some $\tilde{v}$, and $\tilde{w}'$ is a truncation of $\tilde{w}$, then $\tilde{w}'$ also corresponds to some truncation $\tilde{v}'$ of $\tilde{v}$. The proof follows immediately by induction on the length of $\tilde{w}$, and we will omit it here.

3) Finally, let $\mathrm{Vert}_{\mathcal{G}}$ and $\mathrm{Vert}_{\mathcal{H}}$ be the sets of all the vertices of $\mathcal{G}$, respectively $\mathcal{H}$, and let $\mathrm{Vert}_{\mathcal{G},\Phi} \subseteq \mathrm{Vert}_{\mathcal{G}}$ be the set of all the vertices $\tilde{v} \in \mathrm{Vert}_{\mathcal{G}}$ such that there exists some vertex $\tilde{w}_{\tilde{v}} \in \mathrm{Vert}_{\mathcal{H}}$ which corresponds to $\tilde{v}$. Then the correspondence relation introduced at 1) above gives rise to a well defined map $\varphi_\Phi : \mathrm{Vert}_{\mathcal{G},\Phi} \to \mathrm{Vert}_{\mathcal{H}}, \quad \tilde{v} \mapsto \varphi_\Phi(\tilde{v}) = \tilde{w} := \tilde{w}_{\tilde{v}}$.

If $\varphi_\Phi(\tilde{v}) = \tilde{w}$, we also say that Φ maps $\tilde{v}$ to $\tilde{w}$, or that $\tilde{w}$ is the image of $\tilde{v}$ under Φ.

Definition 3.2. In the above notations, let δ be an integer satisfying $0 \le \delta \le \delta_{\mathcal{G}}, \delta_{\mathcal{H}}$. We define a level δ morphism $\Phi : \mathcal{G} \to \mathcal{H}$ inductively on δ and $\delta_{\mathcal{G}}$ as follows:

1) A level $\delta = 0$ morphism $\Phi : \mathcal{G} \to \mathcal{H}$ is any group homomorphism $\Phi : G \to H$ under which w_0 corresponds to v_0. Equivalently, Φ is open.

2) A level $\delta > 0$ morphism $\Phi : \mathcal{G} \to \mathcal{H}$ is any group homomorphism $\Phi : G \to H$ which is open, and inductively on $\delta_{\mathcal{G}}$ and on δ has further properties as follows:

- i) Almost all 1-vertices of $\mathcal{H}$ correspond to some 1 vertices of $\mathcal{G}$, and every 1-vertex of $\mathcal{H}$ corresponds to only finitely many (maybe to none) of the 1-vertices of $\mathcal{G}$.

ii) If w_0 corresponds to a 1-vertex v, then $\delta_{\mathcal{G}_v} = \delta_{\mathcal{G}} - 1 \geq \delta$, and the canonical group homomorphism $\Phi_v : G_v = Z_v/T_v \to H_0$ defines a level δ morphism of the corresponding residual abstract decomposition graphs $\mathcal{G}_v$ and $\mathcal{H}$.

iii) If w is a 1-vertex corresponding to the 1-vertex v, then the canonical group homomorphism $\Phi_v : G_v = Z_v/T_v \to Z_w/T_w = H_w$ defines a level $(\delta - 1)$ morphism of the corresponding residual abstract decomposition graphs $\mathcal{G}_v$ and $\mathcal{H}_w$.

Remarks 3.3. In the above context, let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level δ morphism of abstract decomposition graphs.

1) Then Φ gives rise to a Kummer homomorphism

$$\widehat{\mathcal{L}}_{\mathcal{H}} := \mathrm{Hom}\big(H, \mathbb{Z}_\ell\big) \xrightarrow{\hat{\phi}} \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big) =: \widehat{\mathcal{L}}_{\mathcal{G}}, \quad \varphi \mapsto \varphi \circ \Phi\,.$$

Note that since Φ has an open image, and $\widehat{\mathcal{L}}_{\mathcal{G}}$ and $\widehat{\mathcal{L}}_{\mathcal{H}}$ are torsion free, $\hat{\phi}$ is *injective.*

From now on *suppose that* $\delta > 0$, and that $\tilde{v}$ and $\tilde{w}$ are the multi-indices of $\mathcal{G}$, respectively $\mathcal{H}$ which correspond to each other. Let $\delta_{\tilde{v}}$ and $\delta_{\tilde{w}}$ be their lengths, and suppose that $\delta_{\tilde{w}} < \delta$.

2) $\Phi_{\tilde{v}} : \mathcal{G}_{\tilde{v}} \to \mathcal{H}_{\tilde{w}}$ has level $(\delta - \delta_w)$ and gives rise to the residual Kummer homomorphism $\hat{\phi}_{\tilde{v}} : \widehat{\mathcal{L}}_{\mathcal{H}_{\tilde{w}}} \to \widehat{\mathcal{L}}_{\mathcal{G}_{\tilde{v}}}$, which is injective, by the remark above applied to $\Phi_{\tilde{v}}$.

3) To simplify notations, let us set $\widehat{\mathcal{L}}_{Z_{\tilde{v}}} = \mathrm{Hom}\big(Z_{\tilde{v}}, \mathbb{Z}_\ell\big)$ and $\widehat{\mathcal{L}}_{T_{\tilde{v}}} = \mathrm{Hom}\big(T_{\tilde{v}}, \mathbb{Z}_\ell\big)$, thus in particular, $\widehat{\mathcal{L}}_{T_{\tilde{v}}} \cong \mathbb{Z}_\ell^{\delta_{\tilde{v}}}$. The inclusions $T_{\tilde{v}} \hookrightarrow Z_{\tilde{v}} \hookrightarrow G$ and the canonical exact sequence $1 \to T_{\tilde{v}} \to Z_{\tilde{v}} \to G_{\tilde{v}} \to 1$ give rise in the same way as at Definition/Remark 2.3, points 3) and 6), to morphisms of ℓ-adically complete $\mathbb{Z}_\ell$-modules as follows:

$$\jmath^{\tilde{v}} : \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\mathrm{res}_Z} \widehat{\mathcal{L}}_{Z_{\tilde{v}}} \xrightarrow{\mathrm{res}_T} \widehat{\mathcal{L}}_{T_{\tilde{v}}} \quad \text{and} \quad 0 \to \widehat{\mathcal{L}}_{\mathcal{G}_{\tilde{v}}} \xrightarrow{\mathrm{inf}} \widehat{\mathcal{L}}_{Z_{\tilde{v}}} \xrightarrow{\mathrm{res}_T} \widehat{\mathcal{L}}_{T_{\tilde{v}}} \to 0\,.$$

In particular, setting $\widehat{U}^1_{\tilde{v}} := \ker(\mathrm{res}_Z)$ and $\widehat{U}_{\tilde{v}} = \ker(\jmath^{\tilde{v}})$, we get exact sequences

$$0 \to \widehat{U}_{\tilde{v}} \longrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\tilde{v}}} \widehat{\mathcal{L}}_{T_{\tilde{v}}} \to 0 \quad \text{and} \quad 0 \to \widehat{U}^1_{\tilde{v}} \hookrightarrow \widehat{U}_{\tilde{v}} \xrightarrow{\jmath_{\tilde{v}}} \widehat{\mathcal{L}}_{\mathcal{G}_{\tilde{v}}} \to 0\,.$$

The surjective morphism $\jmath_{\tilde{v}} : \widehat{U}_{\tilde{v}} \to \widehat{\mathcal{L}}_{\mathcal{G}_{\tilde{v}}}$ is called the canonical $\tilde{v}$-reduction homomorphism.

By induction on $\delta_{\tilde{v}}$ and $\delta_{\tilde{w}}$, one gets the following: $\Phi(Z_{\tilde{v}}) \subseteq Z_{\tilde{w}}$ and $\Phi(Z_{\tilde{v}})$ is open in $Z_{\tilde{w}}$, and $\Phi(T_{\tilde{v}}) \subseteq T_{\tilde{w}}$ and $\Phi(T_{\tilde{v}})$ is open in $T_{\tilde{w}}$. Hence since Φ is open and restricts to open homomorphism $Z_{\tilde{v}} \to Z_{\tilde{w}}$ and $T_{\tilde{v}} \to T_{\tilde{w}}$, by taking ℓ-adic duals, we get commutative diagrams with *injective columns and exact rows* as follows:

$$\begin{tikzcd}
\widehat{U}_{\tilde{w}} \arrow[r] \arrow[d, "\hat{\phi}"] & \widehat{\mathcal{L}}_{Z_{\tilde{w}}} \arrow[r] \arrow[d, "\hat{\phi}"] & \widehat{\mathcal{L}}_{T_{\tilde{w}}} \arrow[d, "\hat{\phi}^{\tilde{v}}"] \\
\widehat{U}_{\tilde{v}} \arrow[r] & \widehat{\mathcal{L}}_{Z_{\tilde{v}}} \arrow[r] & \widehat{\mathcal{L}}_{T_{\tilde{v}}}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\widehat{U}^1_{\tilde{w}} \arrow[r, hook] \arrow[d, "\hat{\phi}"] & \widehat{U}_{\tilde{w}} \arrow[r, "\jmath_{\tilde{w}}"] \arrow[d, "\hat{\phi}"] & \widehat{\mathcal{L}}_{\mathcal{H}_{\tilde{w}}} \arrow[d, "\hat{\phi}_{\tilde{v}}"] \\
\widehat{U}^1_{\tilde{v}} \arrow[r, hook] & \widehat{U}_{\tilde{v}} \arrow[r, "\jmath_{\tilde{v}}"] & \widehat{\mathcal{L}}_{\mathcal{H}_{\tilde{v}}}
\end{tikzcd}$$

4) A special case of the above discussion is when $\tilde{v} = v$ and $\tilde{w} = w$ are 1-vertices. If τ_v and τ_w are inertia generators at v, respectively w, then there exists a unique $a_{vw} \in \mathbb{Z}_\ell$ such that the $\Phi(\tau_v) = \tau_w^{a_{vw}}$. And we have commutative diagrams dual to each other:

$$\begin{tikzcd}
T_v \arrow[r] \arrow[d, "\Phi"] & G \arrow[d, "\Phi"] \\
T_w \arrow[r] & H
\end{tikzcd} \qquad \begin{tikzcd}
\widehat{\mathcal{L}}_{\mathcal{H}} \arrow[r, "\jmath^w"] \arrow[d, "\hat{\phi}"] & \mathbb{Z}_\ell \varphi_w \arrow[d, "a_{vw}"] \\
\widehat{\mathcal{L}}_{\mathcal{G}} \arrow[r, "\jmath^v"] & \mathbb{Z}_\ell \varphi_v
\end{tikzcd}$$

where φ_w and φ_v are as in the Construction 2.5. Further, the horizontal maps in the first diagram are the inclusions, and the last vertical map in the second diagram denotes the $\mathbb{Z}_\ell$-morphism defined by $\varphi_w \mapsto a_{vw}\varphi_v$.

Definition 3.4. Let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level δ morphism of abstract decomposition graphs. We will say that:

1) Φ is proper, if each $\tilde{w}$ corresponds to some $\tilde{v}$, and vice-versa, to each $\tilde{v}$ there is some $\tilde{w}$ corresponding to it (which might be the trivial valuation w_0). In particular, if $\tilde{w}$ and $\tilde{v}$ corresponds to each other, then the residual morphism $\Phi_{\tilde{v}} : \mathcal{G}_{\tilde{v}} \to \mathcal{G}_{\tilde{w}}$ is a proper one.

In particular, $\Phi : \mathcal{G} \to \mathcal{H}$ is a proper morphism if and only if in the notations from Definition/Remark 3.1, we have: $\mathrm{Vert}_{\mathcal{G},\Phi} = \mathrm{Vert}_{\mathcal{G}}$, and the map $\varphi_\Phi : \mathrm{Vert}_{\mathcal{G}} \to \mathrm{Vert}_{\mathcal{H}}$ is onto.

2) Φ defines $\mathcal{H}$ as a level δ quotient of $\mathcal{G}$, or that $\mathcal{H}$ is a level δ quotient of $\mathcal{G}$ via Φ, if Φ is proper, and we have $\Phi(G) = H$.

Remarks 3.5. Let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level $\delta > 0$ proper morphism.

1) Let $\tilde{v}$ be a multi-index of $\mathcal{G}$, and $T_{\tilde{v}} \cong \mathbb{Z}_\ell^{\delta_{\tilde{v}}}$ be the inertia group at $\tilde{v}$, as defined in Remark/Definition 2.2, 3). Then $\Phi(T_{\tilde{v}})$ is a free $\mathbb{Z}_\ell$-module of rank $\delta' \leq \delta_{\tilde{v}}$. Suppose that $\delta' \leq \delta$. Then using the *properness* of Φ, one checks by induction on $\delta_{\tilde{v}}$ that there exists a unique multi-index $\tilde{w}$ such that the following hold: $\Phi(Z_{\tilde{v}}) \subseteq Z_{\tilde{w}}$ and $\Phi(T_{\tilde{v}}) \subseteq T_{\tilde{w}}$ are open subgroups. Thus in particular, $\tilde{w}$ corresponds to $\tilde{v}$.

2) Denote by $T_{\mathcal{G}}$ the subgroup of G generated by all the inertia elements of G, and define $T_{\mathcal{H}} \subseteq H$ correspondingly. Then in the notations from Definition/Remark 2.3, 2), Φ gives rise to a commutative diagram as follows:

$$\begin{tikzcd}
1 \arrow[r] & T_{\mathcal{G}} \arrow[r] \arrow[d, "\Phi"] & G \arrow[r] \arrow[d, "\Phi"] & \pi_{1,\mathcal{G}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & T_{\mathcal{H}} \arrow[r] & H \arrow[r] & \pi_{1,\mathcal{H}} \arrow[r] & 1
\end{tikzcd}$$

Next suppose that $\mathcal{G}$ and $\mathcal{H}$ are divisorial, and let $\mathfrak{T}_{\mathcal{G}} = (\tau_v)_v$ and $\mathfrak{T}_{\mathcal{H}} = (\tau_w)_w$ be distinguished systems of generators for $\mathcal{G}$, respectively $\mathcal{H}$, which give rise to abstract divisor groups $\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}}$ and $\mathrm{Div}_{\mathfrak{T}_{\mathcal{H}}}$ for $\mathcal{G}$, respectively $\mathcal{H}$, and abstract divisorial lattices $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{H}}$.

3) For every w, denote by X_w the set of all the v to which w corresponds. Then X_w is finite non-empty (by the fact that Φ is proper). For every w and $v \in X_w$, there exists a unique $a_{vw} \in \mathbb{Z}_\ell$ such that $\Phi(\tau_v) = \tau_w^{a_{vw}}$. Equivalently, if $\mathfrak{B}_{\mathcal{G}} = (\varphi_v)_v$ and $\mathfrak{B}_{\mathcal{H}} = (\varphi_w)_w$ are the dual bases to $\mathfrak{T}_{\mathcal{G}} = (\tau_v)_v$ and $\mathfrak{T}_{\mathcal{H}} = (\tau_w)_w$ as defined/introduced at Construction 2.5, then setting

$$\varphi_w \mapsto \textstyle\sum_{v \in X_w} a_{vw}\varphi_v\,,$$

we get a morphism

$$\mathrm{div}_\Phi : \widehat{\mathrm{Div}}_{\mathcal{H}} \to \widehat{\mathrm{Div}}_{\mathcal{G}}$$

which maps $\mathrm{Div}_{\mathfrak{T}_{\mathcal{H}}} \otimes \mathbb{Z}_\ell$ into $\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}} \otimes \mathbb{Z}_\ell$ and fits into the following commutative diagram:

$$(*)\qquad\begin{tikzcd}
0 \arrow[r] & \widehat{U}_{\mathcal{H}} \arrow[r] \arrow[d, "\hat{\phi}"] & \widehat{\mathcal{L}}_{\mathcal{H}} \arrow[r, "\jmath^{\mathcal{H}}"] \arrow[d, "\hat{\phi}"] & \widehat{\mathrm{Div}}_{\mathcal{H}} \arrow[r] \arrow[d, "\mathrm{div}_\Phi"] & \widehat{\mathfrak{Cl}}_{\mathcal{H}} \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & \widehat{U}_{\mathcal{G}} \arrow[r] & \widehat{\mathcal{L}}_{\mathcal{G}} \arrow[r, "\jmath^{\mathcal{G}}"] & \widehat{\mathrm{Div}}_{\mathcal{G}} \arrow[r] & \widehat{\mathfrak{Cl}}_{\mathcal{G}} \arrow[r] & 0
\end{tikzcd}$$

4) Recall that for every divisorial $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{\mathfrak{T}_{\mathcal{G}}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ we have: $\widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}} = \mathcal{L}_{\mathfrak{T}_{\mathcal{G}}} \otimes \mathbb{Z}_\ell$, and therefore, $\widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}$ it is exactly the preimage of $\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}} \otimes \mathbb{Z}_\ell$ under $\jmath^{\mathcal{G}}$. Hence from the commutative diagram $(*)$ above it follows that

$$(*) \qquad \widehat{\mathcal{L}}_{\mathcal{H},\mathrm{fin}} = \hat{\phi}^{-1}(\widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}), \quad \hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{H},\mathrm{fin}}) \cap \widehat{U}_{\mathcal{G}} = \hat{\phi}(\widehat{U}_{\mathcal{H}}), \quad \hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{H},\mathrm{fin}}) = \hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{H}}) \cap \widehat{\mathcal{L}}_{\mathcal{G},\mathrm{fin}}\,.$$

In particular, $\hat{\phi}$ maps finite co-rank submodules into such, and pre-images of finite co-rank submodules under $\hat{\phi}$ are again such.

5) In the above notations, the following are equivalent:

a) There exist $\mathfrak{T}_{\mathcal{G}} = (\tau_v)_v$ and $\mathfrak{T}_{\mathcal{H}} = (\tau_w)_w$ such that $a_{vw} \in \mathbb{Z}_{(\ell)}$ for all w, $v \in X_w$.

b) $\mathrm{div}_\Phi(\mathrm{Div}_{\mathfrak{T}_{\mathcal{H}}}) \subseteq \mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}}$.

c) $\hat{\phi}(\mathcal{L}_{\mathcal{H}}) \subseteq \mathcal{L}_{\mathcal{G}}$.

Moreover, suppose that the above equivalent conditions are satisfied. Then if $\mathfrak{T}_{\mathcal{G}} = (\tau_v)_v$ is given, $\mathfrak{T}_{\mathcal{H}} = (\tau_w)_w$ is uniquely determined up to multiplication by some $\epsilon \in \mathbb{Z}_{(\ell)}$, and vice-versa. Further, one has equalities as follows:

$$(**) \qquad \mathrm{Div}_{\mathfrak{T}_{\mathcal{H}}} = \mathrm{div}_\Phi^{-1}(\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}}), \quad \mathcal{L}_{\mathfrak{T}_{\mathcal{H}}} = \hat{\phi}^{-1}(\mathcal{L}_{\mathfrak{T}_{\mathcal{G}}}), \quad \hat{\phi}(\mathcal{L}_{\mathfrak{T}_{\mathcal{H}}}) = \hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{H}}) \cap \mathcal{L}_{\mathfrak{T}_{\mathcal{G}}}\,.$$

In particular, $\mathcal{L}_{\mathfrak{T}_{\mathcal{H}}}$ is encoded in $\mathcal{L}_{\mathfrak{T}_{\mathcal{G}}}$ via $\hat{\phi}$.

Proof of 5): The implication a) ⇒ b) follows immediately from the definition of div_Φ, and the implication b) ⇒ c) follows from the definition of $\mathcal{L}_{\mathcal{H}}$ and $\mathcal{L}_{\mathcal{G}}$. In order to prove c) ⇒ a), let $v \in X_w$ be given. Then combining Remark 2.10, 1), with the second diagram from Remark 3.3, 4), we get a commutative diagram of the form

$$\begin{tikzcd} \mathcal{L}_{\mathfrak{T}_{\mathcal{H}}} \arrow[r, "\jmath^w"] \arrow[d, "\hat{\phi}"] & \mathbb{Z}_{(\ell)}\varphi_w \arrow[d, "a_{vw}"] \\ \mathcal{L}_{\mathfrak{T}_{\mathcal{G}}} \arrow[r, "\jmath^v"] & \mathbb{Z}_{(\ell)}\varphi_v \end{tikzcd}$$

hence it follows that $a_{vw} \in \mathbb{Z}_{(\ell)}$, as claimed. Finally, let us show that in the case the equivalent conditions a), b), c), are satisfied, the equalities $(**)$ hold. First remark that since $\hat{\phi}$ and div_Φ are injective, all the above equalities are equivalent. Thus it is enough to prove one of them, say the first one: Recall that by point 3) above, $\mathrm{Div}_{\mathfrak{T}_{\mathcal{H}}} \otimes \mathbb{Z}_\ell$ and $\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}} \otimes \mathbb{Z}_\ell$ are free $\mathbb{Z}_\ell$ modules on the bases $\mathfrak{B}_{\mathcal{H}} = (\varphi_w)_w$, respectively $\mathfrak{B}_{\mathcal{G}} = (\varphi_v)_v$, and that div_Φ maps the former one into the latter one. Hence $\mathrm{div}_\Phi^{-1}(\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}}) \subseteq \mathrm{Div}_{\mathfrak{T}_{\mathcal{H}}} \otimes \mathbb{Z}_\ell$. Now let $x = \sum_w b_w \varphi_w$ with $b_w \in \mathbb{Z}_\ell$ be an element of $\mathrm{div}_\Phi^{-1}(\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}})$. Then $\mathrm{div}_\Phi(x) = \sum_w \sum_{v \in X_w} b_w a_{vw} \varphi_v$ lies in $\mathrm{Div}_{\mathfrak{T}_{\mathcal{G}}}$, hence $b_w a_{vw} \in \mathbb{Z}_{(\ell)}$ for all w and $v \in X_w$. On the other hand, since $a_{vw} \in \mathbb{Z}_{(\ell)}$, it follows that b_w are rational numbers. Since they lie in $\mathbb{Z}_\ell$ too, it follows that $b_w \in \mathbb{Z}_{(\ell)}$. But then we finally get: $x = \sum_w b_w \varphi_w$ lies in $\mathcal{L}_{\mathfrak{T}_{\mathcal{H}}}$ as claimed.

Definition 3.6. Let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level δ proper morphism of divisorial abstract decomposition graph. We call Φ divisorial, if all residual morphisms $\Phi_{\tilde{v}} : \mathcal{G}_{\tilde{v}} \to \mathcal{H}_{\tilde{w}}$ with $\tilde{w}$ of length $< \delta$ satisfy the equivalent conditions a), b), c) from 5) above. In particular, the residual Kummer morphism $\hat{\phi}_{\tilde{v}}$ maps divisorial lattices for $\mathcal{H}_{\tilde{w}}$ into divisorial lattices for $\mathcal{G}_{\tilde{v}}$.

It is not too difficult to give examples of proper morphisms Φ of divisorial abstract decomposition graphs such that Φ are not divisorial. Indeed, one can give such examples even in the case where both $\mathcal{G}$ and $\mathcal{H}$ are complete curve like, and Φ is a proper morphism of level $\delta = 1$. The next Proposition shows that actually the case $\delta = 1$ is the "generic" source for proper non-divisorial morphisms.

Proposition 3.7. *Let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level $\delta = \delta_{\mathcal{H}} > 0$ proper morphism of divisorial abstract decomposition graphs. Let $\tilde{v}$, $\tilde{w}$ be all pairs of the multi-indices of $\mathcal{G}$, respectively of $\mathcal{H}$, such that $\tilde{w}$ has length $\delta_{\mathcal{H}} - 1$ and corresponds to $\tilde{v}$. Suppose that* for all such pairs $\tilde{v}, \tilde{w}$ *the residual morphism $\Phi_{\tilde{v}} : \mathcal{G}_{\tilde{v}} \to \mathcal{H}_{\tilde{w}}$ is divisorial. Then $\Phi : \mathcal{G} \to \mathcal{H}$ is divisorial.*

Proof. One makes induction on $\delta_{\mathcal{G}}$:

Case 1) $\delta_{\mathcal{G}} = 1$. Then $\delta = \delta_{\mathcal{H}} = 1$, and the assertion follows from/by the definitions and the hypothesis of the Proposition.

Case 2) $\delta > 1$ arbitrary: Let v be some 1-index of $\mathcal{G}$, and w the image of v under Φ. Note that w is either the trivial valuation w_0, or otherwise w is a 1-index of $\mathcal{H}$. We show that the resulting residual morphism $\Phi_v : \mathcal{G}_v \to \mathcal{H}_w$ satisfies the hypothesis of the Proposition: First $\Phi_v : \mathcal{G}_v \to \mathcal{H}_w$ is a proper morphism, as $\Phi : \mathcal{G} \to \mathcal{H}$ was so by hypothesis. Second, let $\tilde{w}_w$ be a multi-index of $\mathcal{H}_w$ of length $\delta_{\mathcal{H}_w} - 1$, and $\tilde{v}_w$ a multi-index of $\mathcal{G}_v$ such that $\tilde{w}_w$ corresponds to $\tilde{v}_w$ under Φ_v.

Claim 1. $\Phi_{\tilde{v}_w} : \mathcal{G}_{\tilde{v}_w} \to \mathcal{H}_{\tilde{w}_w}$ is divisorial.

Indeed, let us first suppose that $w = w_0$ is the trivial valuation. Then $\mathcal{H}_w = \mathcal{H}$, and $\tilde{w} := \tilde{w}_w$ is a multi-index of $\mathcal{H}$ of length $\delta - 1$. Further, $(\tilde{v}_w, v)$ is a multi-index of $\mathcal{G}$ which corresponds to $\tilde{w}$. And we have: $\mathcal{G}_{\tilde{v}_w} = \mathcal{G}_{(\tilde{v}_w,v)}$, $\mathcal{H}_{\tilde{w}_w} = \mathcal{H}_{\tilde{w}}$, and $\Phi_{\tilde{v}_w} : \mathcal{G}_{\tilde{v}_w} \to \mathcal{H}_{\tilde{w}_w}$ is actually the same as $\Phi_{(\tilde{v}_w,v)} : \mathcal{G}_{(\tilde{v}_w,v)} \to \mathcal{H}_{\tilde{w}}$. But then the Claim follows from the hypothesis of the Proposition. Next suppose that $w \neq w_0$ is a 1-index of $\mathcal{H}$. Then $\Phi_{\tilde{v}}$ has level $\delta - 1 = \delta_{\mathcal{H}} - 1 = \delta_{\mathcal{H}_w}$. Moreover, if $\tilde{w}_w$ is a multi-index of $\mathcal{H}_w$ of length $\delta_{\mathcal{H}_w} - 1$, then $(\tilde{w}_w, w)$ is a multi-index of $\mathcal{H}$ of length

$$(\delta_{\mathcal{H}_w} - 1) + 1 = \delta_{\mathcal{H}_w} = \delta_{\mathcal{H}} - 1\,.$$

And since $\tilde{w}_w$ corresponds to $\tilde{v}_w$, and w to v, it follows that $(\tilde{w}_w, w)$ corresponds to $(\tilde{v}_w, v)$. But then by the hypothesis of the Proposition we have: The residual morphism

$$\Phi_{(\tilde{v}_w,v)} : \mathcal{G}_{(\tilde{v}_w,v)} \to \mathcal{H}_{(\tilde{w}_w,w)}$$

is divisorial. On the other hand, by definitions we have identifications $\mathcal{G}_{(\tilde{v}_w,v)} = \mathcal{G}_{\tilde{v}_w}$ and $\mathcal{H}_{(\tilde{w}_w,w)} = \mathcal{H}_{\tilde{w}_w}$, and $\Phi_{(\tilde{v}_w,v)} = \Phi_{\tilde{v}_w}$.

This completes the proof of the Claim 1.

Coming back to the proof of the Proposition, let $\mathcal{L}_{\mathcal{H}}$ and $\mathcal{L}_{\mathcal{G}}$ be divisorial lattices in $\widehat{\mathcal{L}}_{\mathcal{H}}$, respectively $\widehat{\mathcal{L}}_{\mathcal{G}}$. For $\Gamma \subset \widehat{\mathcal{L}}_{\mathcal{H}}$ of finite co-rank and satisfying $\Gamma \cap \widehat{U}_{\mathcal{H}} = (0)$, set $\Delta := \hat{\phi}(\Gamma)$.

Claim 2. $\hat{\phi} : \widehat{\mathcal{L}}_{\mathcal{H}} \to \widehat{\mathcal{L}}_{\mathcal{G}}$ maps Γ isomorphically onto its image Δ, and further one has: $\Delta \cap \widehat{U}_{\mathcal{G}} = (0)$, and Δ is a finite co-rank $\mathbb{Z}_\ell$-submodule of $\widehat{\mathcal{L}}_{\mathcal{G}}$.

Indeed, by the diagram $(*)$ from Remark 3.5, 3), and in the notations from there, we have: $\jmath^{\mathcal{H}}$ is injective on Γ, as $\Gamma \cap \widehat{U}_{\mathcal{H}} = (0)$. Since $\widehat{\mathrm{div}}_{\Phi}$ is injective, it finally follows that $\jmath^{\mathcal{H}}(\Gamma)$ is mapped injectively into $\widehat{\mathrm{Div}}_{\mathcal{G}}$. Therefore, $\hat{\phi}$ maps Γ injectively into $\widehat{\mathcal{L}}_{\mathcal{G}}$, and $\Delta := \hat{\phi}(\Gamma)$ has trivial intersection with $\widehat{U}_{\mathcal{G}}$. Now let us check that Δ has finite co-rank in $\widehat{\mathcal{L}}_{\mathcal{G}}$: First let v

be a valuation of $\mathcal{G}$ such that $\jmath^v(\Gamma) \neq (0)$. Equivalently, $\Delta = \hat{\phi}(\Gamma)$ has a non-trivial image under

$$\jmath^v \circ \hat{\phi} : \widehat{\mathcal{L}}_{\mathcal{H}} \xrightarrow{\hat{\phi}} \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^v} \mathbb{Z}_\ell \varphi_v.$$

Hence $\jmath^v \circ \hat{\phi}(\Delta)$ is non-trivial. Since the above sequence is ℓ-adically dual to $T_v \hookrightarrow G \xrightarrow{\Phi} H$, it follows that $\Phi(T_v) \neq 1$ in H. Since Φ is proper by hypothesis, it follows that there exists w such that $\Phi(T_v) \subseteq T_w$, and $\Phi(T_v)$ is open in T_w. Hence finally w corresponds to v. Therefore, if $\jmath^v(\Delta) \neq (0)$, then there exists some $w \neq w_0$ corresponding to v.

Next, by the commutativity of the second diagram in Remark 3.3, 4), it follows that $\jmath^v(\Delta) \neq (0)$ if an only if $\jmath^w(\Gamma) \neq (0)$. Now since Γ has finite co-rank, there exist only finitely many valuations w of H such that $\jmath^w(\Gamma) \neq (0)$. Finally, for each such w there exist only finitely many v's such that w corresponds to one of the v's. Thus finally there are only finitely many valuations v of G such that $\jmath^v(\Delta) \neq (0)$.

This completes the proof of Claim 2.

Now suppose that Γ is non-trivial. Then we have the situation: Γ and its isomorphic image Δ are non-trivial finite co-rank submodules of $\widehat{\mathcal{L}}_{\mathcal{H}}$, respectively $\widehat{\mathcal{L}}_{\mathcal{G}}$. Since $\Delta \cap \widehat{U}_{\mathcal{G}} = (0)$, it follows that $\Delta \cap \mathcal{L}_{\mathcal{G}}$ is a lattice in Δ which completely determines the divisorial lattice $\mathcal{L}_{\mathcal{G}}$ in the ℓ-adic equivalence class of all the divisorial lattices of $\mathcal{G}$. Correspondingly, the same is true for $\Gamma \cap \widehat{U}_{\mathcal{H}}$ and $\mathcal{L}_{\mathcal{H}}$, etc. On the other hand, since Δ has finite co-rank, by the ampleness of $\mathcal{G}$, there exist valuations v of G such that the following are satisfied:

j) $\Delta \subset \widehat{U}_v$ and $\jmath_v$ maps Δ injectively into $\widehat{\mathcal{L}}_{\mathcal{G}_v}$.

jj) $\Delta_v := \jmath_v(\Delta)$ satisfies: $\Delta_v \cap \widehat{U}_{\mathcal{G}_v} = (0)$, as $\Delta \cap \widehat{U}_{\mathcal{G}} = (0)$ by the discussion above.

For such a valuation v, the lattice $\Delta \cap \mathcal{L}_{\mathcal{G}}$ is mapped by $\jmath_v$ isomorphically onto a lattice in Δ_v. Hence by the properties i), ii), from Fact 2.8, we get: There exists a *unique* divisorial $\widehat{U}_{\mathcal{G}_v}$-lattice $\mathcal{L}_{\mathcal{G}_v}$ of $\mathcal{G}_v$ such that $\jmath_v(\Delta \cap \mathcal{L}_{\mathcal{G}}) = \Delta_v \cap \mathcal{L}_{\mathcal{G}_v}$. For v as above we analyze the cases:

Case a) $\Phi(T_v) = 1$. Then the trivial valuation w_0 corresponds to v, and for the residual morphism $\Phi_v : \mathcal{G}_v \to \mathcal{H}$ we have: First, Φ_v is divisorial by Claim 1. Hence by Remark 3.5, 5), there exists a unique divisorial $\widehat{U}_{\mathcal{H}}$-lattice $\mathcal{L}_{\mathcal{H}}$ of $\mathcal{H}$ such that the Kummer homomorphism $\hat{\phi}_v : \widehat{\mathcal{L}}_{\mathcal{H}} \to \widehat{\mathcal{L}}_{\mathcal{G}_v}$ maps $\mathcal{L}_{\mathcal{H}}$ into $\mathcal{L}_{\mathcal{G}_v}$.

Case b) $\Phi(T_v) \neq 1$. Then there is a non-trivial valuation w corresponding to v, and for the corresponding residual morphism $\Phi_v : \mathcal{G}_v \to \mathcal{H}_w$ we have: First, Φ_v is divisorial by Claim 1. Hence by Remark 3.5, 5), there exists a unique divisorial $\widehat{U}_{\mathcal{H}_w}$-lattice $\mathcal{L}_{\mathcal{H}_w}$ of $\mathcal{H}_w$ such that the Kummer homomorphism $\hat{\phi}_v : \widehat{\mathcal{L}}_{\mathcal{H}_w} \to \widehat{\mathcal{L}}_{\mathcal{G}_v}$ maps $\mathcal{L}_{\mathcal{H}_w}$ into $\mathcal{L}_{\mathcal{G}_v}$. Moreover, since $\hat{\phi}$ maps Γ isomorphically onto its image Δ, and $\jmath_v$ maps Δ isomorphically onto its image Δ_v, we get: Since $\jmath_v \circ \hat{\phi}$ and $\jmath_w \circ \hat{\phi}_v$ coincide on Γ, it follows that $\jmath_w$ maps Γ isomorphically onto its image Γ_w, and that $\hat{\phi}_v$ maps Γ_w isomorphically onto Δ_v. Therefore, w satisfies *mutatis mutandis* the conditions j), jj), above with respect to Γ. Hence $\Gamma_w \cap \mathcal{L}_{\mathcal{H}_w}$ is a lattice in Γ_w. Hence there exists a unique divisorial $\widehat{U}_{\mathcal{H}}$-lattice $\mathcal{L}_{\mathcal{H}}$ of $\mathcal{H}$ such that $\Gamma \cap \mathcal{L}_{\mathcal{H}}$ is mapped isomorphically onto $\Gamma_w \cap \mathcal{L}_{\mathcal{H}_w}$.

Claim 3. In both cases above, $\hat{\phi}$ maps $\mathcal{L}_{\mathcal{H}}$ into $\mathcal{L}_{\mathcal{G}}$.

First, in the notations from above, it is clear by the discussion above, that $\hat{\phi}$ maps $\Gamma \cap \mathcal{L}_{\mathcal{H}}$ isomorphically onto $\Delta \cap \mathcal{L}_{\mathcal{G}}$. Now let Γ' be a finite co-rank $\mathbb{Z}_\ell$-module such that $\Gamma' \cap \widehat{U}_{\mathcal{H}} = 1$ and $\Gamma \subseteq \Gamma'$. Let $\mathcal{L}'_{\mathcal{H}}$ be the divisorial $\widehat{U}_{\mathcal{H}}$-lattice given by the construction above when starting with Γ' instead of Γ. Then we have: $\Gamma \cap \mathcal{L}'_{\mathcal{H}}$ is a lattice in Γ, which is ℓ-adically equivalent to $\Gamma \cap \mathcal{L}_{\mathcal{H}}$. Hence $\hat{\phi}(\Gamma \cap \mathcal{L}_{\mathcal{H}})$ and $\hat{\phi}(\Gamma \cap \mathcal{L}'_{\mathcal{H}})$ are ℓ-adically equivalent lattices in $\Delta = \hat{\phi}(\Gamma)$, and both of them are contained in $\mathcal{L}_{\mathcal{G}}$. Hence $\hat{\phi}(\Gamma \cap \mathcal{L}_{\mathcal{H}})$ and $\hat{\phi}(\Gamma \cap \mathcal{L}'_{\mathcal{H}})$ are equal; and therefore $\Gamma \cap \mathcal{L}_{\mathcal{H}}$ and $\Gamma \cap \mathcal{L}'_{\mathcal{H}}$ are equal. Hence finally $\mathcal{L}_{\mathcal{H}}$ and $\mathcal{L}'_{\mathcal{H}}$ are equal. In other words, for every finite co-rank $\mathbb{Z}_\ell$-module Γ' of $\mathcal{H}$ as above we have: If $\Gamma \subseteq \Gamma'$, then $\Gamma' \cap \mathcal{L}_{\mathcal{H}}$ is mapped into $\mathcal{L}_{\mathcal{G}}$. But then $\mathcal{L}_{\mathcal{H}} = \widehat{U}_{\mathcal{H}} + \cup_{\Gamma'}(\Gamma' \cap \mathcal{L}_{\mathcal{H}})$ is mapped into $\mathcal{L}_{\mathcal{G}}$, as claimed. $\square$

B) *Geometric like abstract decomposition graphs*

We begin by first defining rational quotients of divisorial abstract decomposition graphs. The point is that (divisorial) abstract decomposition graphs which arise from geometry posses "sufficiently many" rational quotients; and morphisms of (divisorial) abstract decomposition graphs arising from geometry are compatible with the rational quotients. This suggests that for applications, one should consider/study divisorial abstract decomposition graphs endowed with "sufficiently many" rational quotients, and morphisms of such enriched structures.

To begin with, let $\mathcal{G}_\alpha$ be a level $\delta = 1$ complete curve like abstract decomposition graph. Recall the notations from the Construction 2.5, Case $\delta = 1$: For every distinguished system of generators $\mathfrak{T}_\alpha = (\tau_v)_v$ of $\mathcal{G}_\alpha$, we have an exact sequence of the form:

$$0 \to \widehat{U}_{\mathcal{G}_\alpha} \hookrightarrow \mathcal{L}_{\mathfrak{T}_\alpha} \xrightarrow{\jmath^{\mathcal{G}_\alpha}} \mathrm{Div}_{\mathfrak{T}_\alpha} \xrightarrow{\mathrm{can}} \mathfrak{Cl}_{\mathfrak{T}_\alpha} \cong \mathbb{Z}_{(\ell)} \to 0\,.$$

Definition/Remark 3.8. In the notations from above we define:

1) A level $\delta = 1$ divisorial abstract decomposition graph $\mathcal{G}_\alpha$ is called projective line like, if $\widehat{U}_{\mathcal{G}_\alpha} = (0)$ for some (thus every) distinguished system of inertia generators $\mathfrak{T}_\alpha$ of $\mathcal{G}_\alpha$.

We remark the following: Since $\widehat{U}_{\mathcal{G}_\alpha} = (0)$, every $\widehat{U}_{\mathcal{G}_\alpha}$-lattice in $\mathcal{G}_\alpha$ is actually a lattice in $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$. Let $\mathfrak{T}_\alpha = (\tau_v)_v$ be a distinguished system of inertia generators, and $\mathfrak{B}_\alpha = (\varphi_v)_v$ the corresponding $\mathbb{Z}_{(\ell)}$-basis of $\mathrm{Div}_{\mathfrak{T}_\alpha}$, see loc.cit. for notations. An element of the form

$$\boldsymbol{x} = \varphi_{v'} - \varphi_v$$

is called a generating element of $\mathcal{L}_{\mathcal{G}_\alpha}$. We set $(\boldsymbol{x})_0 := v'$ and $(\boldsymbol{x})_\infty := v$, and call these the zero, respectively the pole, of $\boldsymbol{x}$. Further we denote

$$\mathcal{P}_v = \{\, \boldsymbol{x} \in \mathcal{L}_{\mathfrak{T}_\alpha} \mid \boldsymbol{x} \text{ generating, and } (\boldsymbol{x})_\infty = v \,\} = \{\, \varphi_{v'} - \varphi_v \mid \text{ all } v' \neq v \,\}\,,$$

and call it a generating set at v for $\mathcal{L}_{\mathfrak{T}_\alpha}$. Clearly, $\mathcal{P}_v$ defines a $\mathbb{Z}_{(\ell)}$-basis of $\mathcal{L}_{\mathfrak{T}_\alpha}$ for every v.

Note that if $\mathfrak{T}'_\alpha = \mathfrak{T}^\epsilon_\alpha$ is another distinguished system of inertia generators, and $\mathcal{P}'_v$ is correspondingly defined, then $\epsilon \in \mathbb{Z}_\ell^\times$ is the unique ℓ-adic unit such that $\epsilon \cdot \mathcal{P}'_v = \mathcal{P}_v$.

Let $\mathcal{G}$ be a level $\delta > 0$ divisorial abstract decomposition graph. We will consider quotients $\Phi_\alpha : \mathcal{G} \to \mathcal{G}_\alpha$ of $\mathcal{G}$ such that $\mathcal{G}_\alpha$ is projective line like, and let $\hat{\phi}_\alpha : \widehat{\mathcal{L}}_{\mathcal{G}_\alpha} \to \widehat{\mathcal{L}}_{\mathcal{G}}$ be the corresponding Kummer homomorphism. In order to simplify notations we set

$$\widehat{\mathcal{L}}_\alpha := \hat{\phi}_\alpha(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) \subseteq \widehat{\mathcal{L}}_{\mathcal{G}}\,.$$

Further, for every multi-index $\tilde{v}$ of $\mathcal{G}$, let $\jmath_{\tilde{v}} : \widehat{U}_{\tilde{v}} \to \widehat{\mathcal{L}}_{\tilde{v}}$ be the canonical reduction homomorphism, see Remark 3.3 for definitions.

2) In the above notations, we say that Φ_α is a rational quotient of $\mathcal{G}$, if Φ_α is divisorial, and for all multi-indices $\tilde{v}$ the following hold: If $\jmath_{\tilde{v}}$ is non-trivial on $\widehat{\mathcal{L}}_\alpha \cap \widehat{U}_{\tilde{v}}$, then $\widehat{\mathcal{L}}_\alpha \subset \widehat{U}_{\tilde{v}}$, and $\jmath_{\tilde{v}}$ is injective on $\widehat{\mathcal{L}}_\alpha$, or equivalently, $\jmath_{\tilde{v}} \circ \hat{\phi}_\alpha$ is injective on $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$.

3) Let $\mathcal{L}_{\mathcal{G}}$ be a divisorial $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}$. Since Φ_α is divisorial, by Remark 3.5, 5), it follows that there exists a unique divisorial lattice $\mathcal{L}_{\mathcal{G}_\alpha}$ in $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$ such that $\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \mathcal{L}_{\mathcal{G}}$. Moreover, since Φ_α defines $\mathcal{G}_\alpha$ as a quotient of $\mathcal{G}$, and $\pi_{1,\mathcal{G}_\alpha} = 1$ is trivial, by Remark 3.5, 5), it follows that $\mathcal{L}_\alpha := \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha})$ can be recovered from $\widehat{\mathcal{L}}_\alpha = \hat{\phi}_\alpha(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha})$ and $\mathcal{L}_{\mathcal{G}}$ as follows:

$$(*) \qquad \mathcal{L}_\alpha := \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) = \widehat{\mathcal{L}}_\alpha \cap \mathcal{L}_{\mathcal{G}}\,.$$

Definition 3.9. Let $\mathcal{G}$ be a divisorial abstract decomposition graph, and let $\mathcal{L}_{\mathcal{G}} \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ be a fixed divisorial $\widehat{U}_{\mathcal{G}}$-lattice. Let $\mathfrak{A}_0 = \{\Phi_\alpha\}_\alpha$ be the set of rational quotients of $\mathcal{G}$. For every subset $\mathfrak{A} \subseteq \mathfrak{A}_0$, we denote

$$\mathcal{L}_{\mathfrak{A}} = < \mathcal{L}_\alpha \mid \text{all } \alpha \text{ with } \Phi_\alpha \in \mathfrak{A} >$$

the $\mathbb{Z}_{(\ell)}$-submodule of $\mathcal{L}_{\mathcal{G}} \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ generated by all the $\mathcal{L}_\alpha$ with $\Phi_\alpha \in \mathfrak{A}$.

1) We say that $\mathfrak{A}$ is an ample set of rational quotients of $\mathcal{G}$, if the following hold:

i) For all α, α' one has: If $\Phi_\alpha \neq \Phi_{\alpha'}$, then $\widehat{\mathcal{L}}_\alpha \cap \widehat{\mathcal{L}}_{\alpha'} = (0)$.
ii) $\mathcal{L}_{\mathfrak{A}} \cap \widehat{U}_{\mathcal{G}} = (0)$ and $\mathcal{L}_{\mathfrak{A}}$ is ℓ-adically dense in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

2) Suppose that $\mathfrak{A}$ is an ample set of rational quotients of $\mathcal{G}$. We will say that $\mathcal{G}$ is geometric like with respect to $\mathfrak{A}$, if for every α, α' there exists a multi-index $\tilde{v}$ of $\mathcal{G}$ such that:

j) $\widehat{\mathcal{L}}_\alpha$ and $\widehat{\mathcal{L}}_{\alpha'}$ are contained in $\widehat{U}_{\tilde{v}}$.

jj) $\jmath_{\tilde{v}}$ maps $\widehat{\mathcal{L}}_\alpha$ and $\widehat{\mathcal{L}}_{\alpha'}$ injectively into $\mathcal{L}_{\mathcal{G}_{\tilde{v}}}$, and $\jmath_{\tilde{v}}(\widehat{\mathcal{L}}_\alpha) = \jmath_{\tilde{v}}(\widehat{\mathcal{L}}_{\alpha'})$.

3) In the above context, we will call $\mathcal{L}_{\mathfrak{A}}$ an $\mathfrak{A}$-arithmetical lattice. Its ℓ-adic equivalence class depends in general on $\mathfrak{A}$, and not only on equivalence class of $\mathcal{L}_{\mathcal{G}}$. Further,

$$\widehat{U}_{\mathcal{G}} + \mathcal{L}_{\mathfrak{A}} \subseteq \mathcal{L}_{\mathcal{G}}$$

is a $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$, and therefore $\mathcal{L}_{\mathcal{G}} \,/\, (\widehat{U}_{\mathcal{G}} + \mathcal{L}_{\mathfrak{A}})$ is a torsion free divisible group, hence a $\mathbb{Q}$-vector space. But in general, $\widehat{U}_{\mathcal{G}} + \mathcal{L}_{\mathfrak{A}}$ is not necessarily a divisorial $\widehat{U}_{\mathcal{G}}$-lattice.

Definition/Remark 3.10. Let $\mathcal{G}$ and $\mathcal{H}$ be geometric like abstract decomposition graphs with respect to some sets of rational quotients $\mathfrak{A}_0 = \{\Phi_\alpha\}_\alpha$, respectively $\mathfrak{B}_0 = \{\Phi_\beta\}_\beta$, and let a proper morphism $\Phi : \mathcal{G} \to \mathcal{H}$ of level $\delta := \delta_H$ be given.

1) We say that Φ is compatible with rational quotients, if there exist ample subsets $\mathfrak{A} \subseteq \mathfrak{A}_0$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$ satisfying the following: First, $\mathcal{G}$ and $\mathcal{H}$ are geometric like with respect to $\mathfrak{A}$, respectively $\mathfrak{B}$. Second, for each $\Psi_\beta \in \mathfrak{B}$ there exists $\Phi_\alpha \in \mathfrak{A}$ and an isomorphism $\Phi_{\alpha\beta} : \mathcal{G}_\alpha \to \mathcal{H}_\beta$ such that the following diagram is commutative:

$$(*) \qquad \begin{tikzcd} \mathcal{G} \arrow[r, "\Phi"] \arrow[d, "\Phi_\alpha"] & \mathcal{H} \arrow[d, "\Psi_\beta"] \\ \mathcal{G}_\alpha \arrow[r, "\Phi_{\alpha\beta}"] & \mathcal{H}_\beta \end{tikzcd}$$

2) We remark that in the above context, for every $\Psi_\beta \in \mathfrak{B}$ there exists a unique Φ_α satisfying hypothesis $(*)$. Indeed, let $\Phi_{\alpha'}$ together with $\Phi_{\alpha'\beta}$ also satisfy hypothesis $(*)$. Then we have $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha} = \hat{\phi}_{\alpha\beta}(\widehat{\mathcal{L}}_{\mathcal{H}_\beta})$, and by the commutativity of the diagram $(*)$, we get:

$$\widehat{\mathcal{L}}_\alpha := \hat{\phi}_\alpha(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) = \hat{\phi}_\alpha\big(\hat{\phi}_{\alpha\beta}(\widehat{\mathcal{L}}_{\mathcal{H}_\beta})\big) = \hat{\phi}\big(\hat{\psi}_\beta(\widehat{\mathcal{L}}_{\mathcal{H}_\beta})\big) = \hat{\phi}(\widehat{\mathcal{L}}_\beta).$$

Since the same is true correspondingly for α', we finally get $\widehat{\mathcal{L}}_\alpha = \hat\phi(\widehat{\mathcal{L}}_\beta) = \widehat{\mathcal{L}}_{\alpha'}$. But then by Definition 3.9, 1), i) it follows that $\Phi_\alpha = \Phi_{\alpha'}$, as claimed.

In the above context, we say that α corresponds to β, if the hypothesis $(*)$ is satisfied for Ψ_β and Φ_α. The discussion above shows that α corresponds to β if and only if $\hat\phi(\widehat{\mathcal{L}}_\beta) = \widehat{\mathcal{L}}_\alpha$.

Proposition 3.11. *In the context of the Definition above, let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level $\delta := \delta_H$ proper morphism of geometric like abstract decomposition graphs which is compatible with the rational quotients $\mathfrak{A}$ and $\mathfrak{B}$. Then Φ is divisorial.*

1) *More precisely, let $\hat\phi : \widehat{\mathcal{L}}_{\mathcal{H}} \to \widehat{\mathcal{L}}_{\mathcal{H}}$ be the Kummer homomorphism of Φ. Let $\mathcal{L}_{\mathfrak{B}}$ be an arithmetical lattice for $\mathcal{H}$ defined by $\mathfrak{B}$. Then there exists a unique arithmetical lattice $\mathcal{L}_{\mathfrak{A}}$ for $\mathcal{G}$ defined by $\mathfrak{A}$ such that $\hat\phi(\mathcal{L}_{\mathfrak{B}}) \subseteq \mathcal{L}_{\mathfrak{A}}$, and moreover, one actually has:*

$$\hat\phi(\mathcal{L}_{\mathfrak{B}}) = \hat\phi(\widehat{\mathcal{L}}_{\mathcal{H}}) \cap \mathcal{L}_{\mathfrak{A}}\,.$$

2) *In the above context, suppose that $\hat\phi(\mathcal{L}_{\mathfrak{B}}) \subseteq \mathcal{L}_{\mathfrak{A}}$. For each β, let $\mathcal{L}_{\mathcal{H}_\beta} \subset \widehat{\mathcal{L}}_{\mathcal{H}_\beta}$ be the unique divisorial lattice such that $\mathcal{L}_\beta := \hat\phi_\beta(\mathcal{L}_{\mathcal{H}_\beta}) = \hat\phi_\beta(\widehat{\mathcal{L}}_{\mathcal{H}_\beta}) \cap \mathcal{L}_{\mathfrak{B}}$. And for each α, let $\mathcal{L}_{\mathcal{G}_\alpha} \subset \widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$ be the unique divisorial lattice such that $\mathcal{L}_\alpha := \hat\phi_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) = \hat\phi_\alpha(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) \cap \mathcal{L}_{\mathfrak{A}}$. Then for all α, β one has: $\Phi_\alpha \in \mathfrak{A}$ corresponds to $\Psi_\beta \in \mathfrak{B}$ if and only if*

$$\hat\phi_{\alpha\beta}(\mathcal{L}_{\mathcal{H}_\beta}) = \mathcal{L}_{\mathcal{G}_\alpha}, \quad \hat\phi(\mathcal{L}_\beta) = \mathcal{L}_\alpha\,.$$

Proof. It is clear that assertion 2) follows from assertion 1) and previous discussion. Therefore we will concentrate on the proof of assertion 1).

First recall that by Definition/Remark 3.10, 2), we have: α corresponds to β if and only if $\hat\phi(\widehat{\mathcal{L}}_\beta) = \widehat{\mathcal{L}}_\alpha$. Using this we deduce:

- $\hat\phi$ maps $\widehat{\mathcal{L}}_{\mathfrak{B}} := \sum_{\Psi_\beta \in \mathfrak{B}} \widehat{\mathcal{L}}_\beta$ into $\widehat{\mathcal{L}}_{\mathfrak{A}} := \sum_{\Phi_\alpha \in \mathfrak{A}} \widehat{\mathcal{L}}_\alpha$.

- Let $\mathcal{L}_{\mathfrak{B}}$ and $\mathcal{L}_{\mathfrak{A}}$ be fixed arithmetical lattices of $\mathcal{H}$, respectively $\mathcal{G}$. For a given β, choose α corresponding to it. By the Definition/Remark 3.8, 3) above, and in the notations from there we have: There exists a unique divisorial lattice $\mathcal{L}_{\mathcal{G}_\alpha}$ in $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$ such that $\hat\phi_\alpha$ maps $\mathcal{L}_{\mathcal{G}_\alpha}$ into $\mathcal{L}_{\mathfrak{A}}$, and actually $\hat\phi_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) = \widehat{\mathcal{L}}_\alpha \cap \mathcal{L}_{\mathfrak{A}}$. And correspondingly, the same is true for β, i.e.: There exists a unique $\mathcal{L}_{\mathcal{H}_\beta}$ in $\widehat{\mathcal{L}}_{\mathcal{H}_\beta}$ such that $\hat\psi_\beta(\mathcal{L}_{\mathcal{H}_\beta}) = \widehat{\mathcal{L}}_\beta \cap \mathcal{L}_{\mathfrak{B}}$.

Since α corresponds to β, in the notations from Definition 3.10, let $\hat\phi_{\alpha\beta}$ be the Kummer isomorphism defined by $\Phi_{\alpha\beta}$. Then $\hat\phi_{\alpha\beta}(\mathcal{L}_{\mathcal{H}_\beta})$ is a divisorial lattice in $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$. Thus there exists an ℓ-adic unit $\epsilon_{\alpha\beta}$ such that

$$\hat\phi_{\alpha\beta}(\mathcal{L}_{\mathcal{H}_\beta}) = \epsilon_{\alpha\beta} \cdot \mathcal{L}_{\mathcal{G}_\alpha}\,.$$

On the other hand, the commutativity of the diagram $(*)$ from Definition/Remark 3.10 translated in terms of Kummer homomorphisms means that the above equality is equivalent to the following: For all β and its corresponding α one has:

$$(\alpha\beta) \qquad \hat\phi(\mathcal{L}_\beta) = \epsilon_{\alpha\beta} \cdot \mathcal{L}_\alpha\,.$$

Let β and β', and the corresponding α and α' be given. Hence $\hat\phi$ maps $\widehat{\mathcal{L}}_\beta$ and $\widehat{\mathcal{L}}_{\beta'}$ isomorphically onto $\widehat{\mathcal{L}}_\alpha$, respectively $\widehat{\mathcal{L}}_{\beta'}$. Since $\mathcal{G}$ is geometric like with respect to the family of rational projections $\mathfrak{A}$, it follows that there exists some multi-index $\tilde v$ of $\mathcal{G}$ which has the properties j), jj), of Definition 3.9, 2).

Before moving on, we recall that by Fact 2.8, 2), the fixed divisorial $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{\mathcal{G}}$ of $\mathcal{G}$ defines uniquely a $\tilde{v}$-residual $\widehat{U}_{\mathcal{G}_{\tilde{v}}}$-lattice $\mathcal{L}_{\mathcal{G}_{\tilde{v}}}$ by setting

$$\mathcal{L}_{\mathcal{G}_{\tilde{v}}} := \widehat{U}_{\mathcal{G}_{\tilde{v}}} + \jmath_{\tilde{v}}(\mathcal{L}_{\mathcal{G}} \cap \widehat{U}_{\tilde{v}}).$$

We further remark that condition j) from Definition 3.9, 2) implies that $\Phi_\alpha(T_{\tilde{v}}) = 1$. Hence Φ_α gives rise to a residual morphism $\Phi_{\tilde{v}\alpha} : \mathcal{G}_{\tilde{v}} \to \mathcal{G}_\alpha$. And if $\hat{\phi}_{\tilde{v}\alpha} : \widehat{\mathcal{L}}_{\mathcal{G}_\alpha} \to \mathcal{L}_{\mathcal{G}_{\tilde{v}}}$ is the Kummer homomorphism of $\Phi_{\tilde{v}\alpha}$, then $\jmath_{\tilde{v}} \circ \hat{\phi}_\alpha = \hat{\phi}_{\tilde{v}\alpha}$. Therefore we have

$$\jmath_{\tilde{v}}(\widehat{\mathcal{L}}_\alpha) = \hat{\phi}_{\tilde{v}\alpha}(\mathcal{L}_{\mathcal{G}_\alpha}), \quad \jmath_{\tilde{v}}(\mathcal{L}_\alpha) = \hat{\phi}_{\tilde{v}\alpha}(\mathcal{L}_{\mathcal{G}_\alpha}).$$

Now since Φ_α is divisorial, $\Phi_{\tilde{v}\alpha}$ is so by definition. Hence by Remark 3.5, 5), we have:

$$\hat{\phi}_{\tilde{v}\alpha}(\mathcal{L}_{\mathcal{G}_\alpha}) = \hat{\phi}_{\tilde{v}\alpha}(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) \cap \mathcal{L}_{\mathcal{G}_{\tilde{v}}}\,.$$

Thus combining the assertions above, we finally get

$$\jmath_{\tilde{v}}(\mathcal{L}_\alpha) = \jmath_{\tilde{v}}(\widehat{\mathcal{L}}_\alpha) \cap \mathcal{L}_{\mathcal{G}_{\tilde{v}}}\,.$$

On the other hand, both α and α' satisfy condition j) from Definition 3.9, 2). Hence by symmetry, the equalities above hold correspondingly for α' too. And since by condition jj) of Definition 3.9, 2), one has $\hat{\phi}_{\tilde{v}\alpha}(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) =: \widehat{\mathcal{L}}_{\tilde{v},\alpha\alpha'} := \hat{\phi}_{\tilde{v}\alpha'}(\widehat{\mathcal{L}}_{\alpha'})$, we finally get:

$$\jmath_{\tilde{v}}(\mathcal{L}_\alpha) = \widehat{\mathcal{L}}_{\tilde{v},\alpha\alpha'} \cap \mathcal{L}_{\mathcal{G}_{\tilde{v}}} = \jmath_{\tilde{v}}(\mathcal{L}_{\alpha'})\,. \tag{α}$$

On the other hand, since Φ is proper, there exists some $\tilde{w}$ corresponding to $\tilde{v}$. Recall the second diagram in Remark 3.3, 3), from which we bring forward:

$$\begin{tikzcd}
\widehat{U}_{\tilde{w}} \arrow[r, "\jmath_{\tilde{w}}"] \arrow[d, "\hat{\phi}"] & \widehat{\mathcal{L}}_{\mathcal{H}_{\tilde{w}}} \arrow[d, "\hat{\phi}_{\tilde{v}}"] \\
\widehat{U}_{\tilde{v}} \arrow[r, "\jmath_{\tilde{v}}"] & \widehat{\mathcal{L}}_{\mathcal{G}_{\tilde{v}}}
\end{tikzcd}$$

and recall that $\hat{\phi}$ and $\hat{\phi}_{\tilde{v}}$ are injective. Since $\widehat{\mathcal{L}}_\beta = \hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha})$, and $\widehat{\mathcal{L}}_{\beta'} = \hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{G}_{\alpha'}})$, we get:

c) $\widehat{\mathcal{L}}_\beta, \widehat{\mathcal{L}}_{\beta'} \subset \widehat{U}_{\tilde{w}}$.

d) $\widehat{\mathcal{L}}_\beta$ and $\widehat{\mathcal{L}}_{\beta'}$ are mapped by $\jmath_{\tilde{w}} : \widehat{U}_{\tilde{w}} \to \widehat{\mathcal{L}}_{\mathcal{H}_{\tilde{w}}}$ injectively into $\widehat{\mathcal{L}}_{\mathcal{H}_{\tilde{w}}}$, and have equal images $\jmath_{\tilde{w}}(\widehat{\mathcal{L}}_\beta) =: \widehat{\mathcal{L}}_{\tilde{w},\beta\beta'} := \jmath_{\tilde{w}}(\widehat{\mathcal{L}}_{\beta'})$.

And note that $\hat{\phi}_{\tilde{v}}$ maps $\mathcal{L}_{\tilde{w},\beta\beta'}$ isomorphically onto $\mathcal{L}_{\tilde{v},\alpha\alpha'}$. Then going through the same steps as above and using notations correspondingly, we finally get as above:

$$\jmath_{\tilde{w}}(\mathcal{L}_\beta) = \jmath_{\tilde{w}}(\mathcal{L}_{\beta'})\,. \tag{β}$$

We conclude the proof of the Proposition as follows: For β, β' and α, α' corresponding to them, in the notations from above we have by relation (α) above:

$$\epsilon_{\alpha\beta} \cdot \mathcal{L}_\alpha = \hat{\phi}(\mathcal{L}_\beta) \quad \text{and} \quad \epsilon_{\alpha'\beta'} \cdot \mathcal{L}_{\alpha'} = \hat{\phi}(\mathcal{L}_{\beta'})$$

for some ℓ-adic units $\epsilon_{\alpha\beta}$ and $\epsilon_{\alpha'\beta'}$. Applying $\jmath_{\tilde{v}}$ to the above equalities, and taking into account that by the commutativity of the diagram above one has $\jmath_{\tilde{v}} \circ \hat{\phi} = \hat{\phi}_{\tilde{v}} \circ \jmath_{\tilde{w}}$ on $\widehat{U}_{\tilde{w}}$, thus on $\mathcal{L}_\beta$, $\mathcal{L}_{\beta'} \subset \widehat{U}_{\tilde{w}}$, we finally get:

$$\jmath_{\tilde{v}}(\epsilon_{\alpha\beta} \cdot \mathcal{L}_\alpha) = \jmath_{\tilde{v}}\big(\hat{\phi}(\mathcal{L}_\beta)\big) = (\jmath_{\tilde{v}} \circ \hat{\phi})(\mathcal{L}_\beta)\big) = (\hat{\phi}_{\tilde{v}} \circ \jmath_{\tilde{w}})(\mathcal{L}_\beta)\big) = \hat{\phi}_{\tilde{v}}\big(\jmath_{\tilde{w}}(\mathcal{L}_\beta)\big)$$

and correspondingly

$$\jmath_{\tilde v}(\epsilon_{\alpha'\beta'}\cdot\mathcal{L}_{\alpha'}) = \jmath_{\tilde v}\big(\hat\phi(\mathcal{L}_{\beta'})\big) = (\jmath_{\tilde v}\circ\hat\phi)(\mathcal{L}_{\beta'})\big) = (\hat\phi_{\tilde v}\circ\jmath_{\tilde w})(\mathcal{L}_{\beta'})\big) = \hat\phi_{\tilde v}\big(\jmath_{\tilde w}(\mathcal{L}_{\beta'})\big).$$

On the other hand, $\jmath_{\tilde w}(\mathcal{L}_\beta) = \jmath_{\tilde w}(\mathcal{L}_{\beta'})$ by remark (β) above, hence the last two terms of the equalities above are equal. Thus we get

$$\jmath_{\tilde v}(\epsilon_{\alpha\beta}\cdot\mathcal{L}_\alpha) = \jmath_{\tilde v}(\epsilon_{\alpha'\beta'}\cdot\mathcal{L}_{\alpha'}), \quad \text{hence} \quad \epsilon_{\alpha\beta}\cdot\jmath_{\tilde v}(\mathcal{L}_\alpha) = \epsilon_{\alpha'\beta'}\cdot\jmath_{\tilde v}(\mathcal{L}_{\alpha'}).$$

On the other hand, we also have $\jmath_{\tilde v}(\mathcal{L}_\alpha) = \jmath_{\tilde v}(\mathcal{L}_{\alpha'})$, by equalities (α) above. Thus finally

$$\epsilon_{\alpha\beta}\cdot\jmath_{\tilde v}(\mathcal{L}_\alpha) = \epsilon_{\alpha'\beta'}\cdot\jmath_{\tilde v}(\mathcal{L}_\alpha)\,.$$

Now recall that if $\hat\phi_{\tilde v\alpha} : \widehat{\mathcal{L}}_{\mathcal{G}_\alpha} \to \widehat{\mathcal{L}}_{\mathcal{G}_{\tilde v}}$ is the Kummer homomorphism of $\Phi_{\tilde v\alpha} : \mathcal{G}_{\tilde v} \to \mathcal{G}_\alpha$, then we have $\jmath_{\tilde v}(\mathcal{L}_\alpha) = \hat\phi_{\tilde v\alpha}(\mathcal{L}_{\mathcal{G}_\alpha})$, and the latter is a $\widehat{U}_{\mathcal{G}_{\tilde v}}$-sublattice of $\mathcal{L}_{\mathcal{G}_{\tilde v}}$. Hence finally $\epsilon_{\alpha\beta}/\epsilon_{\alpha'\beta'}$ must be a rational ℓ-adic unit. Since β, β' were arbitrary, we conclude that for every fixed β_0 and the corresponding α_0, after setting $\epsilon := \epsilon_{\alpha_0\beta_0}$, one has

$$\hat\phi(\mathcal{L}_\beta) = \epsilon\cdot\mathcal{L}_\alpha\,.$$

Equivalently, $\hat\phi$ maps $\mathcal{L}_{\mathfrak{B}} = \sum_\beta \mathcal{L}_\beta$ into $\epsilon\cdot\mathcal{L}_{\mathfrak{A}} = \epsilon\cdot\sum_\beta\mathcal{L}_\alpha$. □

C) *Morphisms arising from algebraic geometry*

Let k and l be algebraically closed fields of characteristic $\neq \ell$. Let $K|k$ and $L|l$ be function fields, and let

$$\imath : L|l \hookrightarrow K|k$$

be an embedding of function fields such that l is mapped isomorphically onto k, and $K|\imath(L)$ is a separable field extension, see e.g. Lang [L2] for a thorough discussion of this situation.

As introduced in the Introduction, let $\mathcal{D}_K^{\rm tot}$ and $\mathcal{D}_L^{\rm tot}$ be the total graphs of prime divisors on K, respectively on L. Then $\imath$ gives rise in a canonical way to a morphism of graphs prime divisors

$$\varphi_\imath : \mathcal{D}_K^{\rm tot} \to \mathcal{D}_L^{\rm tot}\,.$$

The precise definition of $\varphi_\imath$ is as follows: First let v be a prime divisor of $K|k$. Then the restriction $v_L := v|_L$ of v to $L|l$ is either the trivial valuation w_0 of $L|l$, or v_L is a prime divisor of $L|l$ otherwise. In both cases, $\imath$ gives rise to an embedding of the residue function fields

$$\imath_v : (Lv_L)|l \hookrightarrow (Kv)|k\,.$$

Inductively, we deduce from this: If $\tilde v = v_r\circ\cdots\circ v_1$ is a prime r-divisor of $K|k$ as defined in Introduction, then $\tilde w := \tilde v|_L$ is a prime s-divisor of $L|l$ for some $0 \leq s \leq r$. Moreover, by general valuation theory, it follows that every generalized prime divisor of $L|l$ is the restriction of some generalized prime divisor of $K|k$, hence $\varphi_\imath$ is surjective, etc.

The situation will become clearer after analyzing in more detail how *geometric prime divisor graphs* $\mathcal{D}_K$ of $K|k$ behave under $\varphi_\imath$.

First, remark that if $K|\imath(L)$ is finite, then for every generalized prime divisor $\tilde w$ of $L|l$, its fiber is finite of cardinality bounded by $[K : \imath(L)]$. From this one immediately deduces that the image of every geometric decomposition graph for $K|k$ under $\varphi_\imath$ is a geometric decomposition graph for $L|l$, etc.

Therefore, let us assume from now on that $K|\imath(L)$ is not algebraic. Then denoting by $K_1|k$ the relative algebraic closure of $\imath(L)$ in K, we have: $K_1|\imath(L)$ is finite separable, and $K|K_1$ is a regular function field extension. The situation of $L|l \hookrightarrow K_1|k$ was explained above. Thus *mutatis mutandis,* we can suppose that $K|\imath(L)$ is *regular field extension.*

Lemma 3.12. *Let $X \to k$ be a projective normal normal model for $K|k$, and let $D \supseteq D_X$ be any set of prime divisors with $D\backslash D_X$ finite. Then there exists a projective normal model $\tilde{X} \to k$ and a dominant k morphism $\phi : \tilde{X} \to X$ such that $D \subset D_{\tilde{X}}$.*

Proof. Clear. □

Using the Lemma above, we have: There exist projective normal models $X \to k$ for $K|k$ such that D_X contains the 1-edges of $\mathcal{D}_K$ and is complete regular like. And correspondingly, the same is thus for $L|l$ and $\mathcal{D}_L$. On the other hand, the regular embedding of function fields $\imath : L|l \to K|k$ is the generic fiber of a dominant rational map $f : X \dashrightarrow Y$ which factors through $\imath : l \to k$. And note that since X and Y are normal, f is defined at all points x_1 of co-dimension 1 of X. Moreover, replacing $X \to k$ by a properly chosen blowup, and normalizing the resulting k-variety, we can suppose that $f : X \to Y$ is a k-morphism of projective normal varieties. And since $K|\imath(L)$ is a regular field extension, it follows that $f : X \to Y$ has geometric generic integral fibers. Hence by the characterization of (the dimension of) the fibers the following hold:

- At almost all points x_1 of codimension 1, $f(x_1)$ is either the generic point of Y, or $y_1 = f(x_1)$ is a point of codimension 1 of Y otherwise.
- On a Zariski open subset $V \subset Y$, the fiber X_y at y is irreducible, and one has

$$\mathrm{codim}(y) + \dim(X_y) = \dim(X)\,.$$

Hence for almost all points y_1 of codimension 1 in Y, the fiber X_{y_1} is irreducible and has $\dim(X_{y_1}) = \dim(X) - 1$. Equivalently, X_{y_1} is a Weil prime divisor of X, and its generic point x_1 has codimension 1 in X.

In birational terms this means the following: For every prime divisor $v = v_{x_1} \in D_X$ let $w := v|_L = \varphi_\imath(v)$ be its restriction to L. Then the center of w on Y is exactly $f(x_1)$, and we have the possibilities:

a) w is the trivial valuation iff $f(x_1)$ is the generic point of Y.

b) w is a prime divisor of $L|l$ and $f(x_1)$ has codimension 1 in Y. Then w is the Weil prime divisor defined by $y_1 := f(x_1)$.

c) w is a prime divisor of $L|l$ and $f(x_1)$ has codimension > 1.

In particular, we see that the following hold: First, all $w \in D_Y$ have preimages v in D_X, and for almost all w the preimage v is unique. Second, there are at most finitely many "exceptional" $v \in D_X$ for which $\varphi_\imath(v)$ does lie in D_Y. Let Σ_f be that set.

We now claim that for the given models $X \to k$ and $Y \to l$ as above, there exist quasi projective normal models $X_0 \to k$ and $Y_0 \to l$ dominating $X \to k$ and $Y \to l$, and a morphism $f_0 : X_0 \to Y_0$ extending $f : X \to Y$, and having the following property:

$$(*) \qquad \varphi_\imath(D_{X_0} \cup \{v_0\}) = D_{Y_0} \cup \{w_0\}\,.$$

Note that in particular, $D_X \subseteq D_{X_0}$ and $D_Y \subseteq D_{Y_0}$.

Indeed, if $\varphi_\imath(D_X \cup \{v_0\}) = D_Y \cup \{w_0\}$, i.e., if the exceptional set Σ_f is empty, then there is nothing to prove. Hence consider some $v := v_{x_1} \in \Sigma_f$, such that the center y_v of $w = \varphi_\imath(v)$ has co-dimension > 1. Let $Y_v \subset Y$ be the closure of y_v in Y. Then setting $Y_1 := Y$, and $Z_1 := Y_v$, we consider a sequence of blowups $\ldots \to Y_{n+1} \to Y_n \to \ldots$ as follows: $Z_n \subset Y_n$ is the closure of the center of w on Y_n. We stop if Z_n has codimension 1, and blow up Z_n otherwise. Then the above sequence is finite. Moreover, if $\operatorname{codim}(Z_n) > 1$, then $Y_{n+1} \to Y_n$ is an isomorphism outside Z_n. But then if the process above stops say at Y_n, it follows that Z_n is the center of w on Y_n, and $\operatorname{codim}(Z_n) = 1$. An easy Noether induction shows that one gets models Y_0' dominating Y such that $\varphi_\imath(D_X) \subseteq D_{Y_0'} \cup \{w_0\}$. On the other hand, $f : X \to Y$ can be interpreted as a dominant rational map $f_0 : X \dashrightarrow Y_0'$. Since X is normal, and Y_0' is complete, f_0 is defined at all points $v \in D_X$, and map these points into D_{Y_0} by the discussion above. To conclude, let $S_Y \subset Y_0'$ be the Zariski closure of the (finite) complement of $D_{Y_0'} \backslash \varphi_\imath(D_X)$, and S_X the preimage of S_Y under f. Finally, set $Y_0 := Y_0' \backslash S_Y$, and $X_0 = U(f) \backslash S_X$, where $U(f)$ is the the domain of f. Then by the choices made, it follows that f defines a dominant morphism $f_0 : X_0 \to Y_0$ which has the required property $(*)$.

Now proceeding by induction on the transcendence degree of the residual function fields $L\tilde{w}|l \hookrightarrow K\tilde{v}|k$, and using the fact $(*)$ above, we finally get the following:

Proposition 3.13. *In the above context, let $\mathcal{D}_K \subset \mathcal{D}_K^{\rm tot}$ and $\mathcal{D}_L \subset \mathcal{D}_L^{\rm tot}$ be geometric graphs of prime divisors for $K|k$, respectively $L|l$. Then there exists a unique maximal geometric subgraph $\mathcal{D}'_K \subset \mathcal{D}_K$ such that $\varphi_\imath$ defines by restriction a morphism of graphs of prime divisors*

$$\varphi_\imath : \mathcal{D}'_K \to \mathcal{D}_L\,.$$

Moreover, for given geometric graphs $\mathcal{D}_K \subset \mathcal{D}_K^{\rm tot}$ and $\mathcal{D}_L \subset \mathcal{D}_L^{\rm tot}$ as above, there exist geometric graphs of prime divisors $\mathcal{D}_K^0 \supseteq \mathcal{D}_K$ and $\mathcal{D}_L^0 \supseteq \mathcal{D}_L$ for $K|k$, respectively $L|l$, such that $\varphi_\imath$ defines by restriction a surjective morphism of graphs of prime divisors

$$\varphi_\imath : \mathcal{D}_K^0 \to \mathcal{D}_L^0\,.$$

Using the Galois theory and decomposition theory of valuations, the above facts have the following translation in terms of abstract decomposition graphs:

Let $\imath' : L' \to K'$ be a prolongation of $\imath : L|l \to K|k$ to L', and let

$$\Phi_\imath : G_K^{'} \to G_L^{'}$$

be the corresponding canonical projection of Galois groups. Then since $\imath : L|l \to K|k$ is a morphism of function fields, it follows that the relative algebraic closure L_1 of $L|l$ in $K|k$ is a finite extension of L, thus a function field over l. But then it follows that $\Phi_\imath$ is an open homomorphism.

Moreover, if $\varphi_\imath(v) = w$, and v' is a prolongation of v to K', then the restriction w' of v' to L' satisfies: First, w' is a prolongation of w to L'. Second, let $T_v \subset Z_v$ and $T_w \subset Z_w$ be the corresponding decomposition groups. Then $\Phi_\imath(Z_v) \subset Z_w$ and $\Phi_\imath(T_v) \subset T_w$ are open subgroups. (This discussion includes the case where w is the trivial valuation of L.) Moreover, if w is non-trivial, then $wL \subset vK$ has finite index $e(v|w)$. Hence we have commutative diagrams of the form:

$$\begin{tikzcd}
L \arrow[r, "w"] \arrow[d, "\imath"] & wL \subset \widehat{wL} = \mathrm{Hom}\big(T_w, \mathbb{Z}_\ell\big) \arrow[d, "e(v|w)"] \\
K \arrow[r, "v"] & vK \subset \widehat{vK} = \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big)
\end{tikzcd}$$

Therefore, if γ_w and γ_v are the unique positive generators of vK, respectively wL, then γ_w is mapped to $e(v|w)\cdot\gamma_v$. Thus if $\tau_v \in T_v$ and $\tau_w \in T_w$ are the arithmetical inertia generators as defined/introduced at Remark 2.14, 2), then from the commutativity of the above diagrams and definitions it follows that $\Phi_\imath(\tau_v) = \tau_w^{e(v|w)}$.

Now combining this observation with Proposition 3.13 above, we obtain the following by merely applying the definitions:

Proposition 3.14. *In the notations from Proposition 3.13, the embedding of function fields $\imath : L|l \hookrightarrow K|k$ and the resulting canonical homomorphism $\Phi_\imath : G_K^{'} \to G_L^{'}$ give rise in a natural way to a level $\delta = \mathrm{tr.deg}(L|l)$ morphism $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_L}$ of the corresponding abstract decomposition graphs.*

1) *Moreover, if $\varphi_\imath : \mathcal{D}_K \to \mathcal{D}_L$ is a proper morphism of graphs of prime divisors, then the corresponding $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_L}$ is a proper morphism of abstract decomposition graphs.*

2) *The Kummer homomorphism $\hat{\phi} : \widehat{L} \to \widehat{K}$ of Φ is actually the ℓ-adic completion of the embedding of function fields $\imath : L|l \hookrightarrow K|k$. In particular, $\imath$ defines $\Phi_\imath$ uniquely.*

3) *Moreover, $\Phi_\imath$ defines $\imath$ uniquely up to Frobenius twists.*

Proof. We already proved 1), and 2) above.

To 3): Recall that in Introduction we considered an identification $\imath_K : \mathbb{T}_{\ell,K} \to \mathbb{Z}_\ell$ of the ℓ-adic Tate module of K with $\mathbb{Z}_\ell$, and via that identification one gets the identification $\widehat{K} = \mathrm{Hom}_{\mathrm{cont}}(G_K^{'}, \mathbb{Z}_\ell)$. Explicitly, this identification works as follows: For each $x \in K^\times$, let $\delta(x) : G_K^{'} \to \mathbb{T}_{\ell,K}$ be the corresponding character defined in Kummer Theory. Then $\delta_x := \imath_K \circ \delta(x)$ is the homomorphism $\delta_x : G_K^{'} \to \mathbb{Z}_\ell$ defined by x. Given the embedding $\imath : L|l \hookrightarrow K|k$, by the functoriality of Kummer Theory one has $\delta\big(\imath(y)\big) = \imath \circ \delta(x) \circ \Phi$. Therefore, if we choose the identifications $\imath_K : \mathbb{T}_{\ell,K} \to \mathbb{Z}_\ell$, $\imath_L : \mathbb{T}_{\ell,L} \to \mathbb{Z}_\ell$ compatible with $\imath$, i.e., such that $\imath_L = \imath_K \circ \imath$, it follows that $\delta_{\imath(y)} = \delta_u \circ \Phi$, hence the Kummer homomorphism defined by $\Phi_\imath$ is:

$$\hat{\phi} : \widehat{L} = \mathrm{Hom}(G_L^{'}, \mathbb{Z}_\ell) \to \mathrm{Hom}(G_K^{'}, \mathbb{Z}_\ell) = \widehat{K}, \quad \delta_y \mapsto \delta_{\imath(y)},$$

and therefore, $\hat{\phi}$ is exactly the ℓ-adic completion of the embedding $\imath : L^\times \to K^\times$.

Now let $\imath' : L|l \hookrightarrow K|k$ be a further embedding of function fields such that $\Phi_{\imath'} = \Phi_\imath$. Then choosing $\imath'_K : \mathbb{T}_{\ell,L} \to \mathbb{Z}_\ell$ such that $\imath'_L = \imath_K \circ \imath'$, it follows that the Kummer homomorphism $\hat{\phi}'$ of $\Phi_{\imath'} = \Phi_\imath$ in this new setting is the ℓ-adic completion of $\imath'$. On the other hand, there exists an ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell^\times$ such that $\imath'_L = \epsilon \cdot \imath_L$. If so, then we have $\hat{\phi}' = \epsilon \cdot \hat{\phi}$ on $\widehat{L}$. Since $\hat{\phi}$ is the ℓ-adic completion of $\imath$, and $\hat{\phi}'$ is the ℓ-adic completion of $\imath'$, if we denote by $\jmath_K : K^\times \to \widehat{K}$ the ℓ-adic completion homomorphisms, we have:

$$\jmath_K\big(\imath'(y)\big) = \epsilon \cdot \jmath_K\big(\imath(y)\big), \quad y \in L^\times .$$

Therefore, ϵ must be a rational ℓ-adic unit, say $\epsilon = m/n$ with n, m natural numbers relatively prime to ℓ. But then for every $y \in L$ there exists $a_y \in k$ such that $\imath'(y) = a_y \imath(y)^{m/n}$ in K, hence $\imath'(y) = u^{m/n}$ in K, as k is algebraically closed. But then $\imath'(y)$ is a n^{th} power in K. Since this is the case for all $\imath'(y) \in \imath'(L)$, it finally follows that $n = p^k$ is a power of the characteristic exponent p of k and l. By symmetry, the same is true for m. Hence finally ϵ is a power of the characteristic exponent of k and l. Equivalently, $\imath'$ is a Frobenius twist of $\imath$. □

D) *Rational quotients from algebraic geometry*

Next we turn our attention to rational projections of abstract decomposition graphs $\mathcal{G}_{\mathcal{D}_K}$ as above. Let $t \in K$ be an arbitrary non-constant function, and let K_t be the relative algebraic closure of $k(t)$ in K. Then $K_t|k$ is a function field in one variable. We endow $K_t|k$ with its unique complete normal model $X_t \to k$ –which is also projective, and consider the corresponding graph of prime divisors $\mathcal{D}_{Kt}$ on K_t, and the resulting abstract decomposition graph $\mathcal{G}_{K_t}$ on G'_{K_t}. Note that by Proposition 2.17, $\mathcal{G}_{K_t}$ has level $\delta = 1$, and is complete curve like. Hence in particular, $\mathcal{G}_{K_t}$ is divisorial. Moreover, if g_t is the geometric genus of X_t, then we have:

- $\widehat{\mathfrak{Cl}}_{\mathcal{G}_{K_t}} \cong \mathbb{Z}_\ell$.
- $\widehat{U}_{\mathcal{G}_{K_t}} \cong \mathbb{Z}_\ell^{2g_t}$ as being the ℓ-adic dual of $\pi_1'(X_t) \cong \mathbb{Z}_\ell^{2g}$. In particular, g_t is encoded in $\mathcal{G}_{K_t}$.

Finally, the inclusion $\imath_t : K_t \hookrightarrow K$ gives rise to a surjective morphism $\Phi_{K_t} : G'_K \to G'_{K_t}$ of pro-ℓ groups, which in turn defines a level $\delta = 1$ divisorial morphism of abstract decomposition graphs $\Phi_{K_t} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{K_t}$.

Fact 3.15. Suppose that $\mathcal{G}_{\mathcal{D}_K}$ is a level $\delta = \text{tr.}\deg(K|k)$ divisorial abstract decomposition graph. Then for $x \in K$ the following are equivalent:

i) $\Phi_{K_x} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{K_x}$ is a rational quotient of $\mathcal{G}_{\mathcal{D}_K}$.

ii) K_x is a rational function field.

Proof. To i) $\Rightarrow$ ii): First recall that $\widehat{U}_{\mathcal{G}_{K_x}} \cong \mathbb{Z}_\ell^{2g_x}$, where g_x is the genus of X_x. Hence $\mathcal{G}_{K_x}$ is projective line like if and only if $g = 0$, or equivalently, K_x is a rational function field.

To ii) $\Rightarrow$ i), we have to check property 2) from Remark/Definition 3.8.

Step 1: $\mathcal{G}_{K_x}$ is projective line like. Indeed, by the discussion above, we have $\widehat{U}_{\mathcal{G}_{K_x}} = 0$ and $\widehat{\mathfrak{Cl}}_{\mathcal{G}_{K_x}} \cong \mathbb{Z}_\ell$, thus the claim.

Step 2: $\Phi_{K_x} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{K_x}$ defines $\mathcal{G}_{K_x}$ as a quotient of $\mathcal{G}_{\mathcal{K}}$. Indeed, first $\Phi_{K_x} : G'_K \to G'_{K_x}$ is surjective by the definition of K_x. Thus it is left to show that Φ_{K_x} is proper, i.e., to show: If $\Phi_{\tilde{v}} : \mathcal{G}_{\tilde{v}} \to \mathcal{G}_{K_x}$ is a level $\delta = 1$ residual morphism for Φ_{K_x}, then the following hold:

a) Each 1-index of $\mathcal{G}_{\tilde{v}}$ is mapped under $\Phi_{\tilde{v}}$ to some multi-index of $\mathcal{G}_{K_x}$.

b) Each multi-index of $\mathcal{G}_{K_x}$ corresponds to some multi-index of $\mathcal{G}_{\tilde{v}}$.

To prove a), let $X_{\tilde{v}} \to k$ be a normal model of $K_{\tilde{v}}|k$ such that $D_{X_{\tilde{v}}}$ is the set of all the 1-vertices of $\mathcal{G}_{\tilde{v}}$. Then $\imath : K_x \hookrightarrow K_{\tilde{v}}$ is defined by some dominant rational map $f : X_{\tilde{v}} \dashrightarrow X_x$. Since $X_{\tilde{v}}$ is normal and X_x is complete, it follows that f is defined at all points of codimension 1. This means that for every $v \in D_{X_{\tilde{v}}}$, we have: If v is trivial on K_x, then $f(v)$ is the generic point of X_x, hence v is mapped to the trivial index of $\mathcal{G}_{K_x}$; and if v is non-trivial on K_x, then $f(v)$ is a closed point in X_x.

To prove b), we proceed by induction on $d_{\tilde v} = \operatorname{tr.deg}(K_{\tilde v}|k)$. If $d_{\tilde v} = 1$, then $X_{\tilde v}$ is a normal curve. Since $\mathcal{G}_{\mathcal{D}_K}$ was assumed to be divisorial, $\mathcal{G}_{\tilde v}$ is divisorial too by definition. Hence by Proposition 2.17, 1), $X_{\tilde v} \to k$ is a complete normal curve. But then the dominant rational map $f : X_{\tilde v} \dashrightarrow X_x$ is a surjective morphism. Finally, if $d_{\tilde v} > 1$, then there exist "many" $w \in D_{X_{\tilde v}}$ which are trivial on K_x. But then $\Phi_{\tilde v}$ gives rise to a level $\delta = 1$ residual morphism $\Phi_w : \mathcal{G}_w \to \mathcal{G}_{K_x}$ of $\mathcal{G}_{\tilde v}$. Since $\operatorname{tr.deg}(Kw|k) < d_{\tilde v}$, by induction we have: Φ_w is proper. On the other hand, the set of vertices $\operatorname{Vert}_{\mathcal{G}_w}$ of $\mathcal{G}_w$ is contained in the set of vertices $\operatorname{Vert}_{\mathcal{G}_{\tilde v}}$. Hence finally every vertex of $\mathcal{G}_{K_x}$ corresponds to some vertex of $\mathcal{G}_{\tilde v}$.

Step 3: If $\jmath_{\tilde v}(\widehat{U}_v \cap \widehat{K_x})$ is non-trivial, then $\jmath_{\tilde v}$ maps $\widehat{K_x}$ injectively into $\widehat{K_{\tilde v}}$. Indeed, let $\tilde v = v_r \circ \cdots \circ v_1$ with v_i prime divisors. Then one has: If $\tilde v$ is not trivial on $K_x^\times$, then $K_x\tilde v = k$, hence $\jmath_{\tilde v}$ is trivial on $\widehat{K_x} \cap \widehat{U}_{\tilde v}$. Second, if $\tilde v$ is trivial on $K_x^\times$, then $K_x\tilde v = K_x$, hence $\widehat{K_x} \subseteq \widehat{U}_{\tilde v}$, and $\jmath_{\tilde v}$ is injective on $\widehat{K_x}$. □

Notations 3.16.

1) Let $\{\kappa_x = k(x) \mid x \text{ general element of } K\}$ be the set of all the subfields of K generated by general elements $x \in K$. Note that $\kappa_x = \kappa_{x'}$ if and only if $x' = (ax + b)/(cx + d)$ is a linear transformation of x.

Further let $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$ be the set of the corresponding rational quotients of $\mathcal{G}_{\mathcal{D}_K}$, and note that $\Phi_{\kappa_x} = \Phi_{\kappa_{x'}}$ if and only if $x' = (ax + b)/(cx + d)$ is a linear transformation of x.

2) In order to simplify notations, we identify κ_x with the corresponding subfield of K. This identification defines a canonical embedding $\widehat{\kappa}_x \hookrightarrow \widehat{K}$ which turns out to be the inflation map defined by the canonical projection $\Phi_{\kappa_x} : G'_K \to G'_{\kappa_x}$. Therefore, the ℓ-adic completion homomorphism $\jmath_K : K^\times \to \widehat{K}$ identifies then $\jmath_{\kappa_x}(\kappa_x^\times)$ with $\jmath_K(\kappa_x^\times)$ inside $\widehat{K}$.

3) Let $\imath : L|l \to K|k$ be an embedding of function fields such that $\imath(l) = k$ and $K|\imath(L)$ a separable field extension. Denote as above by $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$ and $\mathfrak{B}_L = \{\Psi_{\kappa_y}\}_{\kappa_y}$ the set of all the rational quotients of $K|k$, respectively $L|l$. Finally let $\mathfrak{B}_\imath \subseteq \mathfrak{B}_L$ be the set of all the Ψ_{κ_y} such that $\imath(\kappa_y)$ is relatively algebraically closed in K. Thus in the context of Proposition 3.14, by taking into account Fact 3.15, we get: For $\Psi_{\kappa_y} \in \mathfrak{B}_\imath$ and the corresponding $\Phi_{\kappa_x} \in \mathfrak{A}_K$, one has commutative diagrams in which $\Phi_{\kappa_x\kappa_y}$ is an isomorphism:

$$\begin{tikzcd}
G'_K \arrow[r, "\Phi_\imath"] \arrow[d, "\Phi_{\kappa_x}"] & G'_L \arrow[d, "\Psi_{\kappa_y}"] & & \mathcal{G}_{\mathcal{D}_K} \arrow[r, "\Phi_\imath"] \arrow[d, "\Phi_{\kappa_x}"] & \mathcal{G}_{\mathcal{D}_L} \arrow[d, "\Psi_{\kappa_y}"] \\
G'_{\kappa_x} \arrow[r, "\Phi_{\kappa_x\kappa_y}"] & G'_{\kappa_y} & & \mathcal{G}_{\kappa_x} \arrow[r, "\Phi_{\kappa_x\kappa_y}"] & \mathcal{G}_{\kappa_y}
\end{tikzcd}$$

Fact/Definition 3.17. In the context from Notations 3.16 above, the following hold:

Birational Bertini: Let $x, t \in K$ be algebraically independent over k, and x is separable in K, i.e., x is not a p-power in K, where $p = \operatorname{cahr}(k)$. Then for all but finitely many $a \in k$ one has: $ax + t$ is a general element of K, i.e., $k(ax + t)$ is relatively algebraically closed in K, see e.g. Lang [L2], Ch. VIII, Lemma, in proof of Theorem 7, or Roquette [R2], §4.

1) We will use the above "birational Bertini" repeatedly in the following form: Let $x, t \in K$ be fixed algebraically independent functions over k, with x separable, e.g., general.

a) $t_a := ax + t$ is a general element of K for almost all $a \in k$.

b) $t_{a',a} := t/(a'x + a)$ is a general element of K for all $a' \in k^\times$ and almost all $a \in k$.

c) $t_{a'',a',a} := (a''t + a'x + a + 1)/(t + a'x + a)$ is a general element of K for all $a'' \in k$ and almost all $a', a \in k$.

2) For $x, t \in K$ as above, the general elements of the form t_a, $t_{a',a}$, $t_{a'',a',a}$, will be called **general elements of Bertini type** defined by x, t. Further, a set $\Sigma \subset K^\times$ will be called a **Bertini set**, if for all $x, t \in K$ which are algebraically independent over k, and x is separable, one has: $t_a, t_{a',a}, t_{a'',a',a} \in \Sigma$ for all $a'' \in k$, and almost all $a', a \in k$. Clearly, Σ generates the multiplicative group $K^\times$ by 1b) above.

We say that a set of rational quotients $\mathfrak{A} \subseteq \mathfrak{A}_K$ is of **Bertini type**, if $\mathfrak{A}$ has a subset of the form $\mathfrak{A}_\Sigma := \{\Phi_{k(x_0)} \mid x_0 \in \Sigma\}$ for some Bertini set $\Sigma \subset K^\times$.

3) Next let $\imath : L|l \to K|k$ be an embedding of function fields such that $\imath(l) = k$ and $K|\imath(L)$ separable. Then for every separable element $y \in L$ one has: $x := \imath(y)$ is a separable element of $K|k$. Further, directly from the definition of a general element of Bertini type one gets: Let $u_b, u_{b',b}, u_{b'',b',b} \in L$ be general elements of Bertini type defined by some $y, u \in L$. Then for all $b'' \in l$, and almost $b', b \in l$ one has: The images $t_b := \imath(y_b)$, $t_{b',b} := \imath(u_{b',b})$, $t_{b'',b',b} = \imath(u_{b'',b',b})$ are general elements of Bertini type in $K|k$ defined by $x := \imath(y)$, $t := \imath(u)$.

4) From this we deduce that there exist Bertini sets $\Delta \subset L^\times$ and $\Sigma \subset K^\times$ such that $\imath(\Delta) \subseteq \Sigma$. Therefore, for the corresponding Bertini type sets of rational quotients $\mathfrak{B}_\Delta$ and $\mathfrak{A}_\Sigma$ we have: If $\kappa_y \in \mathfrak{B}_\Delta$, then $\kappa_x := \imath(\kappa_y)$ lies in $\mathfrak{A}_\Sigma$, etc.

Proof. The only assertions which are maybe not obvious are 1b) and 1c).

To 1b): $t_{a',a}$ is general if and only if $1/t_{a',a} = a'(x/t) + a(1/t)$ is general. Now note that if $x/t, 1/t \in K$ are algebraically independent over k, and because x is separable, it follows that at least one of the two elements is separable. Finally apply the "birational Bertini".

To 1c): Setting $\alpha := 1 - a''$, we have: $t_{a'',a',a} = a'' + (\alpha a'x + \alpha a + 1]/(t + a'x + a)$ is general element if and and only if $t' := (\alpha a'x + \alpha a + 1)/(t + a'x + a)$ is so. Note that $t + a'x + a$ is a general element for all $a \in k^\times$ and almost all $a' \in k$ by the "birational Bertini". Hence if $\alpha = 0$, then $t' := 1/(t + a'x + a)$ is general element. Finally, if $\alpha \neq 0$, then x' is a general element if and only if $1/t' = (t - \frac{1}{\alpha})/(\alpha a'x + \alpha a + 1) + \frac{1}{\alpha}$ is a general element, thus if and only if $(t - \frac{1}{\alpha})/(\alpha a'x + \alpha a + 1)$ is a general element. And the latter is a general element for all $\alpha a + 1 \in k^\times$ and almost all $a' \in k$, by Case 1b). $\square$

Proposition 3.18. *In the above notations, the following hold:*

1) *Suppose that* $\operatorname{tr.deg}(K|k) > 1$*, and let* $\mathcal{G}_{\mathcal{D}_K}$ *be a complete regular like geometric decomposition graph, which we view as a divisorial abstract decomposition graph. Then endowing* $\mathcal{G}_{\mathcal{D}_K}$ *with a Bertini type set* $\mathfrak{A} \subseteq \mathfrak{A}_K$ *of rational projections,* $\mathcal{G}_{\mathcal{D}_K}$ *becomes a geometric like abstract decomposition graph satisfying the following:* $K_{(\ell)} := \jmath_K(K^\times) \otimes \mathbb{Z}_{(\ell)}$ *is an* arithmetical lattice *defined by* $\mathfrak{A}$ *inside* $\widehat{K}$*, which we call the* ***canonical arithmetical lattice****.*

2) *Let* $\imath : L|l \to K|l$ *be an embedding of function fields such that* $\imath(l) = k$*, and* $K|\imath(L)$ *is separable. Let* $\mathcal{H}_{\mathcal{D}_L}$ *be a complete regular like abstract decomposition graph for* $L|l$ *such that*

$$\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}_{\mathcal{D}_L}$$

gives rise to a proper morphism of abstract decomposition graphs. Then there exist Bertini type sets $\mathfrak{B}$ *of rational quotients for* $\mathcal{H}_{\mathcal{D}_L}$ *such that* $\Phi_\imath$ *is compatible with the rational projections* $\mathfrak{B}$ *and* $\mathfrak{A}$*.*

Proof. To 1): Let us check that to check that $\mathfrak{A}$ satisfies the conditions from Definition 3.9. Let $X \to K$ be a normal quasi-projective model of $K|k$ such that D_X is the set of all 1-vertices of $\mathcal{G}_\mathcal{K}$

Step 1. $\mathfrak{A}$ is an ample family of rational quotients for $\mathcal{G}_{\mathcal{D}_K}$.

Indeed, first recall that by Fact 3.17, 2), the set $\Sigma_\mathfrak{A}$ generates $K^\times$. Therefore, in the notations from Definition 3.9 we have: $\mathcal{L}_{\mathfrak{A}_\Sigma} = K_{(\ell)}$, hence $\mathcal{L}_\mathfrak{A} = K_{(\ell)}$ too. From this we deduce: First, $\mathcal{L}_\mathfrak{A}$ is ℓ-adically dense in $\widehat{K}$, as $\jmath_K(K^\times)$ itself is so. Second, since $\mathcal{D}_K$ was supposed to be complete regular like, for every non-constant $x \in K$ there exists $v \in D_X$ such that $v(x) \neq 0$. Equivalently, for every non-trivial $x \in K_{(\ell)}$, there exists $v \in D^1_{\mathcal{D}_K}$ such that $v(x) \neq 0$. But this means exactly $K_{(\ell)} \cap \widehat{U}_{\mathcal{G}_\mathcal{K}}$ is trivial. From this discussion, the condition ii) of Definition 3.9 follows. For condition i), remark that $\Phi_{\kappa_x} \neq \Phi_{\kappa_{x'}}$ implies that $\kappa_x \neq \kappa_{x'}$. But then $\kappa_x \cap \kappa_{x'} = k$, hence $\widehat{\kappa}_x$ and $\widehat{\kappa}_{x'}$ have trivial intersection inside $\widehat{K}$.

Step 2. $\mathcal{G}_{\mathcal{D}_K}$ endowed with $\mathfrak{A}$ is geometric like.

Indeed, let κ_x and $\kappa_{x'}$ be given. If $\kappa_x = \kappa_{x'}$, then there is nothing to prove. Hence let $\kappa_x \neq \kappa_{x'}$. Since κ_x and $\kappa_{x'}$ are relatively algebraically closed in K, it follows that x, x' are actually algebraically independent over k. Therefore, by the "birational Bertini", it follows that for almost all $a, a' \in k$ we have: $t := ax - a'x'$ gives rise to a dominant rational map $f : X \dashrightarrow \mathbb{P}^1_t$, satisfying: For general points $t = b$, the fiber X_b is a Weil prime divisor of X, and x, x' are non-constant on X_b. The birational translation of this is the following: If $v := v_{X_b} \in D_X$ is the corresponding prime divisor of K, then x, x' are v-units such that $\jmath_v(x)$, $\jmath_v(x')$ are not constant in the residue field Kv of v. But then $\kappa_x^\times$ and $\kappa_{x'}^\times$ consist of non-principal v-units, and are mapped isomorphically into the residue field Kv. Moreover, since $t = ax - a'x'$ has $v(t) > 0$, it follows that $ax \equiv a'x' \,(\mathrm{mod}\, \mathfrak{m}_v)$, hence $\jmath_v(\kappa_x) = \jmath_v(\kappa_{x'})$. Taking ℓ-adic completions, we deduce from this that conditions j), jj) of Definition 3.9 are satisfied at v.

To 2): Clear by the Fact 3.17, 3) and 4, and the commutative diagrams from Notations 3.16, 3). □

4. PROOF OF THE MAIN RESULT

Here is our main result concerning geometric like abstract decomposition graphs:

Theorem 4.1. *Let $K|k$ be a function field with* $\mathrm{tr.deg}(K|k) > 1$, *and let $\mathcal{G}_{\mathcal{D}_K}$ be a complete regular like geometric decomposition graph for $K|k$. We endow $\mathcal{G}_{\mathcal{D}_K}$ with a Bertini type set $\mathfrak{A}$ of rational quotients, and view it as a* geometric like abstract decomposition graph.

1) *Let $\mathcal{H}$ endowed with a family of rational quotients $\mathfrak{B}$ be a geometric like abstract decomposition graph. Then up to multiplication by ℓ-adic units, and composition with automorphisms $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_K}$ defined by embedding of function fields $\imath : K|l \to K|k$ such that $K|\imath(K)$ is purely inseparable, there exists at most one isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}$ of abstract decomposition graphs which is compatible with the rational quotients $\mathfrak{A}$ and $\mathfrak{B}$.*

2) *Let $L|l$ be a further function field with* $\mathrm{tr.deg}(L|l) > 1$, *and let $\mathcal{H}_{\mathcal{D}_L}$ be a complete regular like abstract decomposition graph for $L|l$. We endow $\mathcal{H}_{\mathcal{D}_L}$ with a Bertini type set $\mathfrak{B}$ of rational quotients, and view it as a* geometric like abstract decomposition graph. *Let*

$$\Phi : G'_K \to G'_L$$

be an open group homomorphism which defines a proper morphism $\Phi : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}_{\mathcal{D}_L}$ of abstract decomposition graphs compatible with the rational quotients $\mathfrak{B}$ and $\mathfrak{A}$. Then there exist an ℓ-adic unit ϵ and an embedding of function fields

$$\imath : L|l \to K|k$$

such that $\Phi = \epsilon \cdot \Phi_\imath$, where $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}_{\mathcal{D}_L}$ is the canonical morphism defined by $\imath$ as indicated above.

Further, $\imath(l) = k$, and $\imath$ is unique up to Frobenius twists.

Proof. Since (1) follows from (2), it suffices to prove assertion (2).

Recall that Proposition 3.18, $K_{(\ell)} := \jmath_K(K^\times) \otimes \mathbb{Z}_{(\ell)}$ and $L_{(\ell)} := \jmath_L(L^\times) \otimes \mathbb{Z}_{(\ell)}$ are arithmetical lattices for $\mathcal{G}_{\mathcal{D}_K}$ endowed with $\mathfrak{A}$, respectively for $\mathcal{H}_{\mathcal{D}_L}$ endowed with $\mathfrak{B}$. Now by Proposition 3.11, it follows that $\hat{\phi}(L_{(\ell)})$ is contained in a unique arithmetical lattice of $\mathcal{G}_{\mathcal{D}_K}$. Since the arithmetical lattices of $\mathcal{G}_{\mathcal{D}_K}$ are ℓ-adically equivalent to $K_{(\ell)}$, there exists an ℓ-adic unit ϵ such that

$$\hat{\phi}(L_{(\ell)}) \subseteq \epsilon \cdot K_{(\ell)} \,.$$

Therefore, after replacing Φ by $\epsilon \cdot \Phi$, without loss of generality we can suppose that

Hypothesis I. *$\hat{\phi}$ maps $L_{(\ell)}$ isomorphically into $K_{(\ell)}$.*

We further recall that $\jmath_K(K^\times) = K^\times/k^\times$ and $\jmath_L(L^\times) = L^\times/l^\times$ are true lattices in $K_{(\ell)}$, respectively $L_{(\ell)}$. In order to simplify notations, we denote by:

$$\mathsf{x} = \jmath_K(x) = k^\times x, \quad \mathsf{y} = \jmath_L(y) = l^\times y$$

the image of $x \in K^\times$ under $\jmath_K$, respectively that of $y \in L^\times$ under $\jmath_L$. Further, we will always denote elements of $K_{(\ell)}$, respectively of $L_{(\ell)}$, in boldface:

$$\boldsymbol{x} \in K_{(\ell)}, \quad \boldsymbol{y} \in L_{(\ell)} \,.$$

Next we want to understand the following: Let $\Phi_{\kappa_x} \in \mathfrak{A}$ correspond to some $\Phi_{\kappa_y} \in \mathfrak{B}$, and let $\kappa_{x,(\ell)} \subset \widehat{\kappa}_x$ and $\kappa_{y,(\ell)} \subset \widehat{\kappa}_y$ be the unique divisorial lattices such that $\hat{\phi}_{\kappa_x}(\kappa_{x,(\ell)}) \subset K_{(\ell)}$, respectively $\hat{\phi}_{\kappa_y}(\kappa_{y,(\ell)}) \subset L_{(\ell)}$. Then by Proposition 3.11, 2), we get: $\hat{\phi}_{\kappa_x\kappa_y}(\kappa_{y,(\ell)}) = \kappa_{x,(\ell)}$, $\hat{\phi}\circ\hat{\phi}_{\kappa_y}(\kappa_{y,(\ell)}) = \hat{\phi}_{\kappa_x}(\kappa_{x,(\ell)})$. Hence taking into account the identifications from Notations 3.16, i.e., $\kappa_{x,(\ell)} = K_{(\ell)} \cap \hat{\phi}_{\kappa_x}(\widehat{\kappa}_x)$ inside $\widehat{K} = \widehat{\mathcal{L}}_{\mathcal{G}_{\mathcal{D}_K}}$, and $\kappa_{y,(\ell)} = L_{(\ell)} \cap \hat{\phi}_{\kappa_y}(\widehat{\kappa}_y)$ inside $\widehat{L} = \widehat{\mathcal{L}}_{\mathcal{H}_{\mathcal{D}_L}}$, the above assertion is equivalent to: If $\Phi_{\kappa_x} \in \mathfrak{A}$ corresponds to $\Phi_{\kappa_y} \in \mathfrak{B}$, then

$$\hat{\phi}(\kappa_{y,(\ell)}) = \kappa_{x,(\ell)} \,.$$

Hence we have $\jmath_L(\kappa_y^\times) \subset \kappa_{y,(\ell)}$ and $\jmath_K(\kappa_x^\times) \subset \kappa_{x,(\ell)}$, and the task now is to understand the precise relation between $\hat{\phi} \circ \jmath_L(\kappa_y{}^\times)$ and $\jmath_K(\kappa_x^\times)$ inside $\kappa_{x,(\ell)}$.

Lemma 4.2. *Let $\Phi_{\kappa_x} \in \mathfrak{A}$ correspond to some $\Phi_{\kappa_y} \in \mathfrak{B}$. Then there exist unique relatively prime and prime to ℓ integers $m, n > 0$ such that the following hold:*

1) *$\hat{\phi}\big(n \cdot \jmath_L(ly + l)^\times\big) = m \cdot \jmath_K(kx + k)^\times$, and $\hat{\phi}\big(n \cdot \jmath_L(\kappa_y^\times)\big) = m \cdot \jmath_K(\kappa_x^\times)$ inside $\kappa_{x,(\ell)}$.*
2) *$\hat{\phi}\big(n \cdot \jmath_L(y)\big) = m \cdot \jmath_K(x)$, provided $\mathbb{Z}_{(\ell)} \cdot \jmath_L(y)$ is mapped by $\hat{\phi}$ into $\mathbb{Z}_{(\ell)} \cdot \jmath_K(x)$, and such a choice of a generator x for κ_x is always possible.*

Proof. We begin by considering the systems of *arithmetical inertia generators* $\mathfrak{T}_{\kappa_y}$ of $\mathcal{G}_{\kappa_y}$, respectively $\mathfrak{T}_{\kappa_x}$ of $\mathcal{G}_{\kappa_x}$, as introduced at Definition/Remark 2.14, 2). Let $(y) = w' - w$ be the divisor of $y \in \kappa_y = l(y)$, and $(x) = v' - v$ be the divisor of $x \in \kappa_x = k(x)$. Then $\kappa_x = k(x)$, and in the notations from Definition/Remark 3.8, 1), we have:

a) $\mathsf{y} := \jmath_L(y) = \varphi_{w'} - \varphi_w$, and $\mathcal{P}_{\mathsf{y}} := \mathcal{P}_w = \jmath_L(ly + l)^\times \subset \kappa_{y,(\ell)}$ is the generating set at w with respect to $\mathfrak{T}_{\kappa_y}$.

b) $\mathsf{x} := \jmath_K(x) = \varphi_{v'} - \varphi_v$, and $\mathcal{P}_{\mathsf{x}} := \mathcal{P}_v = \jmath_K(kx + k)^\times \subset \kappa_{x,(\ell)}$ is the generating set at v with respect to $\mathfrak{T}_{\kappa_y}$.

Moreover, we can choose x from the beginning in such a way that v', v are the pre-images of w', w under $\Phi_{\kappa_x\kappa_y}$. Equivalently, we then have $\hat\phi(\mathbb{Z}_{(\ell)}\,\mathsf{y}) = \mathbb{Z}_{(\ell)}\,\mathsf{x}$. Therefore there exists unique relatively prime integers $m, n > 0$ such that

$$(*) \qquad \hat\phi(n\,\mathsf{y}) = m\,\mathsf{x}\,.$$

On the other hand, the image $\Phi_{\kappa_x\kappa_y}(\mathfrak{T}_{\kappa_y})$ of $\mathfrak{T}_{\kappa_y}$ under the isomorphism $\Phi_{\kappa_x\kappa_y}$ is a distinguished systems of inertia generators for κ_x, such that $\hat\phi(\mathcal{P}_w)$ is the generating set at v with respect to $\Phi_{\kappa_x\kappa_y}(\mathfrak{T}_{\kappa_y})$. By the uniqueness up to ℓ-adic equivalence of the distinguished systems of inertia generators we have: $\Phi_{\kappa_x\kappa_y}(\mathfrak{T}_{\kappa_y}) = \mathfrak{T}^{\epsilon}_{\kappa_x}$ for a unique ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell^\times$. Hence $\hat\phi(\mathcal{P}_w) = \epsilon^{-1}\cdot\mathcal{P}_v$ and in particular, $\hat\phi(\mathsf{y}) = \epsilon^{-1}\cdot\mathsf{x}$ inside $\kappa_{x,(\ell)}$. Then by the fact $(*)$ above, it follows that $\epsilon = n/m$, hence both m, n are relatively prime to ℓ. Finally, we get:

$$(*)' \qquad \hat\phi(n\,\mathcal{P}_{\mathsf{y}}) = m\,\mathcal{P}_{\mathsf{x}}\,.$$

Clearly, if m', n' are relatively prime integers such that $\hat\phi(n'\,\mathcal{P}_{\mathsf{y}}) = m'\,\mathcal{P}_{\mathsf{x}}$, then we must have $\hat\phi(n'\,\mathsf{y}) = m'\,\mathsf{x}$. Therefore, $(m, n) = (m', n')$ by the uniqueness of m, n.

Finally, since $\mathcal{P}_{\mathsf{y}}$ and $\mathcal{P}_{\mathsf{x}}$ generate $\kappa_y^\times/l^\times$ inside $\kappa_{y,(\ell)}$, respectively $\kappa_x/k^\times$ inside $\kappa_{x,(\ell)}$, we deduce that for the unique m, n above one has:

$$(*)'' \qquad \hat\phi\big(n\cdot\jmath_L(\kappa_y^\times)\big) = m\cdot\jmath_K(\kappa_x^\times)\,.$$

This completes the proof of the Lemma. $\square$

Norming 4.3. In the context of Lemma 4.2 above, suppose that $\mathbb{Z}_{(\ell)}\cdot\jmath_L(y)$ is mapped by $\hat\phi$ into $\mathbb{Z}_{(\ell)}\cdot\jmath_K(x)$. Then we will say that $\hat\phi$ is y-normed, if $\hat\phi\circ\jmath_L(y) = \jmath_K(x)$.

Clearly, *a priori,* $\hat\phi$ might not be normed with respect to any $\Phi_{\kappa_y} \in \mathfrak{B}$ and the corresponding $\Phi_{\kappa_x} \in \mathfrak{A}$. Nevertheless, we can "artificially" remedy this as follows: In the notations from Lemma 4.2 above, suppose that we have chosen the generator x such that $\mathbb{Z}_{(\ell)}\cdot\jmath_L(y)$ is mapped by $\hat\phi$ into $\mathbb{Z}_{(\ell)}\cdot\jmath_K(x)$. Hence we have $\hat\phi\circ\jmath_L(y) = (m/n)\,\jmath_K(x)$. Further, remark that $\eta_{\hat\phi} := m/n$ is an ℓ-adic unit. And replacing the morphism $\Phi : G_K' \to G_L'$ by its $\eta_{\hat\phi}$-multiple $\Phi' := \eta_{\hat\phi}\cdot\Phi$, amounts to replacing $\hat\phi$ by its $(1/\eta_{\hat\phi})$-multiple $\hat\phi' := (1/\eta_{\hat\phi})\cdot\hat\phi$. In particular, we have $\eta_{\hat\phi'} = 1$, hence $\hat\phi'$ is y-normed.

Hence we have: Let $\kappa_y \in \mathfrak{B}$ and its corresponding $\kappa_x \in \mathfrak{A}$ be given such that $\mathbb{Z}_{(\ell)}\cdot\jmath_L(y)$ is mapped by $\hat\phi$ into $\mathbb{Z}_{(\ell)}\cdot\jmath_K(x)$. Then after replacing Φ by a properly chosen multiple $\eta\cdot\Phi$ with $\eta \in \mathbb{Z}_{(\ell)}$, the resulting Kummer homomorphism $(1/\eta)\,\hat\phi$ is y-normed. Hence *mutatis mutandis,* we can suppose that $\hat\phi$ satisfies the following norming hypothesis:

HYPOTHESIS II. $\kappa_x \in \mathfrak{A}$ corresponds to $\kappa_y \in \mathfrak{B}$, and $\hat\phi\circ\jmath_L(y) = \jmath_K(x)$, hence $\hat\phi$ is y-normed.

Remark/Notation 4.4. If $\hat{\phi}$ is y-normed, then by Lemma 4.2, $\hat{\phi}$ defines bijections:

$$(\dagger) \qquad \hat{\phi} : \jmath_L(ly+l)^\times \to \jmath_L(ky+k)^\times, \quad \hat{\phi} : \jmath_L({\kappa_y}^\times) \to \jmath_L(\kappa_x)^\times .$$

We set $M_K := \hat{\phi}\big(\jmath_L(L^\times)\big) \cap \jmath_K(K^\times)$, and let $M_L \subseteq \jmath_L(L^\times)$ be the pre-image of M_K under $\hat{\phi}$. Then $\jmath_L({\kappa_y}^\times) \subset M_L$ and $\jmath_K({\kappa_x}^\times) \subset M_K$ by the fact (†) above, and

$$\hat{\phi} : M_L \to M_K$$

is an isomorphism which maps $\jmath_L({\kappa_y}^\times)$ isomorphically onto $\jmath_K({\kappa_x}^\times)$.

We will say that $u \in L^\times$ and $t \in K^\times$ correspond to each other, if the following hold:

$$\jmath_L(u) \in M_L, \quad \jmath_K(t) \in M_K, \quad \text{and} \quad \hat{\phi} \circ \jmath_L(y) = \jmath_K(x) .$$

Finally we remark that $M_L \otimes \mathbb{Z}_{(\ell)} = L_{(\ell)}$ inside $\widehat{L}$.

Lemma 4.5. *Suppose that $t \in K$ and $u \in L$ correspond to each other via $\hat{\phi}$. Then $\mathcal{P}_t := (kt+k)^\times/k^\times = \jmath_K(kt+k)^\times \subset M_K$, and $\mathcal{P}_u := (lu+l)^\times/l^\times = \jmath_L(lu+l)^\times \subset M_L$.*

Proof. Case 1) $u \in \kappa_y$. Then $t \in \kappa_x$, and we are in the situation of Lemma 4.2 above with $m = n = 1$, from which the assertion follows.

Case 2. $u \notin \kappa_y$. Since $\kappa_y = l(y)$ is relatively algebraically closed in L, it follows that u, y are algebraically independent over l. Correspondingly, the same is true for t, x, i.e., t, x are algebraically independent over k. Then by the Fact 3.17, 1), we have:

i) $t_{a',a} := t/(a'x+a)$ is a general element of K for almost all $a', a \in k$.

ii) $u_{b',b} := u/(b'x+b)$ is a general element of L for almost all $b', b \in l$.

Hence by condition (†) of Remark/Notation 4.4, we conclude the following: For a', a as at i), let $y_{a',a} \in (ly+l)^\times$ be such that $\hat{\phi} \circ \jmath_L(y_{a',a}) = \jmath_K(a'x+a)$. Then by ii), $u_{a',a} := u/y_{a',a}$ is a general element of L for almost all $a', a \in k$. And note that

$$\hat{\phi} \circ \jmath_L(u_{a',a}) = \hat{\phi} \circ \jmath_L(u/y_{a',a}) = \jmath_K\big(t/(a'x+a)\big) = \jmath_K(t_{a',a}).$$

In particular, since $\mathfrak{A}$ and $\mathfrak{B}$ contain some Bertini type subsets, we can suppose that $\kappa_{t_{a',a}} \in \mathfrak{A}$ and $\kappa_{u_{a',a}} \in \mathfrak{B}$, and $\kappa_{t_{a',a}}$ corresponds to $\kappa_{u_{a',a}}$ under Φ.

On the other hand, since by hypothesis we have $\jmath_K(t) \in M_K$ and $\jmath_L(u) \in M_L$, and by Remarks/Notation 4.4 above, $\jmath_K(kx+k) \subset M_K$ and $\jmath_L(ly+l) \subset M_L$, it follows that for almost $a, a' \in k$, the following hold:

a) $t_{a',a} \in K$ and $u_{a',a} \in L$, respectively $t_{a',a+1} \in K$, and $u_{a',a+1} \in L$, are general elements which correspond to each other under $\hat{\phi}$.

a)′ Hence $\hat{\phi}$ is normed with respect to both $u_{a',a}$ and $u_{a',a+1}$.

For b, b' as at ii), let $x_{b',b} \in kx+k$ be such that $\hat{\phi}\big(\jmath_L(b'y+b)\big) = \jmath_K(x_{b',b})$. Then by i), for all b and almost all b', one has: $t_{b',b} := t/x_{b',b}$ is a general element of K. And note that:

$$\jmath_K(t_{b',b}) = \jmath_K(t/x_{b',b}) = \hat{\phi} \circ \jmath_L\big(u/(b'y+b)\big) = \hat{\phi} \circ \jmath_L(u_{b',b}) .$$

In particular, $\kappa_{t_{b',b}} \in \mathfrak{A}$ and $\kappa_{u_{b',b}} \in \mathfrak{B}$, and $\kappa_{t_{b',b}}$ corresponds to $\kappa_{u_{b',b}}$ under Φ, and $\hat{\phi}$ is normed with respect $u_{b',b}$. Reasoning as above, it follows that for almost $b', b \in l$ one has:

b) $t_{b',b} \in K^\times$ and $u_{b',b} \in L$, respectively $t_{b',b+1} \in K$ and $u_{b',b+1} \in L$, are general elements which correspond to each other under $\hat{\phi}$.

b)′ Hence $\hat{\phi}$ is normed with respect to both $u_{b',b}$ and $u_{b',b+1}$.

But then by the fact (†) from Remark/Notation 4.4 applied to the functions $t_{a',a} \in K^\times$ and $u_{a',a} \in L$, it follows that $\jmath_K(\kappa^\times_{t_{a',a}}) = \hat{\phi} \circ \jmath_L(\kappa^\times_{u_{a',a}}) \subset \hat{\phi} \circ \jmath_L(L^\times)$, hence $\jmath_K(\kappa^\times_{t_{a',a}}) \subset \hat{\phi} \circ \jmath_L(L^\times) \cap \jmath_K(K^\times) = M_K$; and the same correspondingly for the other three pairs of functions which correspond to each other under $\hat{\phi}$. Thus finally we get:

$$\jmath_K(\kappa^\times_{t_{a',a}}) \subset M_K,\ \ \jmath_K(\kappa^\times_{t_{a',a+1}}) \subset M_K, \quad \text{and} \quad \jmath_L(\kappa^\times_{u_{b',b}}) \subset M_L,\ \ \jmath_K(\kappa^\times_{u_{b',b+1}}) \subset M_L\,.$$

Finally, for $a, a', a'' \in k$, consider the functions

$$t_{a'',a',a} = (a''t + a'x + a + 1)/(t + a'x + a)\,.$$

Then by Fact 3.17, 1), it follows that for all a'', and almost all a', a, the function $t_{a'',a',a}$ is a general element of K too. On the other hand, a direct computation shows that:

$$t_{a'',a',a} = \frac{a'x + a + 1}{a'x + a} \cdot \frac{a''t_{a',a+1} + 1}{t_{a',a} + 1}\,.$$

Since the images via $\jmath_K$ of both the denominators and the numerators of the fractions above lie M_K, we get: $\jmath_K(t_{a'',a',a}) \in M_K$. Reasoning as previously in the case of $t_{a',a}$, we find general elements $u_{a'',a',a} \in L$ such that $\kappa_{t_{a'',a',a}}$ corresponds to $\kappa_{u_{a'',a',a}}$, etc. And we further define correspondingly functions

$$u_{b'',b',b} = \frac{b'x + b + 1}{b'x + b} \cdot \frac{b''u_{b',b+1} + 1}{u_{b',b} + 1}\,,$$

and find functions $t_{b'',b',b} \in K$, etc. Finally one gets: $\jmath_K(\kappa_{t_{a'',a',a}}) \subset M_K$ for all $a'' \in k$, and almost all $a', a \in k$. And correspondingly $\jmath_L(\kappa_{u_{b'',b',b}}) \subset M_L$ for all $b'' \in l$, and almost all $b', b \in l$.

Now we conclude the proof of the fact that $\jmath_K(kt + k)^\times \subseteq M_K$ as follows: First, since $\jmath_K(\kappa^\times_{t_{a'',a',a}}) \subset M_K$, we have $\jmath_K(t_{a'',a',a} - 1) \in M_K$. On the other hand,

$$t_{a'',a',a} - 1 = [(a'' - 1)t + 1]/(t + a'x + a)\,.$$

Now remark that $t + a'x + a = (a'x + a + 1)/t_{0,a',a}$. Hence $\jmath_K(t + a'x + a) \in M_K$, as $\jmath_K(t_{0,a',a}), \jmath_K(a'x + a + 1) \in M_K$. Thus we finally deduce that $\jmath_K\big((a'' - 1)t + 1\big) \in M_K$ for all $a'' \in k$. Hence $\mathcal{P}_t = \jmath_K(kt + k)^\times \subset M_K$, as $\jmath_K(t) \in M_K$ by hypothesis.

In a completely similar way, one concludes that $\mathcal{P}_u = \jmath_L(lu + l)^\times \subset M_L$.

□

Lemma 4.6. *Let $K_0 = \jmath_K^{-1}(M_K) \cup \{0\} \subseteq K$ and $L_0 = \jmath_L^{-1}(M_L) \cup \{0\} \subseteq L$ be the preimages of M_K, respectively M_L, in K, respectively L, together with 0 added. Then $K_0 \subseteq K$ and $L_0 \subseteq L$ are function subfields.*

Proof. Indeed, since M_K is a subgroup of $\widehat{K}$, its pre-image $\jmath_K^{-1}(M_K)$ in $K^\times$ is a subgroup too. We check that K_0 is closed with respect to addition: For $t, t' \in K_0$ non-zero, $t'' = t'/t \in K_0$, and $t + t' = t(t'' + 1)$. On the other hand, by Lemma 4.5 we have $t'' + 1 \in K_0$. Hence finally we get $t + t' = t(t'' + 1) \in K_0$. The proof of the assertion concerning L_0 is similar, and we omit it. □

Now remark that $M_K = \jmath_K(K_0^\times) = K_0^\times/k^\times$ can be viewed in a canonical way as the projectivization $\mathcal{P}(K_0) := K_0^\times/k^\times$ of the infinite dimensional k-vector space $(K_0, +)$. And correspondingly, $M_L = \jmath_L(L_0^\times) = L_0^\times/l^\times =: \mathcal{P}(L_0)$ is the projectivization of the infinite dimensional l-vector space $(L_0, +)$. And since the Kummer homomorphism $\hat{\phi} : \widehat{L} \to \widehat{K}$ maps M_L bijectively onto M_K, the restriction of $\hat{\phi}$ defines a bijection:

$$\phi := \hat{\phi}|_{\mathcal{P}(L_0)} : \mathcal{P}(L_0) = M_L \to M_K = \mathcal{P}(K_0).$$

We remark that the lines in $\mathcal{P}(K_0)$ are subsets of the form $\mathfrak{l}_{t_0,t_1} := (kt_0 + kt_1)^\times/k^\times$ with t_0, t_1 k-linearly independent functions in K_0. In particular, setting $t := t_1/t_0$, we see that $\mathfrak{l}_{t_0,t_1} = t_0 \cdot \mathcal{P}_t$, where $\mathcal{P}_t := (k\,t + k)^\times/k^\times = \jmath_K(kt + k)^\times$. Further note that $\mathfrak{l}_{t_0,t_1}$ depends only on $\mathsf{t}_0 = \jmath_K(t_0)$ and $\mathsf{t}_1 := \jmath_K(t_1)$ only, and not on the functions t_0, t_1 themselves. We will therefore also write $\mathfrak{l}_{\mathsf{t}_0,\mathsf{t}_1}$ for the line $\mathfrak{l}_{t_0,t_1}$, and $\mathcal{P}_{\mathsf{t}}$ for $\mathcal{P}_t$.

Correspondingly, the same holds for lines in $\mathcal{P}(L_0)$.

Lemma 4.7. *The morphism $\phi : \mathcal{P}(L_0) \to \mathcal{P}(K_0)$ respects co-lineations, more precisely ϕ maps each line $\mathfrak{l}_{\mathsf{u}_0,\mathsf{u}_1} \subset \mathcal{P}(L_0)$ bijectively onto $\mathfrak{l}_{\mathsf{t}_0,\mathsf{t}_1} \subset \mathcal{P}(K_0)$, where $\mathsf{t}_0 = \phi(\mathsf{u}_0)$, $\mathsf{t}_1 = \phi(\mathsf{u}_1)$.*

Proof. Setting $\mathsf{t} = \phi(\mathsf{u})$, we get $\mathfrak{l}_{\mathsf{t}_0,\mathsf{t}_1} = \mathsf{t}_0 \cdot \mathfrak{l}_{\mathsf{t}}$, and $\mathfrak{l}_{\mathsf{u}_0,\mathsf{u}_1} = \mathsf{u}_0 \cdot \mathfrak{l}_{\mathsf{u}}$. Hence taking into account that ϕ respects the multiplication, it follows it is sufficient to show that ϕ maps $\mathcal{P}_{\mathsf{u}}$ bijectively onto $\mathcal{P}_{\mathsf{t}}$, provided $\mathsf{t} := \phi(\mathsf{u})$.

Recall that $\hat{\phi}$ is y-normed, and $\hat{\phi}(\mathsf{y}) = \mathsf{x}$, where $\mathsf{y} = \jmath_L(y)$, $\mathsf{x} = \jmath_K(x)$, for x and y corresponding to each other under $\hat{\phi}$. Moreover, by fact (†) from Remark/Notations 4.4, $\hat{\phi}$ maps $\mathcal{P}_{\mathsf{y}} = \mathcal{P}_y$ bijectively onto $\mathcal{P}_{\mathsf{x}} = \mathcal{P}_x$. Recall that for every 1-index v of $\mathcal{G}_{\mathcal{D}_K}$, and the corresponding 1-index w of $\mathcal{H}_{\mathcal{D}_L}$, one has the commutative diagrams of the form, see Remark 3.3, 3), and 4):

$$\begin{tikzcd} \widehat{U}_w \arrow[r, "\jmath_w"] \arrow[d, "\hat{\phi}"] & \widehat{Lw} \arrow[d, "\hat{\phi}_v"] & \text{and} & \widehat{L} \arrow[r, "\jmath^w"] \arrow[d, "\hat{\phi}"] & \mathbb{Z}_\ell \varphi_w \arrow[d, "a_{vw}"] \\ \widehat{U}_v \arrow[r, "\jmath_v"] & \widehat{Kv} & & \widehat{K} \arrow[r, "\jmath^v"] & \mathbb{Z}_\ell \varphi_v \end{tikzcd}$$

Let K_t be the relative algebraic closure of $k(t)$ in K_0. We claim that $\phi(\mathcal{P}_u) \subset \jmath_K(K_t)$. Indeed, let v be such that $v(\mathsf{t}') \neq 0$ for some $\mathsf{t}' = \phi(\mathsf{u}')$ with $\mathsf{u}' \in \mathcal{P}_{\mathsf{u}}$. Then by the commutativity of the second diagram above we get: $w(\mathsf{u}') \neq 0$. But then it follows that $\jmath_w$ is trivial on $l(u)^\times \cap U_w$. Hence by the commutativity of the first diagram above, it follows that $\jmath_v$ is trivial on $\phi \circ \jmath_L\big(l(u)^\times\big)$, in particular on $\phi(\mathcal{P}_{\mathsf{u}})$. By contradiction, suppose that $\phi(\mathcal{P}_u) \not\subset \jmath_K(K_t)$. Then $\exists$ $\mathsf{u}_1 \in \mathcal{P}_u$ and $t_1 \in K_0$ such that t and t_1 are algebraically independent over k, and $\jmath_K(t_1) =: \mathsf{t}_1 = \phi(\mathsf{u}_1)$. On the other hand, since t, t_1 are algebraically independent over k, there exist "many" v satisfying the following: v is not trivial on $k(t)$ and t is a v-unit, and v is trivial on $k(t_1)$. Note that v being non-trivial on $k(t)$, and t being a v-unit implies that the residue of t at v lies in k, hence $\jmath_v(\mathsf{t}) = 0$. Now let w correspond to v under Φ. Then by the commutativity of the above diagrams we have: $\mathsf{u} = \jmath_L(u)$ is a w-unit, and $\jmath_w(\mathsf{u}) = 0$. Therefore, w is non-trivial on $l(u)$. Further, $\mathsf{u}_1 = \jmath_L(u_1)$ is a w-unit, and $\jmath_w(\mathsf{u}_1) \neq 0$. But this contradicts the fact that w is non-trivial on $l(u)$!

Now choose a prime divisor v of $K|k$ such that the following are satisfied:

i) v is trivial on κ_x, and t is a v-unit.

ii) x and t have equal residues in Kv, hence $\jmath_v(x) = \jmath_v(t)$.

Note that ii) implies that v is trivial on K_t too, hence $\jmath_v$ maps both ${\kappa_x}^\times$ and $K_t^\times$ injectively into the residue field Kv.

Let w correspond to v under the proper morphism $\Phi : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_L}$. Then reasoning as in the proof of Proposition 3.11, it follows that the following hold:

j) w is trivial on κ_y.

jj) y and u have equal residues in Lw, hence $\jmath_w(u) = \jmath_w(y)$.

Further, since $\jmath_w$ and $\jmath_v$ respect addition and multiplication, the following hold:

$$\jmath_v(\mathcal{P}_t) = \mathcal{P}_{\jmath_v(t)}, \quad \jmath_v(\mathcal{P}_x) = \mathcal{P}_{\jmath_v(x)}, \quad \text{and} \quad \jmath_w(\mathcal{P}_u) = \mathcal{P}_{\jmath_w(u)}, \quad \jmath_w(\mathcal{P}_y) = \mathcal{P}_{\jmath_w(y)}\,.$$

Hence by i), ii), respectively j), jj), we get $\mathcal{P}_{\jmath_v(t)} = \mathcal{P}_{\jmath_v(x)}$ and $\mathcal{P}_{\jmath_w(u)} = \mathcal{P}_{\jmath_w(y)}$. Therefore, since $\hat{\phi}(\mathcal{P}_y) = \mathcal{P}_x$ by the choice of x and y, it follows from $\hat{\phi}_v \circ \jmath_w = \jmath_v \circ \hat{\phi}$ that:

$$\hat{\phi}_v(\mathcal{P}_{\jmath_w(y)}) = \hat{\phi}_v\big(\jmath_w(\mathcal{P}_y)\big) = \jmath_v\big(\hat{\phi}(\mathcal{P}_y)\big) = \jmath_v(\mathcal{P}_x) = \mathcal{P}_{\jmath_v(x)}\,.$$

Thus taking into account the equalities above and that $\jmath_v \circ \hat{\phi} = \hat{\phi}_v \circ \jmath_w$, we finally get:

$$\jmath_v\big(\hat{\phi}(\mathcal{P}_u)\big) = \hat{\phi}_v\big(\jmath_w(\mathcal{P}_u)\big) = \hat{\phi}_v\big(\mathcal{P}_{\jmath_w(u)}\big) = \hat{\phi}_v(\mathcal{P}_{\jmath_w(y)}) = \mathcal{P}_{\jmath_v(x)} = \mathcal{P}_{\jmath_v(t)} = \jmath_v(\mathcal{P}_t)\,.$$

Hence $\jmath_v\big(\hat{\phi}(\mathcal{P}_u)\big) = \jmath_v(\mathcal{P}_t)$. Since both $\hat{\phi}(\mathcal{P}_u)$ and $\mathcal{P}_t$ are subsets of $\jmath_K(K_t^\times)$, and $\jmath_v$ is injective on $\jmath_K(K_t^\times)$, we get $\hat{\phi}(\mathcal{P}_u) = \mathcal{P}_t$, as claimed. ☐

In order to conclude the proof of the Theorem we proceed as follows:

By Lemma 4.6 above, ϕ respects co-lineations. Therefore, by the *Fundamental theorem of projective geometries,* see e.g. Artin, ϕ is the projectivization $\phi = \mathcal{P}(\phi')$ of some linear $\imath_0$-isomorphism $\phi' : (L_0, +) \to (K_0, +)$, i.e., $\imath_0 : l \to k$ is an isomorphism of fields, and ϕ' is an isomorphism of Abelian groups, such that $\phi'(au) = \imath_0(a)\phi'(u)$ for all $a \in l$ and $u \in L_0$. Moreover, ϕ' is unique up to composition by homotheties of the form $l_a \circ \phi' \circ l_b$ (all $a \in k$, $b \in l$). Further, as $k^\times = \ker(\jmath_K)$ and $l^\times = \ker(\jmath_L)$, it follows that $\phi'(l) = k$. We set

$$\phi_0 := \big(1/\phi'(1)\big)\,\phi'\,,$$

and claim that ϕ_0 is a field isomorphism which maps l isomorphically onto k. Indeed, for a fixed $y \in L_0$, consider $\phi_y : L_0 \to K_0$ defined by $\phi_y(u) := \phi_0(yu)$. Then ϕ_y is a linear $\imath_0$-isomorphism. Set $x = \phi_0(y)$. Then considering projectivisations, and using the fact that $\phi = \mathcal{P}(\phi_0)$ is multiplicative, it follows that for all $u \in L_0$ we have:

$$\mathcal{P}(\phi_y)(u) = \mathcal{P}(\phi_0)(yu) = \mathcal{P}(l_x) \circ \mathcal{P}(\phi_0)(u)\,,$$

where l_x is the multiplication by x on K_0. Therefore, there exist $a \in k^\times$ and $b \in l^\times$ such that $l_b \circ \phi_y \circ l_b = l_x \circ \phi_0$. In other words, $a\,\phi_0(y\,b\,u) = x\,\phi_0(u)$ for all $u \in L_0$. Setting $u = 1$, and taking into account that $\phi_0(1) = 1$, we have: $a\,\imath_0(b)\,x = x$. Thus $a\,\imath_0(b) = 1$, hence the effects of l_a and l_b cancel each other. Hence we have

$$\phi_0(y\,u) = x\,\phi_0(u) = \phi_0(y)\,\phi_0(u), \quad (\text{all } u, y \in L_0).$$

Therefore, ϕ_0 is a field isomorphism as claimed.

Finally, in order to conclude the proof of the Theorem we prove the following:

Lemma 4.8. *$L|L_0$ is a purely inseparable field extension.*

Proof. First, recall that by the last fact in Remark/Notations 4.4, we have $M_L \otimes \mathbb{Z}_{(\ell)} = L_{(\ell)}$ inside $\widehat{L}$. Equivalently, for every $u \in L^\times$ there exists a positive prime to ℓ integer $n_u > 0$ such that $u^{n_u} \in L_0$. This means in particular that $L|L_0$ is an algebraic extension. Since $L|l$ is a function field over the (algebraically closed) field l, it follows that $L_0|l$ is so, and $L|L_0$ is actually a finite field extension, say $[L : L_0] = n > 0$. From this we deduce that if n_u is minimal such that $u^{n_u} \in L$, then $n_u|n$. In particular, all the n^{th} powers u^n, $u \in L$, are contained in L_0. But then $n = [L : L_0]$ must be a power of the characteristic, as claimed. ☐

For the uniqueness of $\imath$ up to Frobenius twists, one uses the last Lemma above, and applies Proposition 3.14.

The Theorem 4.1 is proved. ☐

REFERENCES

[Ar] E. Artin, *Geometric Algebra*, Interscience Publishers Inc., New York 1957. ↑

[Bo] F. A. Bogomolov, *On two conjectures in birational algebraic geometry*, in Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, ed A. Fujiki et all, Springer Verlag Tokyo 1991. ↑

[B–T1] F. A. Bogomolov and Y. Tschinkel, *Commuting elements in Galois groups of function fields* (2002). ↑

[B–T2] ______, *Reconstruction of function fields*, Manuscript, 2003. ↑

[BOU] N. Bourbaki, *Algèbre commutative*, Hermann Paris 1964. ↑

[D] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, in: Galois groups over **Q**, Math. Sci. Res. Inst. Publ. **16**, 79–297, Springer 1989. ↑

[E–E] O. Endler and A. J. Engler, *Fields with Henselian Valuation Rings*, Math. Z. **152** (1977), 191–193. ↑

[Fa] G. Faltings, *Curves and their fundamental groups (following Grothendick, Tamagawa and Mochizuki)*, Astérisque **252** (1998), Exposé 840. ↑

[GGA] Geometric Galois Actions I, LMS LNS Vol **242**, eds L. Schneps – P. Lochak, Cambridge Univ. Press 1998. ↑

[G1] A. Grothendieck, *Letter to Faltings, June 1983*, See [GGA]. ↑

[G2] ______, *Esquisse d'un programme, 1984*, See [GGA]. ↑

[Ki] M. Kim, *The motivic fundamental group of* $\mathbb{P}^1\backslash\{0,1,\infty\}$ *and the theorem of Siegel*, Invent. mathematicae **161** (2005), 629–656. ↑

[Ko] J. Koenigsmann, *On the "Section Conjecture" in anabelian geometry*, J. reine angew. Mathematik **588** (2005), 221–235. ↑

[Ku] F.-V. Kuhlmann, *Book on Valuation Theory*, See: `http://math.usask.ca/~fvk/Fvkbook.htm` (in preparation). ↑

[L1] Serge Lang, *Algebra*, Revised third edition. Graduate Texts in Mathematics 211. Springer-Verlag, New York, 2002. ↑

[L2] ______, *Introduction to Algebraic geometry*, Third printing, with corrections. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1972. ↑

[Mz] Sh. Mochizuki, *Absolute Anabelian Cuspidalizations of Proper Hyperbolic Curves*, See http://www.kurims.kyoto-u.ac.jp/ motizuki/papers-english.html. ↑

[M] D. Mumford, *The ref book of varieties and schemes*, LNM 1358, 2nd expanded edition, Springer Verlag 1999. ↑

[N1] J. Neukirch, *Über eine algebraische Kennzeichnung der Henselkörper*, J. reine angew. Math. **231** (1968), 75–81. ↑

[N2] ———, *Kennzeichnung der p-adischen und endlichen algebraischen Zahlkörper*, Inventiones math. **6** (1969), 269–314. ↑

[N3] ———, *Kennzeichnung der endlich-algebraischen Zahlkörper durch die Galoisgruppe der maximal auflösbaren Erweiterungen*, J. reine angew. Math. **238** (1969), 135–147. ↑

[Pa] A. N. Parshin, *Finiteness Theorems and Hyperbolic Manifolds*, in: The Grothendieck Festschrift III, ed P. Cartier et all, PM Series Vol 88, Birkhäuser Boston Basel Berlin 1990. ↑

[P1] F. Pop, *Recovering $K|k$ from $G_K(\ell)$*, MSRI Talk notes, Fall 1999, see:. ↑

[P2] ———, *The birational anabelian conjecture —revisited—*, Manuscript, Princeton/Bonn 2002. See `http://www.math.leidenuniv.nl/gtem/view.php`. ↑

[P3] ———, *Pro-ℓ birational anabelian geometry over alg. closed fields I*, Manuscript Bonn 2003. See `http://arxiv.org/pdf/math.AG/0307076`. ↑

[P4] ———, *Pro-ℓ abelian-by-central Galois theory of Zariski prime divisors*, Manuscript, Philadelphia 2006. See: `http://arxiv.org/pdf/math.AG/0307076`. ↑

[R1] P. Roquette, *Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten*, J. reine angew. Math. **200** (1958), 1–44. ↑

[R2] ———, *Nonstandard aspects of Hilbert's irreducibility theorem*, 231–275, in: Model theory and algebra (A memorial tribute to Abraham Robinson), LNM Vol. **498**, Springer, Berlin, 1975. ↑

[S–T] S. Saidi and A. Tamagawa, *Prime to p version of Grothendieck's anabelian conjecture in characteristic p*, Manuscript, Kyoto 2006. ↑

[Sz] T. Szamuely, *Groupes de Galois de corps de type fini (d'après Pop)*, Astérisque **294** (2004), 403–431. ↑

[S] J.-P. Serre, *Cohomologie Galoisienne*, LNM 5, Springer 1965. ↑

[Sp] M. Spiess, *An arithmetic proof of Pop's Theorem concerning Galois groups of function fields over number fields*, J. reine angew. Math. **478** (1996), 107–126. ↑

[St] J. Stix, *Projective anabelian curves in positive characteristic and descent theory for log-étale covers*, Thesis, Univ. of Bonn, 2002. See. ↑

[U1] K. Uchida, *Isomorphisms of Galois groups of solvably closed Galois extensions*, Tôhoku Math. J. **31** (1979), 359–362. ↑

[U2] ———, *Homomorphisms of Galois groups of solvably closed Galois extensions*, J. Math. Soc. Japan **33** (1981). ↑

[Z–S] O. Zariski and P. Samuel, *Commutative Algebra*, Vol II, Springer-Verlag, New York, 1975. ↑

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