Recovering function fields from their decomposition graphs

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In memory of Serge Lang

Abstract We develop the *global theory* of a strategy to tackle a program initiated by Bogomolov in 1990. That program aims at giving a group theoretical recipe by which one can reconstruct function fields $K|k$ with $\mathrm{td}(K|k) > 1$ and k algebraically closed from the maximal pro-ℓ abelian-by-central Galois group Π_K^c of K, where ℓ is any prime number $\neq \mathrm{char}(k)$.

Key words: anabelian geometry, pro-ℓ groups, Galois theory, function fields, valuations theory, (Riemann) space of prime divisors, Hilbert decomposition theory, Parshin chains, decomposition graphs

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1 Introduction

Recall that the birational anabelian conjecture originating in ideas presented in Grothendieck's *Esquisse d'un Programme* [11] and *Letter to Faltings* [12] asserts roughly the following: First, there should exist a group-theoretical recipe by which one can recognize the absolute Galois groups G_K of finitely generated infinite fields K among all the profinite groups. Second, if $G = G_K$ is such an absolute Galois group, then the group-theoretical recipe should recover the field K from G_K in a functorial way. Third, the recipe should be invariant under open homomorphisms of absolute Galois groups. In particular, the category of finitely generated infinite fields (up to Frobenius twist) should be equivalent to the category of their absolute Galois groups and open outer homomorphisms between these groups. A first instance of this situation is the celebrated Neukirch–Uchida theorem, which says that global fields are characterized by their absolute Galois groups. I will not go into further detail about the results concerning Grothendieck's (birational) anabelian geometry, but the interested reader can find more about this in Szamuely's Bourbaki Séminaire talk [34], Faltings' Séminaire Bourbaki talk [9], Stix [35], and newer results by Mochizuki [19], Saidi–Tamagawa [33], Minhyong Kim [13], and Koenigsmann [15] concerning the (birational) section conjecture.

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The idea behind Grothendieck's anabelian geometry is that the *arithmetical Galois action* on rich geometric fundamental groups (such as the geometric absolute Galois group) makes objects very rigid, so that there is no room left for non-geometric open morphisms between such rich fundamental groups endowed with arithmetical Galois action.

On the other hand, Bogomolov [2] advanced at the beginning of the 1990s the idea that one should have anabelian-type results in a total absence of an arithmetical action as follows: Let ℓ be a fixed rational prime number. Consider function fields $K|k$ over algebraically closed fields k of characteristic $\neq \ell$. For each such function field $K|k$, let $\Pi_K^{\mathrm{c}} := \mathrm{Gal}(K''|K)$ be the Galois group of a maximal pro-ℓ abelian-by-central Galois extension $K''|K$. Note that if $G^{(1)} = G_K$ and $G^{(i+1)} := [G^{(i)}, G_K](G^{(i)})^{\ell^\infty}$ for $i \geq 1$ are the central ℓ^∞ terms of the absolute Galois group G_K of K, then we have that $\Pi_K = G^{(1)}/G^{(2)}$ is the Galois group of the maximal pro-ℓ abelian subextension $K'|K$ of $K''|K$, and $\Pi_K^{\mathrm{c}} = G^{(1)}/G^{(3)}$; and denoting by $G^{(\infty)}$ the intersection of all the $G^{(i)}$, it follows that $G_K(\ell) := G_K/G^{(\infty)}$ is the maximal pro-ℓ quotient of G_K. Now the program initiated by Bogomolov [2] has as ultimate goal to recover function fields $K|k$ with $\mathrm{td}(K|k) > 1$ as above from Π_K^{c} in a functorial way. (Note that Bogomolov denotes Π_K^{c} by $\mathrm{PGal}_K^{\mathrm{c}}$.) If successful, this program would go far beyond Grothendieck's birational anabelian conjectures, as k being algebraically closed implies that there is no arithmetical action in the game. The program initiated by Bogomolov is not completed yet, and the present manuscript is a contribution towards trying to settle that program; see the historical note below for more about this.

Since the present manuscript is quite abstract, let me announce the following "concrete" result, whose proof relies in an essential way on the main theorem of the present manuscript (see Pop [30] for a complete proof):

Target Result. *Let $K|k$ be a function field with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite field. Then the following hold:*

(1) *There exists a group-theoretical recipe which recovers $K|k$ from Π_K^{c}.*

(2) *The above group-theoretical recipe is functorial in the following sense: Let $L|l$ be a function field with l an algebraically closed field, and let $\Phi : \Pi_K \to \Pi_L$ be the abelianization of some isomorphism $\Phi^{\mathrm{c}} : \Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$ Then denoting by L^{i} and K^{i} the perfect closures, there exist an isomorphism of field extensions $\imath : L^{\mathrm{i}}|l \to K^{\mathrm{i}}|k$ and an ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\varepsilon \cdot \Phi$ is induced by $\imath$. Moreover, the isomorphism $\imath$ is unique up to Frobenius twists, and the ℓ-adic unit ε is unique up to multiplication by p-powers, where $p = \mathrm{char}(k)$.*

(3) *For a function field $L|l$ as above, let $\mathrm{Isom}^{\mathrm{F}}(L,K)$ be the set of isomorphisms of field extensions $\imath : L^{\mathrm{i}}|l \to K^{\mathrm{i}}|k$ up to Frobenius twists, and let $\mathrm{Isom}^{\mathrm{c}}(\Pi_K, \Pi_L)$ be the set of abelianizations of continuous group isomorphisms $\Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$ up to multiplication by ℓ-adic units $\varepsilon \in \mathbb{Z}_\ell^\times$. Then there is a canonical bijection*

$$\mathrm{Isom}^{\mathrm{F}}(L|l, K|k) \to \mathrm{Isom}^{\mathrm{c}}(\Pi_K, \Pi_L).$$

A sketch of a strategy to functorially recover $K|k$ from pro-ℓ Galois information, in particular to prove the above target result, can be found essentially already in (the notes of) Pop [25], and has as starting point the following simple idea: Let $\widehat{K}$ be the ℓ-adic completion of the multiplicative group $K^\times$ of $K|k$.[1] Since the cyclotomic character of K is trivial, one can identify the ℓ-adic Tate module $\mathbb{T}_{K,\ell}$ of K with $\mathbb{Z}_\ell$ (non-canonically), and let $\imath_K : \mathbb{T}_{K,\ell} \to \mathbb{Z}_\ell$ be a fixed identification. Via Kummer theory, one has isomorphisms of ℓ-adically complete groups:

$$\widehat{K} = \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{T}_{K,\ell}) \xrightarrow{\imath_K} \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell),$$

i.e., $\widehat{K}$ can be recovered from Π_K, hence from Π_K^{c} via the projection $\Pi_K^{\mathrm{c}} \to \Pi_K$. On the other hand, since $k^\times$ is divisible, $\widehat{K}$ equals the ℓ-adic completion of the free abelian group $K^\times/k^\times$. Now the idea of recovering $K|k$ is as follows:

(a) First, give a recipe to recover the image $j_K(K^\times) = K^\times/k^\times$ of the ℓ-adic completion functor $j_K : K^\times \to K^\times/k^\times \subset \widehat{K}$ inside the "known" $\widehat{K} = \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell)$.

(b) Second, interpreting $K^\times/k^\times =: \mathscr{P}(K)$ as the projectivization of the infinite-dimensional k-vector space $(K,+)$, give a recipe to recover the projective lines $\mathfrak{l}_{x,y} := (kx+ky)^\times/k^\times$ inside $\mathscr{P}(K)$, where $x, y \in K$ are k-linearly independent.

(c) Third, apply the *fundamental theorem of projective geometries* of Artin [1], and deduce that $K|k$ can be recovered from $\mathscr{P}(K)$ endowed with all the lines $\mathfrak{l}_{x,y}$.

(d) Finally, show that the recipes above are functorial, i.e., they are invariant under isomorphisms of profinite groups $\Pi_K \to \Pi_L$ which are abelianizations of isomorphisms $\Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$. In particular, such isomorphisms $\Pi_K \to \Pi_L$ originate actually from geometry.

The strategy from Pop [25] to tackle the above problems (a), (b), (c), (d), above is in principle similar to the strategies (initiated by Neukirch and Uchida) for tackling Grothendieck's anabelian conjectures. It has two main parts, as follows, the terminology being as introduced later:

Local theory: It has as input the Galois/group-theoretical information Π_K^{c}. It should be a recipe which in a first approximation recovers from Π_K^{c} the decomposition/inertia groups of prime divisors of $K|k$ in Π_K (N.B., not in Π_K^{c}). The final output of the local theory should be the total decomposition graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ of $K|k$. This recipe should be invariant under isomorphisms $\Pi_K \to \Pi_L$ which are induced by some isomorphisms $\Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$.

Global theory: Its input is the total decomposition graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ of $K|k$. It should be a recipe which in a first approximation recovers the geometric decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ (together with some of their special properties) for $K|k$ from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ together with their sets of rational quotients $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$. In a second approximation, this recipe should recover $\mathscr{P}(K)$ and its projective lines from the $\mathscr{G}_{\mathscr{D}_K}$ endowed with their rational quotients $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$. It thus should finally recover the function field $K|k$. Moreover, this recipe should be functorial, i.e., invariant under isomorphisms of total decomposition graphs $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{G}_{\mathscr{D}_L^{\mathrm{tot}}}$.

The present manuscript deals mainly with questions of the above *global theory,* precisely, recovering the geometric decomposition graphs (together with some of their special properties) from the total decomposition graph, and finally proving the main result of the paper, which is to show that morphisms of (total) decomposition graphs which are compatible with rational projections originate in a precise way from geometry.

Before announcing the main result of the manuscript, let us briefly introduce the main concepts and objects, which will discussed/studied in detail later.

• **Prime divisor graphs** (see Section 3 for more details)

Recall that in the context above, a (Zariski) prime divisor of a function field $K|k$ is a discrete valuation v of K whose valuation ring is the local ring $\mathscr{O}_{X,x_1}$ of the generic point x_1 of some Weil prime divisor of some normal model $X \to k$ of $K|k$. If so, then the residue field Kv of v is the function field $Kv = \kappa(x_1)$, and therefore, $\mathrm{td}(Kv|k) = \mathrm{td}(K|k) - 1$. A set of Zariski prime divisors D of $K|k$ is called a geometric set if there exists a quasi-projective normal model $X \to k$ of $K|k$ such that $D = D_X$ is the set of valuations v_{x_1} defined by the generic points x_1 of all the Weil prime divisors of X. We next generalize the prime divisors of $K|k$ as follows: First, for a valuation $\mathfrak{v}$ of K the following are equivalent:

(i) $\mathfrak{v}$ is trivial on k, and the residue field has $\mathrm{td}(K\mathfrak{v}|k) = \mathrm{td}(K|k) - r$, and there exists a chain of valuations $\mathfrak{v}_1 < \cdots < \mathfrak{v}_r := \mathfrak{v}$.

(ii) $\mathfrak{v}$ is the valuation-theoretical composition $\mathfrak{v} = v_r \circ \cdots \circ v_1$, where v_1 is a prime divisor of K, and inductively, v_{i+1} is a prime divisor of the residue function field $\kappa(v_i)\,|\,k$.

A valuation $\mathfrak{v}$ of K which satisfies the above equivalent conditions is called a prime r-divisor of $K|k$; and a sequence of prime divisors $(v_r, \dots, v_1)$ as above will be called a Parshin r-chain of $K|k$. By definition, the trivial valuation will be considered a generalized prime divisor of rank zero, and the corresponding Parshin chain is the trivial Parshin chain. Finally, note that $r \leq \mathrm{td}(K|k)$, and that in the above notation, one has $\mathfrak{v}_i = v_i \circ \cdots \circ v_1$ for all $i \geq 1$.

The total prime divisor graph $\mathscr{D}_K^{\mathrm{tot}}$ of K is the following half-oriented graph:

(a) The vertices of $\mathscr{D}_K^{\mathrm{tot}}$ are the residue fields $K\mathfrak{v}$ of all the generalized prime divisors $\mathfrak{v}$ of $K|k$ viewed as distinct function fields.

(b) For given $\mathfrak{v} = v_r \circ \cdots \circ v_1$ and $\mathfrak{w} = w_s \circ \cdots \circ w_1$, the edges from $K\mathfrak{v}$ to $K\mathfrak{w}$ are as follows:

(i) If $\mathfrak{v} = \mathfrak{w}$, i.e., $K\mathfrak{v} = K\mathfrak{w}$, then the trivial valuation $\mathfrak{v}/\mathfrak{w} = \mathfrak{w}/\mathfrak{v}$ is the only edge from $K\mathfrak{v} = K\mathfrak{w}$ to itself; and it is by definition a non-oriented edge.

(ii) If $K\mathfrak{v} \neq K\mathfrak{w}$, then the set of edges from $K\mathfrak{v}$ to $K\mathfrak{w}$ is non-empty iff $s = r+1$ and $v_i = w_i$ for $1 \leq i \leq r$; and if so, then $w_s = \mathfrak{w}/\mathfrak{v}$ is the only edge from $K\mathfrak{v}$ to $K\mathfrak{w}$, and it is by definition an oriented edge.

A geometric prime divisor graph for $K|k$ is any connected subgraph $\mathscr{D}_K$ of $\mathscr{D}_K^{\mathrm{tot}}$ which satisfies the following conditions: First, for each vertex $K\mathfrak{v}$ of $\mathscr{D}_K$, the set $D_{\mathfrak{v}}$ of all the non-trivial edges of $\mathscr{D}_K$ originating from $K\mathfrak{v}$ is a geometric set of prime divisors of $K\mathfrak{v}\,|\,k$. Second, all maximal branches of non-trivial edges of $\mathscr{D}_K$ originate at K and have length equal to $\mathrm{td}(K|k)$. Equivalently, $\mathscr{D}_K$ is a half-oriented connected graph having $K = K_0$ as origin and satisfying:

(a) The vertices of $\mathscr{D}_K$ are distinct function fields $K_i|k$ over k.

(b) For every vertex K_i, the trivial valuation of K_i is the only edge from K_i to itself. And the set of non-trivial edges v_i originating at K_{i^*} is a geometric set of prime divisors of $K_{i^*}|k$, and if v_i is a non-trivial edge from K_{i^*} to K_i, then $K_i = K_{i^*}v_i$.

(c) The only cycles of the graph are the non-oriented edges, and all the maximal branches consisting of oriented edges only have length equal to $\mathrm{td}(K|k)$.

The functorial behavior of geometric prime divisor graphs is as follows:

(1) *Embeddings.* Let $L|l \hookrightarrow K|k$ be an embedding of function fields which maps l onto k. Then the canonical restriction map of valuations $\mathrm{Val}_K \to \mathrm{Val}_L$, $v \mapsto v|_L$, gives rise to a morphism of the total prime divisor graphs $\varphi_\imath : \mathscr{D}_K^{\mathrm{tot}} \to \mathscr{D}_L^{\mathrm{tot}}$, which moreover is surjective. The relation between *geometric prime divisor graphs* $\mathscr{D}_K$ and $\mathscr{D}_L$ is a little bit more subtle; see Proposition 37: Given geometric prime divisor graphs $\mathscr{D}_K$ and $\mathscr{D}_L$, there exist geometric prime divisor graphs $\mathscr{D}_K^0$ and $\mathscr{D}_L^0$ containing $\mathscr{D}_K$, respectively $\mathscr{D}_L$, such that $\varphi_\imath$ defines a surjective morphism of geometric prime divisor graphs:

$$\varphi_\imath : \mathscr{D}_K^0 \to \mathscr{D}_L^0.$$

(2) *Restrictions.* Given a generalized prime divisor $\mathfrak{v}$ of $K|k$, let $\mathscr{D}_{\mathfrak{v}}^{\mathrm{tot}}$ be the set of all generalized prime divisors $\mathfrak{w}$ of $K|k$ with $\mathfrak{v} \leq \mathfrak{w}$. Then the map

$$\mathscr{D}_{\mathfrak{v}}^{\mathrm{tot}} \to \mathscr{D}_{K\mathfrak{v}}^{\mathrm{tot}}, \quad \mathfrak{w} \mapsto \mathfrak{w}/\mathfrak{v},$$

is an isomorphism of $\mathscr{D}_{\mathfrak{v}}^{\mathrm{tot}}$ onto $\mathscr{D}_{K\mathfrak{v}}^{\mathrm{tot}}$. Moreover, if $K\mathfrak{v}$ is a vertex of some geometric prime divisor graph $\mathscr{D}_K$ for $K|k$, then one has that the maximal subgraph $\mathscr{D}_{K\mathfrak{v}}$ of $\mathscr{D}_K$ whose initial vertex is $K\mathfrak{v}$ is a geometric graph of prime divisors of $K\mathfrak{v}$.

• **Decomposition graphs** (see Section 3 for more details)

Let $K|k$ be as considered above. Then we have the following, see e.g., Pop [28], Introduction, for a discussion of these facts: For every prime divisor v of $K|k$ one has $T_v \cong \mathbb{T}_{\ell,K}$, and for every prime r-divisor $\mathfrak{v}$ one has $T_{\mathfrak{v}} \cong \mathbb{T}_{\ell,K}^r$. Further, for generalized prime divisors $\mathfrak{v}$ and $\mathfrak{w}$ one has $Z_{\mathfrak{v}} \cap Z_{\mathfrak{w}} \neq 1$ if and only if $\mathfrak{v}, \mathfrak{w}$ are not independent as valuations, i.e., $\mathscr{O} := \mathscr{O}_{\mathfrak{v}}\,\mathscr{O}_{\mathfrak{w}} \neq K$; and if so, then $\mathscr{O}$ is the valuation ring of a generalized prime divisor $\mathfrak{u}$ of $K|k$ which turns out to be the unique generalized prime divisor with $T_{\mathfrak{u}} = T_{\mathfrak{v}} \cap T_{\mathfrak{w}}$, and also the unique generalized prime divisor of $K|k$ maximal with the property $Z_{\mathfrak{v}}, Z_{\mathfrak{w}} \subseteq Z_{\mathfrak{u}}$.

In particular, $\mathfrak{v} = \mathfrak{w}$ iff $T_{\mathfrak{v}} = T_{\mathfrak{w}}$ iff $Z_{\mathfrak{v}} = Z_{\mathfrak{w}}$. Further, $\mathfrak{v} < \mathfrak{w}$ iff $T_{\mathfrak{v}} \subset T_{\mathfrak{w}}$ strictly iff $Z_{\mathfrak{v}} \supset Z_{\mathfrak{w}}$ strictly, and $T_{\mathfrak{w}}/T_{\mathfrak{v}} \cong \mathbb{Z}_\ell^{s-r}$ if $\mathfrak{v}$ is a prime r-divisor and $\mathfrak{w}$ is a prime s-divisor.

We conclude that the partial ordering of the set of all the generalized prime divisors $\mathfrak{v}$ of $K|k$ *is encoded in the set* of their inertia/decomposition groups $T_{\mathfrak{v}} \subseteq D_{\mathfrak{v}}$. In particular, the existence of the trivial, respectively a non-trivial, edge from $K\mathfrak{v}$ to $K\mathfrak{w}$ in $\mathscr{D}_K^{\mathrm{tot}}$ is equivalent to $T_{\mathfrak{v}} = T_{\mathfrak{w}}$, respectively to $T_{\mathfrak{v}} \subset T_{\mathfrak{w}}$ and $T_{\mathfrak{w}}/T_{\mathfrak{v}} \cong \mathbb{Z}_\ell$.

Via the Galois correspondence and the functorial properties of the Hilbert decomposition theory for valuations, we attach to the total prime divisor graph $\mathscr{D}_K^{\mathrm{tot}}$ of $K|k$ a graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ whose vertices and edges are in bijection with those of $\mathscr{D}_K^{\mathrm{tot}}$ as follows:

(a) The vertices of $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ are $\Pi_{K\mathfrak{v}}$, viewed as distinct pro-ℓ groups (all $\mathfrak{v}$).

b) If the edge from $K\mathfrak{v}$ to $K\mathfrak{w}$ exists, the corresponding edge from $\Pi_{K\mathfrak{v}}$ to $\Pi_{K\mathfrak{w}}$ is endowed with the pair of groups $T_{\mathfrak{w}/\mathfrak{v}} \subseteq Z_{\mathfrak{w}/\mathfrak{v}}$ viewed as subgroups of $\Pi_{K\mathfrak{v}}$; thus $\Pi_{K\mathfrak{w}} = Z_{\mathfrak{w}/\mathfrak{v}}/T_{\mathfrak{w}/\mathfrak{v}}$.

The graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ will be called the total decomposition graph of $K|k$, or of Π_K. If $\mathscr{D}_K \subseteq \mathscr{D}_K^{\mathrm{tot}}$ is a geometric graph of prime divisors of $K|k$, the corresponding subgraph $\mathscr{G}_{\mathscr{D}_K} \subseteq \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ will be called a geometric decomposition graph for $K|k$, or for Π_K.

Next recall that the isomorphy type of (the maximal abelian pro-ℓ quotient of) the fundamental group $\Pi_1(X) := \pi_1^{\mathrm{ab},\ell}(X)$ of complete regular models $X \to k$, if such models exist, does depend on $K|k$ only, and not on $X \to k$. Moreover, one can recover $\Pi_1(X)$ as being $\Pi_1(X) = \Pi_K/T_K$, where T_K is the subgroup of G_K generated by all the inertia groups T_v with v a prime divisor of $K|k$. This justifies calling the group $\Pi_{1,K} := \Pi_K/T_K$ the birational fundamental group for $K|k$. As discussed at Fact 57, there always exist quasi-projective normal models $X \to k$ for $K|k$ such that $T_K = T_{D_X}$, where T_{D_X} is the closed subgroup of Π_K generated by all the T_v with $v \in D_X$. We will say that a model $X \to k$ of $K|k$ and/or that D_X is complete regular-like if $T_K = T_{D_X}$ and the rational rank $\mathrm{rr}\big(\mathfrak{Cl}(X)\big)$ of the divisor class group $\mathfrak{Cl}(X)$ is positive, and for every normal quasi-projective model $\tilde{X}$ with $D_X \subseteq D_{\tilde{X}}$ one has that $\mathrm{rr}\big(\mathfrak{Cl}(\tilde{X})\big) = \mathrm{rr}\big(\mathfrak{Cl}(X)\big) + |D_{\tilde{X}} \backslash D_X|$. Note that a complete regular like curve is a complete normal curve and vice-versa. We say that a geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$ is complete regular-like if for all vertices $\mathfrak{v}$ of $\mathscr{D}_K$ with $\mathrm{td}(K\mathfrak{v}|k) > 0$ one has that the set $D_{\mathfrak{v}}$ of 1-edges of $\mathscr{G}_{\mathscr{D}_{K\mathfrak{v}|k}}$ is complete regular-like.

As shown in Proposition 22, there exists a group-theoretical recipe by which one can recover the geometric decomposition graphs (and the property of being complete regular-like) from the total decomposition graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$. Further, by Proposition 39, that recipe is invariant under isomorphisms $\Phi : \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{G}_{\mathscr{D}_L^{\mathrm{tot}}}$, i.e., every such isomorphism gives rise by restriction to isomorphisms of the (complete regular-like) decomposition graphs for $K|k$ onto the (complete regular-like) ones for $L|l$.

The functorial properties of the graphs of prime divisors translate to the following functorial properties of the decomposition graphs:

(1) *Embeddings.* Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields which maps l onto k. Then the canonical projection homomorphism $\Phi_\imath : \Pi_K \to \Pi_L$ is an open homomorphism, and for every generalized prime divisor $\mathfrak{v}$ of $K|k$ and its restriction $\mathfrak{v}_L$ to L one has that $\Phi_\imath(Z_{\mathfrak{v}}) \subseteq Z_{\mathfrak{v}_L}$ is an open subgroup, and $\Phi_\imath(T_{\mathfrak{v}}) \subseteq T_{\mathfrak{v}_L}$ satisfies $\Phi_\imath(T_{\mathfrak{v}}) = 1$ iff $\mathfrak{v}_L$ is the trivial valuation. Therefore, $\Phi_\imath$ gives rise to a morphism of total decomposition graphs, which we denote by the same symbol

$$\Phi_\imath : \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{G}_{\mathscr{D}_L^{\mathrm{tot}}}.$$

In turn, for given geometric decomposition graphs $\mathscr{D}_K$ and $\mathscr{D}_L$, for which $\imath$ gives rise to a morphism of geometric decomposition graphs $\mathscr{D}_K \to \mathscr{D}_L$, the above $\Phi_\imath$ morphism of total decomposition graphs gives rise to a morphisms of geometric decomposition graphs $\Phi_\imath : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_L}$, as defined later in Sections 4 and 5.

(2) *Restrictions.* Given a generalized prime divisor $\mathfrak{v}$ of $K|k$, let $\mathrm{pr}_\mathfrak{v} : Z_\mathfrak{v} \to \Pi_{K\mathfrak{v}}$ be the canonical projection. Then for every $\mathfrak{w} \geq \mathfrak{v}$ we have that $T_\mathfrak{w} \subseteq Z_\mathfrak{w}$ are mapped onto $T_{\mathfrak{w}/\mathfrak{v}} \subseteq Z_{\mathfrak{w}/\mathfrak{v}}$. Therefore, the total decomposition graph of $K\mathfrak{v}\,|\,k$ can be recovered from that of $K|k$ in a canonical way via $\mathrm{pr}_\mathfrak{v} : Z_\mathfrak{v} \to \Pi_{K\mathfrak{v}}$.

• **Rational quotients** (see Section 5 for more details) Let $K|k$ be a function field as above satisfying $\mathrm{td}(K|k) > 1$. For every non-constant function $t \in K$, let κ_t be the relative algebraic closure of $k(t)$ in K. Since $\mathrm{td}(\kappa_t|k) = 1$, it follows that κ_t has a unique complete normal model $X_t \to k$, which is a projective smooth curve. Therefore, the set of prime divisors of $\kappa_t|k$ is actually in bijection with the (local rings at the) closed points of X_t, thus with the set of Weil prime divisors of X_t. Therefore, the total prime divisor graph $\mathscr{D}^{\mathrm{tot}}_{\kappa_t}$ for $\kappa_t|k$ is actually the *unique maximal* geometric prime divisor graph for $\kappa_t|k$. We denote $\mathscr{D}^{\mathrm{tot}}_{\kappa_t}$ simply by $\mathscr{D}_{\kappa_t}$.

Let $\imath_t : \kappa_t \to K$ be the canonical embedding, and $\Phi_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ the (surjective) canonical projection. Then by the functoriality of embeddings, Φ_{κ_t} gives rise canonically to a morphism $\Phi_{\kappa_t} : \mathscr{G}_{\mathscr{D}^{\mathrm{tot}}_K} \to \mathscr{G}_{\kappa_t}$. Moreover, if $\mathscr{G}_{\mathscr{D}_K}$ is a geometric decomposition graph for $K|k$, then Φ_{κ_t} restricts to a morphism of geometric decomposition graphs $\Phi_{\kappa_t} : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\kappa_t}$.

In the above context, if $\kappa_t = k(t)$, we say that Φ_{κ_t} is a rational quotient of $\mathscr{G}_{\mathscr{D}^{\mathrm{tot}}_K}$ as well as of every geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$ for $K|k$. We call such $t \in K$ "general elements" of K; and usually denote general elements of K by x, in order to distinguish them from the "usual" non-constant $t \in K$. A "birational" Bertini-type argument shows that there are "many" general elements in K; see Lang [18], Ch. VIII, and/or Roquette [32], § 4, respectively Fact 43 in Section 5: For any given algebraically independent functions $t, t' \in K$, not both inseparable, $t_{a',a} := t/(a't' + a)$ is a general element of K for almost all $a', a \in k$. A set of general elements $\Sigma \subset K$ is a Bertini set if Σ contains almost all elements $t_{a',a}$ for all t, t' as above. We denote by $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$ the set of all the rational quotients of $K|k$, and consider subsets $\mathfrak{A} \subset \mathfrak{A}_K$ containing all the $\Phi_{\kappa_x} \in \mathfrak{A}$, $x \in \Sigma$, with Σ some Bertini set of general elements, and call them, for short, Bertini-type sets of rational quotients.

The relation between rational projections and morphisms of geometric decomposition graphs is as follows: Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields with $\imath(l) = k$, such that $K|\imath(L)$ a separable field extension, and $\mathrm{td}(L|l) > 1$. Then there exists a Bertini-type set $\mathfrak{B} = \{\Phi_{\kappa_y}\}_{\kappa_y}$ for $L|l$ such that $\kappa_x := \imath(\kappa_y)$ is relatively algebraically closed in K for all κ_y. Hence for all $\Phi_{\kappa_y} \in \mathfrak{B}$ and the corresponding $\Phi_{\kappa_x} \in \mathfrak{A}_K$, $\kappa_x := \imath(\kappa_y)$, we get that the isomorphism $\Phi_{\kappa_x\kappa_y} : \mathscr{G}_{\kappa_x} \to \mathscr{G}_{\kappa_y}$ defined by $\imath_{\kappa_x\kappa_y} := \imath|_{\kappa_y}$ satisfies the condition

$$\Phi_{\kappa_y} \circ \Phi_\imath = \Phi_{\kappa_x\kappa_y} \circ \Phi_{\kappa_x}.$$

Because of this property, we will say that $\Phi_\imath$ is compatible with rational quotients.

• Abstract decomposition graphs

It is one of our main tasks in the present manuscript to define and study abstract decomposition graphs, which resemble the geometric decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ (this will be done in Section 2) and to define proper morphisms of such abstract decomposition graphs, in particular their rational quotients (which will be done in Section 4). The abstract decomposition graphs, which endowed with families of rational quotients resemble the complete regular-like geometric decomposition graphs as introduced above, will be called complete regular-like abstract decomposition graphs.

The main result of this manuscript is the following; see Theorem 45 for a more general assertion, and Definition 21, Fact/Definition 43 (2), Definition 33 (and Definitions 12 and 9), and Definition/Remark 34 for the definitions of all the terms:

Main Theorem. *Let $K|k$ and $L|l$ be function fields with* $\mathrm{td}(K|k) > 1$. *Let $\mathscr{G}_{\mathscr{D}_K^{\rm tot}}$ and $\mathscr{H}_{\mathscr{D}_L^{\rm tot}}$ be their total decomposition graphs, which we endow with Bertini-type sets of rational quotients $\mathfrak{A}$, respectively $\mathfrak{B}$. Then the following hold:*

(1) *There exists a group-theoretical recipe which recovers $K|k$ from $\mathscr{G}_{\mathscr{D}_K^{\rm tot}}$ endowed with $\mathfrak{A}$. Moreover, this recipe is invariant under isomorphisms in the following sense: Up to multiplication by ℓ-adic units and composition with automorphisms $\Phi_\imath$ of $\mathscr{G}_{\mathscr{D}_K^{\rm tot}}$ defined by automorphisms $\imath : K^{\rm i}|k \to K^{\rm i}|k$, there exists at most one isomorphism $\Phi : \mathscr{G}_{\mathscr{D}_K^{\rm tot}} \to \mathscr{H}_{\mathscr{D}_L^{\rm tot}}$ of* abstract decomposition graphs *which is compatible with the sets of rational quotients $\mathfrak{A}$ and $\mathfrak{B}$.*

(2) *The following more precise assertion holds: Suppose that* $\mathrm{td}(L|l) > 1$. *Let $\mathscr{G}_{\mathscr{D}_K}$ and $\mathscr{H}_{\mathscr{D}_L}$ be geometric complete regular-like decomposition graphs for $K|k$, which endowed with $\mathfrak{A}$, respectively $\mathfrak{B}$, are viewed as* complete regular-like abstract decomposition graphs. *Then for every morphism*

$$\Phi : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$$

which is compatible with the sets of rational quotients $\mathfrak{A}$ and $\mathfrak{B}$, there exist an ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$ and an embedding of field extensions

$$\imath : L^{\rm i}|l \to K^{\rm i}|k$$

such that $\Phi = \varepsilon \cdot \Phi_\imath$, where $\Phi_\imath : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$ is the canonical morphism defined by $\imath$ as above.

Further, $\imath(l) = k$, and $\imath$ is unique up to Frobenius twists, and ε is unique up to multiplication by powers of p, where $p =$ $\mathrm{char}(k)$.

We notice that the main theorem above (together with Propositions 22 and 39) reduces the problem of functorially recovering $K|k$ from $\Pi_K^{\rm c}$, thus completing the proof of the target result above, to recovering the *total decomposition graph* $\mathscr{G}_{\mathscr{D}_K^{\rm tot}}$ of $K|k$ and its *rational quotients.* In the case that k is an algebraic closure of a finite field, both these problems were solved in Pop [27], but working with the full pro-ℓ Galois group $G_K(\ell)$ instead of $\Pi_K^{\rm c}$. Nevertheless, the methods of Pop [27] to recover $\mathscr{G}_{\mathscr{D}_K^{\rm tot}}$ and its rational quotients used only the set of all the divisorial groups $T_v \subset Z_v$

inside Π_K. Using the local theory developed in Pop [28] instead of the local theory of Pop [27], we complete the proof of the target result above in Pop [30].

Historical note

The idea to recover $K|k$ from Π_K^{c} originates from Bogomolov [2], and a first attempt to do so can be found in his fundamental paper [2]. Although that paper is too sketchy to make clear what the author precisely proposes, a thorough inspection shows that it provides a fundamental tool for recovering inertia elements of valuations v of K (which nevertheless may be non-trivial on k). This is Bogomolov's theory of *commuting liftable pairs*; see Bogomolov–Tschinkel [3] for detailed proofs. On the other hand, it is not at all clear how and whether one could develop a "global theory" along the lines (vaguely) suggested in [2], and there was virtually no progress on the problem for about a decade.

A sketch of a viable global theory —at least in the case that *k is an algebraic closure of a finite field*— was proposed in the notes of my MSRI talk in the fall of 1999; see Pop [25]. In the second part of Pop [26], the technical details concerning the global theory hinted at in Pop [25] were worked out. Actually, the present manuscript is an elaboration of parts of Pop [26], and the main theorem here, more precisely Theorem 45, is the *Hom-form* of the *Isom-form* of Theorem 5.11 of [26]. However, I should mention that in [26] the mixed "arithmetic + geometric situation" was considered as well as non-abelian Galois groups, which was/is of interest in the case that k is not algebraically closed.

In the case that *k is an algebraic closure of a finite field,* let me finally mention:

• In the manuscript Pop [27], a recipe to functorially recover $K|k$ from $G_K(\ell)$, in particular a proof of (a slightly stronger form of) the above target result was given. First, the assertion one proves using $G_K(\ell)$ instead of Π_K^{c} is stronger, namely, if $\Phi : \Pi_K \to \Pi_L$ is the abelianization of an isomorphism $\Phi(\ell) : G_K(\ell) \to G_L(\ell)$, then there exists an isomorphism $\imath : L^{\mathrm{i}}|l \to K^{\mathrm{i}}|k$ (unique up to Frobenius twists) which defines Φ; thus one does not need to "adjust" Φ by multiplying by an ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$. I should also observe that the full $G_K(\ell)$ was used in loc.cit. essentially only in order to recover the divisorial subgroups of Π_K via the canonical projection $G_K(\ell) \to \Pi_K$, whereas all the other steps of the local and global theory are virtually identical with the ones in the case of Π_K^{c}. (The recipe to recover the divisorial subgroups of Π_K via $\Pi_K^{\mathrm{c}} \to \Pi_K$ is given in Pop [28] and uses Bogomolov's theory of commuting liftable pairs as a "black box." That recipe is used in Pop [30].)

• Bogomolov–Tschinkel [4], [5], consider the case $K = k(X)$, where X is a projective smooth surface over k. In the initial variant of their manuscript [4], they considered only the case that $\pi_1(X)$ is finite, and proved that if Π_K^{c} and Π_L^{c} are isomorphic, then $K|k$ and $L|l$ are isomorphic up to pure inseparable closures, provided k and l are algebraic closures of finite fields with char $\neq 2$ (which is less precise than what the above target result gives in this case). Nevertheless, in the published version [5] of their earlier manuscript [4], they announce their main result for surfaces in a form almost identical with the target result above and use a strategy of proof which is in many ways very similar to that announced in Pop [25], and used in Pop [27].

Thanks. First I would like to thank P. Deligne for several useful discussions we had during my visits to IAS Princeton. Actually, the theory of abstract decomposition graphs presented here is inspired by a suggestion of his (made in the letter [8] from September 1995) for an axiomatic approach to the birational anabelian conjecture in the arithmetical case. I also want to thank J.-L. Colliot-Thélène for several discussions we had at MSRI in the fall of 1999, as well as M. Saidi, T. Szamuely, and J. Stix for "discussion sessions" at Bonn and Penn. Finally, I would like to thank a few others for their interest in this work, including D. Harbater, J. Koenigsmann, P. Lochak, J. Mínáč, H. Nakamura, and A. Tamagawa.

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2 Pro-ℓ abstract decomposition graphs

In this section we develop an abelian pro-ℓ prime divisor decomposition theory for "abstract function fields" which is similar in some sense to the abstract class field theory. Throughout ℓ is a fixed prime number, and $\delta \geq 0$ is a non-negative integer.

2.1 Axioms and definitions

Definition 1 A level-δ (pro-ℓ) abstract decomposition graph is a connected half-oriented graph $\mathscr{G}$ whose vertices are endowed with pro-ℓ abelian groups G_i and whose edges v_i are endowed with pairs of pro-ℓ abelian groups $T_{v_i} \subseteq Z_{v_i}$ satisfying the following:

Axiom I): The vertices of $\mathscr{G}$ are pro-ℓ abelian free groups G_i, and $\mathscr{G}$ has an origin, which we denote by $G_0 = G$.

Axiom II): The edges v_i and the corresponding $T_{v_i} \subseteq Z_{v_i}$ satisfy the following:

(i) For every vertex G_i there exists a unique non-oriented edge v_{i0} from G_i to itself, and the corresponding pair of pro-ℓ groups is $\{1\} =: T_{v_{i0}} \subseteq Z_{v_{i0}} := G_i$. For all other vertices $G_{i^*} \neq G_i$ there exists at most one edge v_i from G_{i^*} to G_i. If v_i exists, we say that v_i is the oriented edge from G_{i^*} to G_i, and v_i is endowed with a pair $T_{v_i} \subseteq Z_{v_i}$ of subgroups of G_{i^*} such that $T_{v_i} \cong \mathbb{Z}_\ell$ and $G_i = Z_{v_i}/T_{v_i}$.

The edges of $\mathscr{G}$ are also called valuations of $\mathscr{G}$; in particular, the edges originating from G_i are called valuations of G_i. The non-oriented edge v_{i0} from G_i to itself is called the trivial valuation of G_i, whereas the oriented edges v_i originating from G_{i^*} are called non-trivial valuations of G_{i^*}.

The groups $T_{v_i} \subseteq Z_{v_i}$ are called the inertia, respectively decomposition, groups of v_i; and $G_i := Z_{v_i}/T_{v_i}$ is called the residue group of v_i.

(ii) For distinct non-trivial edges $v_i \neq v_{i'}$ originating from G_{i^*}, one has $Z_{v_i} \cap Z_{v_{i'}} = 1$, hence $T_{v_i} \cap T_{v_{i'}} = 1$ holds as well.

For every cofinite subset $\mathfrak{U}_i$ of the set of non-trivial edges v_i originating from G_{i^*}, let $T_{\mathfrak{U}_i}$ be the closed subgroup of G_{i^*} generated by all the T_{v_i}, $v_i \in \mathfrak{U}_i$. A system $(\mathfrak{U}_{i,\alpha})_\alpha$ of such cofinite subsets is called cofinal, if every finite set of valuations v_i as above is contained in the complement of $\mathfrak{U}_{i,\alpha}$ for some α.

(iii) There exist cofinal systems $(\mathfrak{U}_{i,\alpha})_\alpha$ such that $T_{v_i} \cap T_{\mathfrak{U}_{i,\alpha}} = 1$ for all α and all $v_i \notin \mathfrak{U}_{i,\alpha}$.

Axiom III): The non-oriented edges v_{i0} are the only cycles of the graph $\mathscr{G}$, and all maximal branches of non-trivial edges of $\mathscr{G}$ have length equal to δ.

Definition/Remark 2 Let $\mathscr{G}$ be an abstract decomposition graph of level-$\delta_{\mathscr{G}}$ on a pro-ℓ group $G = G_0$. We will say that $\mathscr{G}$ is a level-$\delta_{\mathscr{G}}$ abstract decomposition graph on G. A valuation of $G = G_0$ will be called a 1-edge of $\mathscr{G}$. If no confusion is possible, we will denote the 1-edges of $\mathscr{G}$ simply by v; thus the corresponding pro-ℓ groups involved are denoted by $T_v \subseteq Z_v$ and $G_v := Z_v/T_v$.

(1) Consider any δ such that $0 \leq \delta \leq \delta_{\mathscr{G}}$. By induction on δ it is easy to see that $\mathscr{G}$ has a unique maximal connected abstract decomposition subgraph containing the origin G of $\mathscr{G}$ and having all branches of oriented edges of length δ.

(2) Let $\mathfrak{v} = (v_r, \dots, v_1)$ be a path of length $\delta_{\mathfrak{v}} := r$ of non-trivial valuations originating at $G = G_0$. This means by definition that v_1 is a non-trivial valuation of G_0, and if $r > 1$, then for all $i < r$ one has inductively that G_i is the residue group of v_i, and v_{i+1} is a non-trivial valuation of G_i. In particular, G_r is the residue group of v_r. Then there exists a unique maximal connected subgraph $\mathscr{G}_{\mathfrak{v}}$ of $\mathscr{G}$ having $G_{\mathfrak{v}} := G_r$ as origin. Clearly, $\mathscr{G}_{\mathfrak{v}}$ is in a natural way an abstract decomposition graph of level $\delta_{\mathscr{G}} - \delta_{\mathfrak{v}}$ on $G_{\mathfrak{v}}$.

We say that $\mathscr{G}_{\mathfrak{v}}$ is an r-residual abstract decomposition graph of $\mathscr{G}$. In particular, the unique 0-residual abstract decomposition graph of $\mathscr{G}$ is $\mathscr{G}$ itself.

(3) For every path $\mathfrak{v} = (v_r, \dots, v_1)$ of length $\delta_{\mathfrak{v}} = r$ as above, we will say that $G_{\mathfrak{v}}$ is an r-residual group of $\mathscr{G}$, precisely that $G_{\mathfrak{v}}$ is the $\mathfrak{v}$-residual group of $\mathscr{G}$. One can further elaborate here as follows: For $r > 1$ we set $\mathfrak{w} = (v_{r-1}, \dots, v_1)$, and suppose that the inertia/decomposition groups $T_{\mathfrak{w}} \subseteq Z_{\mathfrak{w}} \subseteq G_0$ of $\mathfrak{w}$ have been defined inductively such that the residue group $G_{\mathfrak{w}} := Z_{\mathfrak{w}}/T_{\mathfrak{w}}$ of $\mathfrak{w}$ is $G_{\mathfrak{w}} = G_{v_{r-1}}$. We then define the inertia/decomposition groups $T_{\mathfrak{v}} \subseteq Z_{\mathfrak{v}}$ of $\mathfrak{v}$ in G_0 as being the preimages of $T_{v_r} \subseteq Z_{v_r} \subseteq G_{v_{r-1}}$ via $Z_{\mathfrak{w}} \to Z_{\mathfrak{w}}/T_{\mathfrak{w}} = G_{v_{r-1}}$. Note that by definition we have $Z_{\mathfrak{v}}/T_{\mathfrak{v}} =: G_{\mathfrak{v}}$ and $T_{\mathfrak{v}} \cong \mathbb{Z}_\ell^{\delta_{\mathfrak{v}}}$.

We call $\mathfrak{v} = (v_r, \dots, v_1)$ a generalized valuation of $G = G_0$, or a multi-index of length $\delta_{\mathfrak{v}} := r$ of $\mathscr{G}$. And we will say that $\delta_{\mathfrak{v}}$ is the rank of $\mathfrak{v}$ or that $\mathfrak{v}$ is a generalized r-valuation if $r = \delta_{\mathfrak{v}}$.

Given generalized valuations $\mathfrak{v} = (v_r, \ldots, v_1)$, $\mathfrak{w} = (w_s, \ldots, w_1)$, we will say that $\mathfrak{w} \leq \mathfrak{v}$ if $s \leq r$, and $v_i = w_i$ for all $i \leq s$. From the definitions one gets that if $\mathfrak{w} \leq \mathfrak{v}$, then $Z_{\mathfrak{v}} \subseteq Z_{\mathfrak{w}}$ and $T_{\mathfrak{w}} \subseteq T_{\mathfrak{v}}$. On the other hand, by Axiom II (ii), it immediately follows that the converse of (any of) these assertions is also true. We will say that $\mathfrak{v}$ and $\mathfrak{w}$ are dependent if there exists some $q > 0$ such that $v_i = w_i$ for $i \leq q$. For dependent generalized valuations $\mathfrak{v}$ and $\mathfrak{w}$ as above, the following are equivalent:

(a) q is maximal such that $v_i = w_i$ for $i \leq q$.

(b) $T_{\mathfrak{v}} \cap T_{\mathfrak{w}} \cong \mathbb{Z}_\ell^q$.

(c) q is maximal such that $Z_{\mathfrak{v}}, Z_{\mathfrak{w}}$ are both contained in the decomposition group of some generalized q-valuation of $G = G_0$.

(4) In order to have a uniform notation, we take $\mathfrak{v} = v_0$ to be the trivial multi-index, or the trivial path, of $\mathscr{G}$ as the unique one having length equal to 0. We further set $Z_{v_0} := G_0$ and $T_{v_0} = \{1\}$. In particular, one has $G_{v_0} = Z_{v_0}/T_{v_0} = G_0$, which is compatible with the other notations/conventions. Further, $v_0 \leq \mathfrak{v}$ for all multi-indices $\mathfrak{v}$.

Definition/Remark 3 Let $\mathscr{G}$ be a level-$\delta_{\mathscr{G}}$ abstract decomposition graph on the abelian pro-ℓ group $G = G_0$. In notation as above, we consider the following:

(1) Define $\widehat{\Lambda}_{\mathscr{G}} := \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big)$. Since G is a pro-ℓ free abelian group, $\widehat{\Lambda}_{\mathscr{G}}$ is a free ℓ-adically complete $\mathbb{Z}_\ell$-module (in ℓ-adic duality with G).

From now on suppose that $\delta_{\mathscr{G}} > 0$. Recall that $T_v \subset Z_v$ and $G_v = Z_v/T_v$ denote respectively the inertia, the decomposition, and the residue groups at the 1-edges v of $\mathscr{G}$, i.e., at the valuations v of G.

(2) Denote by $T \subseteq G$ the closed subgroup generated by all the inertia groups T_v (all v as above). We set $\Pi_{1,\mathscr{G}} := G/T$ and call it the abstract fundamental group of $\mathscr{G}$. One has a canonical exact sequence

$$1 \to T \to G \to \Pi_{1,\mathscr{G}} \to 1.$$

Taking continuous $\mathbb{Z}_\ell$-Homs, we get an exact sequence of the form

$$0 \to \widehat{U}_{\mathscr{G}} := \mathrm{Hom}\big(\Pi_{1,\mathscr{G}}, \mathbb{Z}_\ell\big) \xrightarrow{\text{can}} \widehat{\Lambda}_{\mathscr{G}} := \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big) \xrightarrow{j^{\mathscr{G}}} \widehat{\Lambda}_T := \mathrm{Hom}\big(T, \mathbb{Z}_\ell\big).$$

We will call $\widehat{U}_{\mathscr{G}} := \mathrm{Hom}\big(\Pi_{1,\mathscr{G}}, \mathbb{Z}_\ell\big)$ the unramified part of $\widehat{\Lambda}_{\mathscr{G}}$. And if no confusion is possible, we will identify $\widehat{U}_{\mathscr{G}}$ with its image in $\widehat{\Lambda}_{\mathscr{G}}$.

(3) Next we have a closer look at the structure of $\widehat{\Lambda}_{\mathscr{G}}$. For every 1-edge v as above, the inclusions $T_v \hookrightarrow Z_v \hookrightarrow G$ give rise to restriction homomorphisms as follows:

$$j^v : \widehat{\Lambda}_{\mathscr{G}} \xrightarrow{\mathrm{res}_{Z_v}} \widehat{\Lambda}_{Z_v} := \mathrm{Hom}\big(Z_v, \mathbb{Z}_\ell\big) \xrightarrow{\mathrm{res}_v} \widehat{\Lambda}_{T_v} := \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big).$$

(a) We set $\widehat{U}_v^1 = \ker(\mathrm{res}_{Z_v})$ and $\widehat{U}_v = \ker(j^v)$ and call them the principal v-units, respectively the v-units, in $\widehat{\Lambda}_{\mathscr{G}}$. And observe that the unramified part of $\widehat{\Lambda}_{\mathscr{G}}$ is exactly $\widehat{U}_{\mathscr{G}} = \cap_v \ker(j^v)$.

(b) The family $(j^v)_v$ gives rise canonically to a continuous homomorphism $\widehat{\oplus}_v\, j^v$ of ℓ-adically complete $\mathbb{Z}_\ell$-modules

$$\widehat{\oplus}_v\, j^v : \widehat{\Lambda}_{\mathscr{G}} \to \widehat{\Lambda}_T \hookrightarrow \widehat{\oplus}_v\, \widehat{\Lambda}_{T_v}$$

Thus identifying $\widehat{\Lambda}_T$ with its image inside $\widehat{\oplus_v}\widehat{\Lambda}_{T_v}$, one has $j^{\mathscr{G}} = \widehat{\oplus_v}\, j^v$ on $\widehat{\Lambda}_{\mathscr{G}}$. We define $\widehat{\mathrm{Div}}_{\mathscr{G}} := \widehat{\oplus_v}\widehat{\Lambda}_{T_v}$ and call it the ℓ-adic abstract divisor group of $\mathscr{G}$.

(c) Finally, we set $\widehat{\mathfrak{Cl}}_{\mathscr{G}} = \mathrm{coker}(j^{\mathscr{G}})$ and call it the ℓ-adic abstract divisor class group of $\mathscr{G}$. And observe that we have a canonical exact sequence

$$0 \to \widehat{U}_{\mathscr{G}} \hookrightarrow \widehat{\Lambda}_{\mathscr{G}} \xrightarrow{j^{\mathscr{G}}} \widehat{\mathrm{Div}}_{\mathscr{G}} \xrightarrow{\mathrm{can}} \widehat{\mathfrak{Cl}}_{\mathscr{G}} \to 0\,.$$

4) Let $\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}} := \{x \in \widehat{\Lambda}_{\mathscr{G}} \mid j^v(x) = 0 \text{ for almost all } v\}$. We notice that by Axiom II (iii), the $\mathbb{Z}_\ell$-module $\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$ is dense in $\widehat{\Lambda}_{\mathscr{G}}$. Indeed, let $(\mathfrak{U}_\alpha)_\alpha$ be a cofinal system of 1-edges v. Then setting $G_\alpha = G/T_{\mathfrak{U}_\alpha}$ and $T_\alpha = T/T_{\mathfrak{U}_\alpha}$, we have a canonical exact sequence

$$1 \to T_\alpha \to G_\alpha \to \Pi_{1,\mathscr{G}} \to 1\,,$$

and T_α is generated by the images $T_{v,\alpha}$ of T_v (all $v \notin \mathfrak{U}_\alpha$) in G_α. Clearly, the image of the inflation homomorphism $\mathrm{inf}_\alpha : \mathrm{Hom}\big(G_\alpha, \mathbb{Z}_\ell\big) \to \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big)$ is exactly

$$\Delta_\alpha := \{x \in \widehat{\Lambda}_{\mathscr{G}} \mid j^v(x) = 0 \text{ for all } v \in \mathfrak{U}_\alpha\} = \cap_{v\in\mathfrak{U}_\alpha} \ker(j^v).$$

Taking inductive limits over the cofinal system $(\mathfrak{U}_\alpha)_\alpha$, the density assertion follows.

We observe that $j^{\mathscr{G}}(\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}) \cong \widehat{\Lambda}_{\mathscr{G}}/\widehat{U}_{\mathscr{G}}$ is a $\mathbb{Z}_\ell$-submodule of the $\mathbb{Z}_\ell$-free module $\oplus_v\widehat{\Lambda}_{T_v} \cong \oplus_v \mathbb{Z}_\ell v$; hence $j^{\mathscr{G}}(\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}})$ is a free $\mathbb{Z}_\ell$-module too. Therefore, for every $\mathbb{Z}_\ell$-submodule $\Delta \subseteq \widehat{\Lambda}_{\mathscr{G}}$, its image $j^{\mathscr{G}}(\Delta)$ under $j^{\mathscr{G}}$ is a free $\mathbb{Z}_\ell$-module. The rank of $j^{\mathscr{G}}(\Delta)$ will be called the corank of Δ.

We notice that a $\mathbb{Z}_\ell$-submodule $\Delta \subset \widehat{\Lambda}_{\mathscr{G}}$ has finite corank iff Δ is contained in $\ker(j^v)$ for almost all v. Clearly, the sum of two finite corank submodules of $\widehat{\Lambda}_{\mathscr{G}}$ is again of finite corank. Thus the set of such submodules is inductive, and one has

$$\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}} = \cup_\Delta \,(\text{all finite corank } \Delta) = \cup_\alpha \Delta_\alpha.$$

(5) We say that $\mathscr{G}$ is complete curve-like if the following holds: There exist generators τ_v of T_v such that $\prod_v \tau_v = 1$, and this is the only pro-relation satisfied by the system of elements $\mathfrak{T} = (\tau_v)_v$. We call such a system $\mathfrak{T} = (\tau_v)_v$ a distinguished system of inertia generators.

We notice the following: Let $\mathscr{G}$ be complete curve-like, and let $\mathfrak{T} = (\tau_v)_v$ and $\mathfrak{T}' = (\tau'_v)_v$ be distinguished systems of inertia generators. Then $\tau'_v = \tau_v^{\varepsilon_v}$ for some ℓ-adic units $\varepsilon_v \in \mathbb{Z}_\ell$, because both τ_v and τ'_v are generators of T_v. Hence we have $1 = \prod_v \tau'_v = \prod_v \tau_v^{\varepsilon_v}$. By the uniqueness of the relation $\prod_v \tau_v = 1$, it follows that $\varepsilon_v = \varepsilon$ for some fixed ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell$.

Next consider some δ with $0 < \delta \le \delta_{\mathscr{G}}$. We say that $\mathscr{G}$ is level-δ complete curve-like if all the $(\delta-1)$-residual abstract decomposition graphs $\mathscr{G}_{\mathfrak{v}}$ are residually complete curve-like. In particular, "level 1 complete curve-like" is the same as "complete curve-like."

(6) For every 1-vertex v consider the exact sequence $1 \to T_v \to Z_v \to G_v \to 1$ given by Axiom II (i). Let $\mathrm{inf}_v : \mathrm{Hom}\big(G_v, \mathbb{Z}_\ell\big) \to \mathrm{Hom}\big(Z_v, \mathbb{Z}_\ell\big)$ be the resulting inflation

homomorphism. Since $T_v = \ker(Z_v \to G_v)$, it follows that $\mathrm{res}_{Z_v}(\widehat{U}_v)$ is the image of the inflation map infl_v. Therefore there exists a canonical exact sequence

$$0 \to \widehat{U}_v^1 \longrightarrow \widehat{U}_v \xrightarrow{j_v} \mathrm{Hom}\big(G_v, \mathbb{Z}_\ell\big) = \widehat{\Lambda}_{\mathscr{G}_v} \to 0,$$

and we call j_v the v-reduction homomorphism.

(7) In particular, if $\delta_{\mathscr{G}} > 1$, then $\delta_{\mathscr{G}_v} = \delta_{\mathscr{G}} - 1 > 0$ for every 1-vertex v, and we have the corresponding exact sequence for the residual abstract decomposition graph $\mathscr{G}_v$

$$0 \to \widehat{U}_{\mathscr{G}_v} \hookrightarrow \widehat{\Lambda}_{\mathscr{G}_v} \xrightarrow{j^{\mathscr{G}_v}} \widehat{\mathrm{Div}}_{\mathscr{G}_v}.$$

We will say that $\mathscr{G}$ is ample if $\delta_{\mathscr{G}} > 0$ and the following conditions are satisfied:

(i) $j^\Sigma : \widehat{\Lambda}_{\mathscr{G}} \longrightarrow \oplus_{v\in\Sigma} \Lambda_{T_v}$ is surjective for every finite set Σ, where $j^\Sigma := \oplus_{v\in\Sigma} j^v$.

(ii) If $\delta_{\mathscr{G}} > 1$, then the following hold:

(a) $j_v(\widehat{U}_{\mathscr{G}}) \subseteq \widehat{U}_{\mathscr{G}_v}$ and $\widehat{U}_{\mathscr{G}_v} + j_v(\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}} \cap \widehat{U}_v) = \widehat{\Lambda}_{\mathscr{G}_v,\mathrm{fin}}$ for every v.

(b) For every finite-corank submodule $\Delta \subseteq \widehat{\Lambda}_{\mathscr{G}}$, there exists v such that $\Delta \subseteq \widehat{U}_v$, and Δ and $j_v(\Delta)$ have equal coranks.

Notice that the condition (ii) above is empty in the case $\delta_{\mathscr{G}} = 1$. Thus if $\delta_{\mathscr{G}} = 1$, then condition (i) is necessary and sufficient for $\mathscr{G}$ to be ample.

Next consider $0 < \delta \le \delta_{\mathscr{G}}$. We say that $\mathscr{G}$ is ample up to level δ if all the residual abstract decomposition graphs $\mathscr{G}_{\mathfrak{v}}$ for $\mathfrak{v}$ such that $0 \le \delta_{\mathfrak{v}} < \delta$ are ample. In particular, “ample up to level 1” is the same as “ample.”

2.2 Abstract $\mathbb{Z}_{(\ell)}$ divisor groups

Definition 4 (1) Let M be the ℓ-adic completion of a free $\mathbb{Z}$-module. A $\mathbb{Z}_{(\ell)}$-submodule $\mathscr{M}_{(\ell)} \subseteq M$ of M is called a $\mathbb{Z}_{(\ell)}$-lattice in M (for short, a lattice) if $\mathscr{M}_{(\ell)}$ is a free $\mathbb{Z}_{(\ell)}$-module, it is ℓ-adically dense in M, and it satisfies the following equivalent conditions:

(a) $M/\ell = \mathscr{M}_{(\ell)}/\ell$

(b) $\mathscr{M}_{(\ell)}$ has a $\mathbb{Z}_{(\ell)}$-basis $\mathfrak{B}$ which is ℓ-adically independent in M.

(c) Every $\mathbb{Z}_{(\ell)}$-basis of $\mathscr{M}_{(\ell)}$ is ℓ-adically independent in M.

(2) Let $N \subset \mathscr{M}_{(\ell)} \subseteq M$ be $\mathbb{Z}_{(\ell)}$-submodules of M such that N and M/N are ℓ-adically complete and torsion-free. We call $\mathscr{M}_{(\ell)}$ an N-lattice in M, if $\mathscr{M}_{(\ell)}/N$ is a lattice in M/N.

(3) In the context above, a true lattice in M is a free abelian subgroup $\mathscr{M}$ of M such that $\mathscr{M}_{(\ell)} := \mathscr{M} \otimes \mathbb{Z}_{(\ell)}$ is a lattice in M in the above sense. And we will say that a $\mathbb{Z}$-submodule $\mathscr{M} \subseteq M$ is a true N-lattice in M if $N \subset \mathscr{M}$ and $\mathscr{M}/N$ is a true lattice in M/N.

(4) Let M be an arbitrary $\mathbb{Z}_\ell$-module. We say that subsets M_1, M_2 of M are ℓ-adically equivalent if there exists an ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell$ such that $M_2 = \varepsilon \cdot M_1$ inside M. Further, given systems $S_1 = (x_i)_i$ and $S_2 = (y_i)_i$ of elements of M, we will say

that S_1 and S_2 are ℓ-adically equivalent if there exists an ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell$ such that $x_i = \varepsilon\, y_i$ (all i).

(5) We define correspondingly the ℓ-adic N-equivalence of N-lattices, etc.

Construction 5 Let $\mathscr{G}$ be an abstract decomposition graph on G which is *level-δ complete curve-like* and *ample up to level* δ for some given $\delta > 0$. Recall the last exact sequence from point (4) from Definition/Remark 3:

$$0 \to \widehat{U}_{\mathscr{G}} \hookrightarrow \widehat{\Lambda}_{\mathscr{G}} \xrightarrow{j^{\mathscr{G}}} \widehat{\mathrm{Div}}_{\mathscr{G}} \xrightarrow{\mathrm{can}} \widehat{\mathfrak{Cl}}_{\mathscr{G}} \to 0.$$

The aim of this subsection is to describe the ℓ-adic equivalence class of a lattice $\mathrm{Div}_{\mathscr{G}}$ in $\widehat{\mathrm{Div}}_{\mathscr{G}}$, in case it exists, which will be called an abstract divisor group of $\mathscr{G}$. In case the lattice $\mathrm{Div}_{\mathscr{G}} \subset \widehat{\mathrm{Div}}_{\mathscr{G}}$ exists, it satisfies

$$\mathrm{Div}_{\mathscr{G}} \otimes \mathbb{Z}_\ell = \oplus_v \Lambda_{T_v}.$$

Further, the existence (of the equivalence class) of the lattice $\mathrm{Div}_{\mathscr{G}}$ will turn out to be equivalent to the existence (of the equivalence class) of a $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$, which will turn out to be the preimage of $\mathrm{Div}_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$. In particular, if $\Lambda_{\mathscr{G}}$ exists, it satisfies

$$\Lambda_{\mathscr{G}} \otimes \mathbb{Z}_\ell = \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}.$$

The case $\delta = 1$, i.e., $\mathscr{G}$ is complete curve-like and ample.

In the notation from Definition/Remark 3 (5) above, let $\mathfrak{T} = (\tau_v)_v$ be a distinguished system of inertia generators. Further, let $\mathscr{F}_{\mathfrak{T}}$ be the abelian pro-ℓ free group on the system $\mathfrak{T}$ (written multiplicatively). Then one has a canonical exact sequence of pro-ℓ groups

$$1 \to \tau^{\mathbb{Z}_\ell} \to \mathscr{F}_{\mathfrak{T}} \to T \to 1,$$

where $\tau = \prod_v \tau_v$ in $\mathscr{F}_{\mathfrak{T}}$ is the pro-ℓ product of the generators τ_v (all v). Observing that $\mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) \cong \widehat{\mathrm{Div}}_{\mathscr{G}}$ in a canonical way, and taking ℓ-adically continuous Homs, we get an exact sequence

$$0 \to \widehat{\Lambda}_T = \mathrm{Hom}\big(T, \mathbb{Z}_\ell\big) \to \widehat{\mathrm{Div}}_{\mathscr{G}} = \mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) \to \mathbb{Z}_\ell = \mathrm{Hom}\big(\tau^{\mathbb{Z}_\ell}, \mathbb{Z}_\ell\big) \to 0,$$

where the last homomorphism maps each φ to its "trace": $\varphi \mapsto \big(\tau \mapsto \sum_v \varphi(\tau_v)\big)$. Thus $\widehat{\Lambda}_T$ consists of all the homomorphisms $\varphi \in \mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big)$ with trivial trace.

Consider the system $\mathfrak{B} = (\varphi_v)_v$ of all the functionals $\varphi_v \in \mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) = \widehat{\mathrm{Div}}_{\mathscr{G}}$ defined by $\varphi_v(\tau_w) = 1$ if $v = w$, and $\varphi_v(\tau_w) = 0$ for all $v \neq w$. We denote by

$$\mathrm{Div}_{\mathfrak{T}} = \langle \mathfrak{B} \rangle_{(\ell)} \subset \widehat{\mathrm{Div}}_{\mathscr{G}}$$

the $\mathbb{Z}_{(\ell)}$-submodule of $\mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) = \widehat{\mathrm{Div}}_{\mathscr{G}}$ generated by $\mathfrak{B}$. Then $\mathrm{Div}_{\mathfrak{T}}$ is a lattice in $\widehat{\mathrm{Div}}_{\mathscr{G}}$, and $\mathfrak{B}$ is an *ℓ-adic basis* of $\widehat{\mathrm{Div}}_{\mathscr{G}}$. We next set

$$\mathrm{Div}^0_{\mathfrak{T}} := \big\{ \textstyle\sum_v a_v \varphi_v \in \mathrm{Div}_{\mathfrak{T}} \mid \sum_v a_v = 0 \big\} = \mathrm{Div}_{\mathfrak{T}} \cap \widehat{\Lambda}_T.$$

Clearly, $\mathrm{Div}^0_{\mathfrak{T}}$ is a lattice in $\widehat{\Lambda}_T$. And moreover, the system $(e_w = \varphi_w - \varphi_v)_{w\neq v}$ is an ℓ-adic $\mathbb{Z}_{(\ell)}$-basis of $\mathrm{Div}^0_{\mathfrak{T}}$ for every fixed v.

The dependence of $\mathrm{Div}_{\mathfrak{T}}$ on $\mathfrak{T} = (\tau_v)_v$ is as follows. Let $\mathfrak{T}' = (\tau'_v)_v = \mathfrak{T}^\varepsilon$ with $\varepsilon \in \mathbb{Z}_\ell^\times$ be another distinguished system of inertia generators. If $\mathfrak{B}' = (\varphi'_v)_v$ is the dual basis to $\mathfrak{T}'$, then $\varepsilon \cdot \mathfrak{B}' = \mathfrak{B}$. Thus $\mathfrak{B}$ and $\mathfrak{B}'$ are ℓ-adically equivalent, and we have $\mathrm{Div}_{\mathfrak{T}} = \varepsilon \cdot \mathrm{Div}_{\mathfrak{T}'}$ and $\mathrm{Div}^0_{\mathfrak{T}} = \varepsilon \cdot \mathrm{Div}^0_{\mathfrak{T}'}$.

Therefore, all the subgroups of $\widehat{\mathrm{Div}}_{\mathscr{G}}$ of the form $\mathrm{Div}_{\mathfrak{T}}$, respectively $\mathrm{Div}^0_{\mathfrak{T}}$, are ℓ-adically equivalent (for all distinguished $\mathfrak{T}$). Hence the ℓ-adic equivalence classes of $\mathrm{Div}_{\mathfrak{T}}$ and $\mathrm{Div}^0_{\mathfrak{T}}$ do not depend on $\mathfrak{T}$, but only on $\mathscr{G}$.

Fact 6 *In the above context, denote by $\Lambda_{\mathfrak{T}}$ the preimage of $\mathrm{Div}^0_{\mathfrak{T}}$, thus of $\mathrm{Div}_{\mathfrak{T}}$, in $\widehat{\Lambda}_{\mathscr{G}}$. Consider all the finite-corank submodules $\Delta \subset \Lambda_{\mathscr{G},\mathrm{fin}}$ with $\widehat{U}_{\mathscr{G}} \subset \Delta$. Then the following hold:*

(i) *$\Lambda_{\mathfrak{T}}$ is a $\widehat{U}_{\mathscr{G}}$-lattice in $\widehat{\Lambda}_{\mathscr{G}}$, and $\Lambda_{\mathfrak{T}} \subset \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$.*

(ii) *$\Delta \cap \Lambda_{\mathfrak{T}}$ is a $\widehat{U}_{\mathscr{G}}$-lattice in Δ (all Δ as above).*

Moreover, $j^v(\Lambda_{\mathfrak{T}}) = \mathbb{Z}_{(\ell)}\varphi_v$ (all v).

Proof. Clear. □

Definition 7 In the context of Fact 6 above, we define objects as follows:

(1) A lattice of the form $\mathrm{Div}_{\mathfrak{T}} \subset \widehat{\mathrm{Div}}_{\mathscr{G}}$ will be called an abstract divisor group of $\mathscr{G}$. We will further say that $\mathrm{Div}^0_{\mathfrak{T}}$ is the abstract divisor group of degree 0 in $\mathrm{Div}_{\mathfrak{T}}$.

(2) The $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathfrak{T}}$ is called a divisorial $\widehat{U}_{\mathscr{G}}$-lattice for $\mathscr{G}$ in $\widehat{\Lambda}_{\mathscr{G}}$. And we will say that $\Lambda_{\mathfrak{T}}$ and $\mathrm{Div}_{\mathfrak{T}}$ correspond to each other, and that $\mathfrak{T}$ defines them.

• Note that $\Lambda_{\mathscr{G}} \subset \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$ and $\Lambda_{\mathscr{G}} \otimes \mathbb{Z}_\ell = \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$. Indeed, if $x \in \Lambda_{\mathscr{G}}$, then $j^v(x) = 0$ for almost all v, etc.

The case $\delta > 1$.

We begin by mimicking the construction from the case $\delta = 1$, and then conclude the construction by induction on δ. Thus let $\mathfrak{T} = (\tau_v)_v$ be any system of generators for the inertia groups T_v (all 1-edges v). Further let $\mathscr{F}_{\mathfrak{T}}$ be the abelian pro-ℓ free group on the system $\mathfrak{T}$ (written multiplicatively). Then T is a quotient $\mathscr{F}_{\mathfrak{T}} \to T \to 1$ in a canonical way. Observing that $\mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) \cong \widehat{\mathrm{Div}}_{\mathscr{G}}$ in a canonical way, by taking ℓ-adic Homs we get an exact sequence

$$0 \to \mathrm{Hom}\big(T, \mathbb{Z}_\ell\big) \to \mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big) = \widehat{\mathrm{Div}}_{\mathscr{G}}.$$

Next let $\mathfrak{B} = (\varphi_v)_v$ be the system of all the functionals $\varphi_v \in \mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big)$ defined by $\varphi_v(\tau_w) = 1$ if $v = w$, and $\varphi_v(\tau_w) = 0$ for all $v \neq w$. We denote by

$$\mathrm{Div}_{\mathfrak{T}} = \langle \mathfrak{B} \rangle_{(\ell)} \subset \mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big)$$

the $\mathbb{Z}_{(\ell)}$-submodule generated by $\mathfrak{B}$. Then $\mathfrak{B}$ *is an ℓ-adic basis* of $\mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big)$, i.e., $\mathrm{Div}_{\mathfrak{T}}$ is ℓ-adically dense in $\widehat{\mathrm{Div}}_{\mathscr{G}} = \mathrm{Hom}\big(\mathscr{F}_{\mathfrak{T}}, \mathbb{Z}_\ell\big)$, and there are no non-trivial

ℓ-adic relations between the elements of $\mathfrak{B}$. We will call $\mathfrak{B} = (\varphi_v)_v$ the "dual basis" to $\mathfrak{T}$, and remark that $\mathrm{Div}_{\mathfrak{T}}$ is a lattice in $\mathrm{Hom}\big(T, \mathbb{Z}_\ell\big)$.

Finally, let $\mathfrak{T}' = (\tau'_v)_v$ be another system of inertia generators, and suppose that $\mathfrak{T}' = \mathfrak{T}^\varepsilon$ for some $\varepsilon \in \mathbb{Z}_\ell$. If $\mathfrak{B}' = (\varphi'_v)_v$ is the dual basis to $\mathfrak{T}'$, then $\varepsilon\,\varphi'_v = \varphi_v$ inside $\mathrm{Hom}\big(T, \mathbb{Z}_\ell\big)$. Thus $\varepsilon \cdot \mathfrak{B}' = \mathfrak{B}$. In other words, $\mathfrak{B}$ and $\mathfrak{B}'$ are ℓ-adically equivalent, and we have $\mathrm{Div}_{\mathfrak{T}} = \varepsilon \cdot \mathrm{Div}_{\mathfrak{T}'}$.

Fact 8 *In the notations from above let a $\widehat{U}_{\mathscr{G}_v}$-lattice $\Lambda_{\mathscr{G}_v} \subset \widehat{\Lambda}_{\mathscr{G}_v}$ with $\widehat{U}_{\mathscr{G}_v} \subset \Lambda_{\mathscr{G}_v}$ be given for every valuation v of $\mathscr{G}$. Then the following hold:*

(1) *Up to ℓ-adic equivalence, there exists at most one $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$ such that first, $\widehat{U}_{\mathscr{G}} \subset \Lambda_{\mathscr{G}} \subset \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$, and second, for every finite-corank submodule Δ of $\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$ with $\widehat{U}_{\mathscr{G}} \subset \Delta$ and the corresponding $\Delta_v := j_v(\Delta \cap \widehat{U}_v) + \widehat{U}_{\mathscr{G}_v} \subset \widehat{\Lambda}_{\mathscr{G}_v,\mathrm{fin}}$ the following hold:*

   (i) *$\Lambda_\Delta := \Delta \cap \Lambda_{\mathscr{G}}$ is a $\widehat{U}_{\mathscr{G}}$-lattice in Δ.*

   (ii) *$j_v(\Lambda_\Delta \cap \widehat{U}_v) + \widehat{U}_{\mathscr{G}_v}$ is a $\widehat{U}_{\mathscr{G}_v}$-lattice in Δ_v, which is ℓ-adically $\widehat{U}_{\mathscr{G}_v}$-equivalent to $\Lambda_{\mathscr{G}_v} \cap \Delta_v$.*

   Moreover, if the $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathscr{G}}$ exists, then its ℓ-adic equivalence class depends only on the ℓ-adic equivalence classes of the $\widehat{U}_{\mathscr{G}_v}$-lattices $\Lambda_{\mathscr{G}_v}$ (all v).

(2) *In the above context, suppose that $\mathscr{G}$ is ample, and that the $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathscr{G}}$ satisfying the conditions* (i), (ii)*, exists. Then $\widehat{U}_{\mathscr{G}_v} + j_v(\Lambda_{\mathscr{G}} \cap \widehat{U}_v)$ is a $\widehat{U}_{\mathscr{G}_v}$-lattice, which moreover is ℓ-adically $\widehat{U}_{\mathscr{G}_v}$-equivalent to $\Lambda_{\mathscr{G}_v}$ (all v).*

Proof. To (1): Let $\Lambda_{\mathscr{G}}, \Lambda'_{\mathscr{G}}$ be $\widehat{U}_{\mathscr{G}}$-lattices in $\widehat{\Lambda}_{\mathscr{G}}$ satisfying the conditions from (1) above. Let $\Delta \in \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$ be have finite non-zero corank, and satisfy $\widehat{U}_{\mathscr{G}} \subset \Delta$. By the ampleness of $\mathscr{G}$, it follows that there exists v such that, first, $\Delta \subseteq \widehat{U}_v$, and second, Δ and $\Delta_v := j_v(\Delta) + \widehat{U}_{\mathscr{G}_v}$ have equal coranks. Therefore, j_v defines an isomorphism of $\Delta/\widehat{U}_{\mathscr{G}}$ onto $\Delta_v/\widehat{U}_{\mathscr{G}_v}$, and one has

$$(*) \qquad \ker(j_v) \cap \Delta \subseteq \widehat{U}_{\mathscr{G}}, \qquad j_v(\Delta) \cap \widehat{U}_{\mathscr{G}_v} \subseteq j_v(\widehat{U}_{\mathscr{G}}).$$

For Δ as above, set $\Lambda'_\Delta = \Delta \cap \Lambda'_{\mathscr{G}}$. Then by hypothesis (i), it follows that Λ_Δ and Λ'_Δ are both $\widehat{U}_{\mathscr{G}}$-lattices in Δ. Further, by hypothesis (ii), both $\Lambda_{\Delta_v} := \widehat{U}_{\mathscr{G}_v} + j_v(\Lambda_\Delta)$ and $\Lambda'_{\Delta_v} := \widehat{U}_{\mathscr{G}_v} + j_v(\Lambda'_\Delta)$ are $\widehat{U}_{\mathscr{G}_v}$ lattices in Δ_v, which are both equivalent to the $\widehat{U}_{\mathscr{G}_v}$-lattice $\Lambda_{\mathscr{G}_v} \cap \Delta_v$. Therefore, there exists $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\Lambda'_{\Delta_v} = \varepsilon \cdot \Lambda_{\Delta_v}$.

Claim. $\Lambda'_\Delta = \varepsilon \cdot \Lambda_\Delta$.

Indeed, $\Lambda'_{\Delta_v} = \varepsilon \cdot \Lambda_{\Delta_v}$ implies that $j_v(\Lambda'_\Delta) \subseteq \varepsilon \cdot j_v(\Lambda_\Delta) + \widehat{U}_{\mathscr{G}_v}$. Hence for every $e' \in \Lambda'_\Delta$ there exist $e \in \Lambda_\Delta$ and $u_v \in \widehat{U}_{\mathscr{G}_v}$ such that $j_v(e') = \varepsilon\, j_v(e) + u_v$. Therefore we have $u_v = j_v(e' - \varepsilon\, e) \in j(\Delta)$, and hence $u_v \in j_v(\Delta) \cap \widehat{U}_{\mathscr{G}_v}$. Hence by assertion $(*)$ above, there exists $u \in \widehat{U}_{\mathscr{G}}$ such that $j_v(u) = u_v$; thus $j_v(u) = j_v(e' - \varepsilon\, e)$. But then we have $e' - (\varepsilon\, e + u) \in \ker(j_v) \cap \Delta$, thus $e' - (\varepsilon\, e + u) \in \widehat{U}_{\mathscr{G}}$ by assertion $(*)$. We conclude that $e' \in \varepsilon\, e + \widehat{U}_{\mathscr{G}}$. Since $e' \in \Lambda'_\Delta$ was arbitrary, we have $\Lambda'_\Delta \subseteq \varepsilon \cdot \Lambda_\Delta + \widehat{U}_{\mathscr{G}}$. On the other hand, by hypothesis we have $\widehat{U}_{\mathscr{G}} \subset \Lambda_\Delta$ and $\widehat{U}_{\mathscr{G}} \subset \Lambda'_\Delta$. Hence the above inclusion is actually equivalent to $\Lambda'_\Delta \subseteq \varepsilon \cdot \Lambda_\Delta$. By symmetry, the other inclusion also holds, and we finally get $\Lambda'_\Delta = \varepsilon \cdot \Lambda_\Delta$.

We also observe that ε is unique up to multiplication by rational ℓ-adic units, because $\Lambda'_\Delta/\widehat{U}_{\mathscr{G}} = \varepsilon \cdot \Lambda'_\Delta/\widehat{U}_{\mathscr{G}}$ are ℓ-adically equivalent lattices in the non-trivial $\mathbb{Z}_\ell$-module $\Delta/\widehat{U}_{\mathscr{G}}$. Hence recalling that $\Lambda_{\mathscr{G}} = \cup_\Delta \Lambda_\Delta$ and $\Lambda'_{\mathscr{G}} = \cup_\Delta \Lambda'_\Delta$, and taking into account the uniqueness of ε, one immediately gets that $\Lambda'_{\mathscr{G}} = \varepsilon \cdot \Lambda'_{\mathscr{G}}$, as claimed.

To (2): First, since $\Lambda_{\mathscr{G}} = \cup_\Delta \Lambda_\Delta$ as mentioned above, it follows from hypotheses (i), (ii), that $\widehat{U}_{\mathscr{G}_v} + j_v(\Lambda_{\mathscr{G}} \cap \widehat{U}_v)$ is ℓ-adically equivalent to some $\widehat{U}_{\mathscr{G}_v}$-sublattice of $\Lambda_{\mathscr{G}_v}$, as this is the case for all the $\widehat{U}_{\mathscr{G}_v} + j_v(\Lambda_\Delta \cap \widehat{U}_v)$. After replacing $\Lambda_{\mathscr{G}_v}$ by some properly chosen ℓ-adic multiple, say $\varepsilon \cdot \Lambda_{\mathscr{G}_v}$ with $\varepsilon \in \mathbb{Z}_\ell^\times$, without loss of generality, we can suppose that $j_v(\Lambda_{\mathscr{G}} \cap \widehat{U}_v) \subseteq \Lambda_{\mathscr{G}_v}$, and thus $\widehat{U}_{\mathscr{G}_v} + j_v(\Lambda_{\mathscr{G}} \cap \widehat{U}_v) \subseteq \Lambda_{\mathscr{G}_v}$. For the converse inclusion, let $\Gamma \subseteq \widehat{\Lambda}_{\mathscr{G}_v}$ be a finite-corank submodule. Then by the ampleness of $\mathscr{G}$, see Definition/Remark 3 (7) (ii), there exists a finite-corank submodule $\Delta \subseteq \widehat{\Lambda}_{\mathscr{G}}$ such that $\Gamma \subseteq \widehat{U}_{\mathscr{G}_v} + j_v(\Delta \cap \widehat{U}_v)$. But then by properties (i), (ii), we get $\Gamma \cap \Lambda_{\mathscr{G}_v} \subseteq \widehat{U}_{\mathscr{G}_v} + j_v(\Lambda_\Delta \cap \widehat{U}_v) \subseteq \widehat{U}_{\mathscr{G}_v} + j_v(\Lambda_{\mathscr{G}} \cap \widehat{U}_v)$. Since Γ was arbitrary and $\Lambda_{\mathscr{G}_v} = \widehat{U}_{\mathscr{G}_v} + \cup_\Gamma(\Gamma \cap \Lambda_{\mathscr{G}_v})$, the converse inclusion follows. □

Let $\mathscr{G}$ be an abstract decomposition graph which is both level-δ complete curve-like and ample up to level δ for some $\delta > 1$. In particular, all residual abstract decomposition graphs $\mathscr{G}_\mathfrak{v}$ to non-trivial indices $\mathfrak{v}$ of length $\delta_\mathfrak{v} < \delta$ are both *level-*$(\delta - \delta_\mathfrak{v})$ *complete curve-like and ample up to level* $(\delta - \delta_\mathfrak{v})$; and if $\delta_\mathfrak{v} = \delta - 1$, then $\mathscr{G}_\mathfrak{v}$ is complete curve-like and ample. Hence if $\delta_\mathfrak{v} = \delta - 1$, then $\mathscr{G}_\mathfrak{v}$ has an abstract divisor group $\mathrm{Div}_{\mathscr{G}_\mathfrak{v}}$ as defined/introduced in Definition 7. In the above context, let us fix notation as follows:

Definition 9 In the above context, we define an abstract divisor group of $\mathscr{G}$ (if it exists) to be the lattice defined by any system $\mathfrak{T}$ of inertia generators as above,

$$\mathrm{Div}_{\mathscr{G}} := \mathrm{Div}_{\mathfrak{T}} \subset \widehat{\mathrm{Div}}_{\mathscr{G}},$$

which together with its preimage $\Lambda_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$ satisfies inductively on δ the following:

(i) Abstract divisor groups $\mathrm{Div}_{\mathscr{G}_v}$ exist for all residual abstract decomposition graphs $\mathscr{G}_v$. Let $\Lambda_{\mathscr{G}_v}$ be the preimage of $\mathrm{Div}_{\mathscr{G}_v}$ in $\widehat{\Lambda}_{\mathscr{G}_v}$ (all v).

(ii) $\Lambda_{\mathscr{G}}$ satisfies conditions (i), (ii) from Fact 8 for all finite corank submodules $\Delta \subset \widehat{\Lambda}_{\mathscr{G}}$ with respect to the preimages $\Lambda_{\mathscr{G}_v}$ defined at (i) above.

- Note that if $\Lambda_{\mathscr{G}}$ exists, then $\Lambda_{\mathscr{G}} \subset \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$ and $\Lambda_{\mathscr{G}} \otimes \mathbb{Z}_\ell = \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$. Indeed, if $x \in \Lambda_{\mathscr{G}}$, then $j^v(x) = 0$ for almost all v, etc.

Remarks 10 Let $\mathscr{G}$ be an abstract decomposition graph which is level-δ complete curve-like and ample up to level δ for some $\delta > 0$. Suppose that an abstract divisor group $\mathrm{Div}_{\mathscr{G}} := \mathrm{Div}_{\mathfrak{T}}$ for $\mathscr{G}$ exists, and let $\Lambda_{\mathscr{G}}$ be its preimage in $\widehat{\Lambda}_{\mathscr{G}}$. Then one has:

(1) The homomorphism $j^v : \widehat{\Lambda}_{\mathscr{G}} = \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big) \xrightarrow{\mathrm{res}_v} \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big) = \mathbb{Z}_\ell \varphi_v$ gives rise by restriction to a surjective homomorphism

$$j^v : \Lambda_{\mathscr{G}} \to \mathbb{Z}_{(\ell)}\, \varphi_v.$$

Indeed, by condition (i) of the ampleness, see Definition/Remark 3 (7), it follows that $j^v(\widehat{\Lambda}_{\mathscr{G}}) = \mathbb{Z}_\ell\,\varphi_v$. Further, since $\Lambda_{\mathscr{G}}$ is ℓ-adically dense in $\widehat{\Lambda}_{\mathscr{G}}$, it follows that $j^v(\Lambda_{\mathscr{G}})$ is dense in $\mathbb{Z}_\ell\,\varphi_v$. Thus the assertion.

(2) Moreover, the $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathscr{G}}$ endowed with all the homomorphisms j^v determines $\mathrm{Div}_{\mathscr{G}}$ as the additive subgroup

$$\mathrm{Div}_{\mathscr{G}} = \textstyle\sum_v \mathbb{Z}_{(\ell)}\varphi_v = \sum_v j^v(\Lambda_{\mathscr{G}}) \subset \widehat{\mathrm{Div}}_{\mathscr{G}}$$

generated by the $j^v(\Lambda_{\mathscr{G}})$ for all the v. Therefore, giving an abstract divisor group $\mathrm{Div}_{\mathscr{G}}$ is *equivalent* to giving a $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$ such that inductively we have:

(i) $\Lambda_{\mathscr{G}}$ satisfies conditions (i), (ii) from Fact 8 with respect to the preimages $\Lambda_{\mathscr{G}_v}$ of some abstract divisor groups $\mathrm{Div}_{\mathscr{G}_v}$ (all v).

(ii) $j^v(\Lambda_{\mathscr{G}}) \cong \mathbb{Z}_{(\ell)}$ (all v), and $\Lambda_{\mathscr{G}}$ is the preimage of $\oplus_v j^v(\Lambda_{\mathscr{G}})$ via $j^{\mathscr{G}}$.

(3) Finally, for an abstract divisor group $\mathrm{Div}_{\mathscr{G}}$ for $\mathscr{G}$ and its preimage $\Lambda_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$, we set $\mathfrak{Cl}_{\Lambda_{\mathscr{G}}} = \mathrm{Div}_{\mathscr{G}}/j^{\mathscr{G}}(\Lambda_{\mathscr{G}})$ and call it the abstract ideal class group of $\Lambda_{\mathscr{G}}$. Thus one has a commutative diagram of the form

$$(*) \qquad \begin{array}{ccccccccc} 0 \to & \widehat{U}_{\mathscr{G}} & \hookrightarrow & \Lambda_{\mathscr{G}} & \xrightarrow{j^{\mathscr{G}}} & \mathrm{Div}_{\mathscr{G}} & \xrightarrow{\text{can}} & \mathfrak{Cl}_{\mathscr{G}} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \widehat{U}_{\mathscr{G}} & \hookrightarrow & \widehat{\Lambda}_{\mathscr{G}} & \xrightarrow{j^{\mathscr{G}}} & \widehat{\mathrm{Div}}_{\mathscr{G}} & \xrightarrow{\text{can}} & \widehat{\mathfrak{Cl}}_{\mathscr{G}} & \to 0 \end{array}$$

where the first three vertical morphisms are the canonical inclusions, and the last one is the ℓ-adic completion homomorphism.

Proposition 11 *Let $\mathscr{G}$ be an abstract decomposition graph which is level-δ complete curve-like and ample up to level $\delta > 0$. Then any two abstract divisor groups* $\mathrm{Div}_{\mathscr{G}}$ *and* $\mathrm{Div}'_{\mathscr{G}}$ *for $\mathscr{G}$ are ℓ-adically equivalent as lattices in* $\widehat{\mathrm{Div}}_{\mathscr{G}}$. *Equivalently, their preimages $\Lambda_{\mathscr{G}}$ and $\Lambda'_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$ are ℓ-adically equivalent $\widehat{U}_{\mathscr{G}}$-lattices in $\widehat{\Lambda}_{\mathscr{G}}$. In particular, there exist distinguished systems of inertia generators $\mathfrak{T}$ and $\mathfrak{T}'$ defining* $\mathrm{Div}_{\mathscr{G}}$, *respectively* $\mathrm{Div}'_{\mathscr{G}}$, *which are ℓ-adically equivalent, i.e., $\mathfrak{T}' = \mathfrak{T}^\varepsilon$ for some ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$.*

Proof. We prove this assertion by induction on δ. For $\delta = 1$, the uniqueness is already shown, see Fact 6, and Definition 7 in case $\delta = 1$. Now suppose that $\delta > 1$. Let $\mathrm{Div}_{\mathscr{G}_v}$ and $\mathrm{Div}'_{\mathscr{G}_v}$ be abstract divisor groups for $\mathscr{G}$ used for the definition of $\mathrm{Div}_{\mathscr{G}}$, respectively $\mathrm{Div}'_{\mathscr{G}}$ (all v). By the induction hypothesis, $\mathrm{Div}_{\mathscr{G}_v}$ and $\mathrm{Div}'_{\mathscr{G}_v}$ are ℓ-adically equivalent. Thus their preimages Λ_v and Λ'_v in $\widehat{\Lambda}_{\mathscr{G},v}$ are ℓ-adically equivalent $\widehat{U}_{\mathscr{G},v}$-lattices. Therefore, by Fact 8, the lattices $\Lambda_{\mathscr{G}}$ and $\Lambda'_{\mathscr{G}}$ (which are the preimages of $\mathrm{Div}_{\mathscr{G}}$ respectively $\mathrm{Div}'_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$) are ℓ-adically equivalent. Finally, use Remark 10 (2), above to conclude. □

Definition 12 Let $\mathscr{G}$ be an abstract decomposition graph which is level-δ complete curve-like and ample up to level δ. We will say that $\mathscr{G}$ is a divisorial abstract decomposition graph if it has abstract divisor groups $\mathrm{Div}_{\mathscr{G}} = \mathrm{Div}_{\mathfrak{T}}$ as introduced above. If this is the case, we will denote by $\Lambda_{\mathscr{G}}$ the preimage of $\mathrm{Div}_{\mathscr{G}}$ in $\widehat{\Lambda}_{\mathscr{G}}$, and call it a divisorial $\widehat{U}_{\mathscr{G}}$-lattice in $\widehat{\Lambda}_{\mathscr{G}}$.

3 Abstract decomposition graphs arising from algebraic geometry

3.1 Some general valuation-theoretical nonsense

Let K be an arbitrary field. The space of all the equivalence classes of valuations Val_K of K is in a canonical way a partially ordered set by $v \leq w$ iff $\mathscr{O}_w \subseteq \mathscr{O}_v$ iff $\mathfrak{m}_v \subseteq \mathfrak{m}_w$, and if so, then $\mathfrak{m}_v \subset \mathscr{O}_w$ is a prime ideal of $\mathscr{O}_w$, and $\mathscr{O}_v$ is the localization $\mathscr{O}_v = (\mathscr{O}_w)_{\mathfrak{m}_v}$. The unique minimal element of Val_K is the trivial valuation v_0 which has $\mathscr{O}_{v_0} = K$ as valuation ring. Further, the minimal *non-trivial* elements of Val_K are exactly the rank one valuation rings of K (which then correspond to the equivalence classes of non-archimedean absolute values of K). Note that if $v \leq w$, then $\mathscr{O}_w/\mathfrak{m}_v$ is a valuation ring in the residue field Kv of v. We denote the corresponding valuation of Kw by w/v, and call it the quotient of w by v. Conversely, given $v \in \mathrm{Val}_K$ and a valuation $\overline{w}$ of Kv, the preimage $\mathscr{O}$ of $\mathscr{O}_{\overline{w}}$ under $\mathscr{O}_v \to Kv$ is a valuation ring of a valuation $w \geq v$ such that $w/v = \overline{w}$. We define $\overline{w} \circ v := w$, and call it the composition of $\overline{w}$ and v. Val_K has in a canonical way the structure of a (half-oriented) graph with origin $K = Kv_0$ as follows:

(a) The vertices are the residue fields Kv indexed by $v \in \mathrm{Val}_K$.

(b) The set of edges from Kv to Kw is non-empty if and only if $v \leq w$ and $\mathrm{rank}(w/v) \leq 1$. If so, then w/v is the unique edge from Kv to Kw. We say that w/v is a non-trivial oriented edge if $\mathrm{rank}(w/v) = 1$, respectively we call w/v a trivial non-oriented edge if $v = w$, i.e., w/v is the trivial valuation of Kv.

We will call the graph defined above the valuation graph for K. There are two functorial constructions one should mention here:

(1) *Embeddings.* Let $\imath : L \hookrightarrow K$ be a field embedding and $\varphi_\imath : \mathrm{Val}_K \to \mathrm{Val}_L$, $v \mapsto v_L := v|_L$, the canonical restriction map. Then $\varphi_\imath$ is surjective and compatible with the ordering of valuations. And if $v \leq w$ in Val_K, then $v_L \leq w_L$ in Val_L, and $\mathrm{rank}(w_L/v_L) \leq \mathrm{rank}(w/v)$. Hence if the edge w/v from Kv to Kw exists, then the edge w_L/v_L from Lv_L to Lw_L exists too. Therefore, $\varphi_\imath$ defines a canonical projection from the valuation graph of K onto the valuation graph of L, under which Kv is mapped to Lv_L, and the edge w/v from Kv to Kw (if it exists) is mapped to the edge w_L/v_L from Lv_L to Lw_L. Note that if w/v is a non-trivial oriented edge such that $w_L = v_L$, then w/v is mapped to the trivial non-oriented edge of $Lw_L = Lv_L$.

(2) *Restrictions.* Let Kv be the residue field of v, and let $\mathrm{Val}_v = \{w \in \mathrm{Val}_K \mid w \geq v\}$ be the set of all refinements of v. Then $\mathrm{Val}_v \to \mathrm{Val}_{Kv}$, $w \mapsto w/v$, is a canonical bijection which respects the ordering, thus defines an isomorphism of the subgraph Val_v of the valuation graph for K onto the valuation graph Val_{Kv} for Kv.

- **The Galois decomposition theoretical side**

Let ℓ be a fixed prime number as above. For every field K which contains the ℓ^∞ roots of unity, let $K'|K$ be a maximal pro-ℓ abelian extension, and we denote by $\Pi_K = \mathrm{Gal}(K'|K)$ its Galois group. For $v \in \mathrm{Val}_K$ and prolongations v' of v to K', we have that the inertia/decomposition groups $T_{v'} \subseteq Z_{v'}$ of the several prolongations $v'|v$ are conjugated under Π_K; hence these groups are equal, as Π_K is commutative. We will denote them by $T_v \subseteq Z_v$, and call them the inertia/decomposition groups at v. Recall that $\Pi_{Kv} = Z_v/T_v$ canonically.

Via the Galois correspondence and using the functorial properties of Hilbert decomposition theory, we attach to Val_K a graph $\mathscr{G}_{\mathrm{Val}_K}$ which is in bijection with Val_K and has vertices and edges as follows: The vertices of $\mathscr{G}_{\mathrm{Val}_K}$ are indexed by the (distinct) pro-ℓ abelian groups Π_{Kv}. Concerning edges, if v/w is the unique edge from some Kw to some Kv (hence, either $w = v$ and v/w is the trivial valuation on $Kv = Kw$, or $v < w$ and $\mathrm{rank}(w/v) = 1$ on Kv), then the unique edge from Π_{Kv} to Π_{Kw} is the pair of groups $T_{w/v} \subseteq Z_{w/v}$ viewed as subgroups of Π_{Kv}. Note that in case w/v is the trivial valuation, we have merely by definition that $T_{w/v} = 1$ and $Z_{w/v} = \Pi_{Kw}$.

We will call $\mathscr{G}_{\mathrm{Val}_K}$ the valuation decomposition graph of K, or of Π_K.

Note that the above functorial constructions concerning embeddings and restrictions give rise functorially to corresponding functorial constructions on the Galois side as follows:

(1) *Embeddings.* Let $\imath : L \hookrightarrow K$ be an embedding of fields, and consider a prolongation $\imath' : L' \hookrightarrow K'$ of $\imath$. Then $\imath'$ gives rise to a projection $\Phi_\imath : \Pi_K \to \Pi_L$, which in turn gives rise canonically to a morphism of valuation decomposition graphs, which we denote by $\Phi_\imath$ again:

$$\Phi_\imath : \mathscr{G}_{\mathrm{Val}_K} \to \mathscr{G}_{\mathrm{Val}_L}.$$

Note that $\Phi_\imath$ maps the profinite group Π_{Kv} at the vertex Kv into the profinite group Π_{Lv_L} at the corresponding vertex Lv_L. And concerning edges, $\Phi_\imath$ maps $T_{w/v} \subseteq Z_{w/v}$ into the pair $T_{w_L/v_L} \subseteq Z_{w_L/v_L}$ of the corresponding inertia/decomposition subgroups of w_L/v_L in Π_{Lv_L}.

(2) *Restrictions.* For $w \in \mathrm{Val}_v$, one has $Z_w \subseteq Z_v$ and $T_v \subseteq T_w$. And under the canonical projection $Z_v \to \Pi_{Kv}$, every $T_w \subseteq Z_w$ is mapped onto $T_{w/v} \subseteq Z_{w/v}$ in Π_{Kv}, etc.

3.2 *Recovering the geometric decomposition graphs from the total decomposition graph*

Let $K|k$ be a function field as introduced/considered in the introduction. We notice that the total graph of prime divisors $\mathscr{D}_K^{\mathrm{tot}}$ of $K|k$, as defined in the introduction, is the subgraph of Val_K whose vertices are the generalized prime divisors of $K|k$ and whose non-trivial edges are of the form $\mathfrak{w}/\mathfrak{v}$ with $\mathfrak{w} > \mathfrak{v}$ generalized prime divisors. (If so, then $\mathfrak{w}/\mathfrak{v}$ is a prime divisor of $K\mathfrak{v}\,|\,k$.) We also recall that a subgraph $\mathscr{D}_K$ of $\mathscr{D}_K^{\mathrm{tot}}$ was called a geometric graph of prime divisors for $K|k$ if for every vertex $\mathfrak{v}$ of $\mathscr{D}_K$, the following hold: First, the trivial edge from $K\mathfrak{v}$ to itself is an edge of $\mathscr{D}_K$, and second, the set of non-trivial edges $D_\mathfrak{v}$ originating from $K\mathfrak{v}$ form a geometric set of prime divisors of $K\mathfrak{v}\,|\,k$.

Concerning the Galois theoretical side, the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ of $K|k$, or of Π_K, is the subgraph of the valuation decomposition graph $\mathcal{G}_{\mathrm{Val}_K}$ which is defined by the total prime divisors graph $\mathcal{D}_K^{\rm tot}$. And a geometric decomposition graph for $K|k$, or for Π_K, is any subgraph $\mathcal{G}_{\mathcal{D}_K}$ of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ which corresponds to a geometric graph $\mathcal{D}_K$ of prime divisors.

In this subsection we give a recipe to recover/describe the geometric decomposition graphs $\mathcal{G}_{\mathcal{D}_K}$ for $K|k$ inside the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ of $K|k$ using only the Galois theoretical information encoded in $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$.

We begin by recalling a criterion for the description of the geometric sets of prime divisors of a function field $K|k$, as presented in Pop [24], Subsection 2 D), an idea which was used in essence already in Pop [27].

Let $K|k$ be a function field over an algebraically closed base field as usual. For every normal model $X \to k$ of $K|k$, we denote by D_X the set of all the prime divisors of $K|k$ defined by the Weil prime divisors of X.

Fact 13 *For a set D of prime divisors of $K|k$, the following conditions are equivalent:*

(i) *For all normal models $X \to k$ of $K|k$ one has that D and D_X are almost equal. (Recall that two sets are almost equal if their symmetric difference is finite.)*

(ii) *D is geometric, i.e., there exists a quasi-projective normal model $X \to k$ of $K|k$ such that $D = D_X$.*

Recall that a line on a k-variety X is an integral k-subvariety of X, which is a curve of geometric genus equal to 0. We denote by $X^{\rm line}$ the union of all the lines on X.

We will say that a variety $X \to k$ is very unruly if the set $X^{\rm line}$ is not dense in X. In particular, a curve X is very unruly iff its geometric genus g_X is positive.

Further recall that being very unruly is a birational notion. In particular, it makes sense to say that a function field $K|k$ with $\mathrm{td}(K|k) = d > 0$ is very unruly if $K|k$ has models $X \to k$ which are very unruly.

Suppose that $d > 1$. We call a prime divisor v of $K|k$ very unruly if $Kv\,|\,k$ is very unruly as a function field over k. A prime divisor v of $K|k$ is very unruly iff there exist a normal model $X \to k$ of $K|k$ and a very unruly prime Weil divisor X_1 of X such that $v = v_{X_1}$.

The following is a more precise form of Proposition 2.6 from Pop [27], but see rather Pop [24], Section 3, for details:

Proposition 14 *With the usual notation, the following hold:*

(1) *A set D of prime divisors of $K|k$ is geometric iff there exists a finite ℓ-elementary subextension $K_0|K$ of $K'\,|K$ of degree ℓ^d such that for every ℓ-elementary subextension $K_1|K$ of $K'\,|K$ of degree ℓ^{2d} containing K_0, one has that D consists of almost all prime divisors v of $K|k$ whose prolongations $v_1|v$ to $K_1|L$ are very unruly prime divisors of K_1.*

(2) *Let $L|K$ be a finite subextension of $K'\,|K$. Then a set D_L of prime divisors of $L|k$ is geometric iff there exists a geometric set of prime divisors D of $K|k$ such that D_L is almost equal to the prolongation of D to L.*

Proof. The proof is more or less identical with the one from Pop [24] thus we refer the reader to that work for the details: Choose some transcendence basis $(t_1,\dots,t_d)$ of $K|k$ and a "sufficiently general" separable polynomial $p(T)\in k[T]$ of degree ≥ 3. For $m=1,\dots,d$, consider $u_m\in K'$ with $u_m^\ell = p(t_m)$. Then $K_0 = K[u_1,\dots,u_d]$ has degree ℓ^d over K, and it does the job; see [24] for details. □

Using the proposition above, one deduces the following inductive procedure on $d=\mathrm{td}(K|k)$ for deciding whether a given set D of prime divisors of $K|k$ is geometric, respectively whether a finite subextension $L|K$ of $K'|K$ viewed as a function field $L|k$ is very unruly.

Criterion 15 By induction on d, we consider the criteria $\mathscr{P}^{(d)}_{\mathrm{geom}}(D)$ for sets of prime divisors D of $K|k$ to be geometric sets of prime divisors, respectively $\mathscr{P}^{(d)}_{\mathrm{v.u.}}(L|K)$ for $L|K$ a finite subextension of $K'\,|K$ to be very unruly, as follows:

(1) Case $d=1$:

- $\mathscr{P}^{(1)}_{\mathrm{geom}}(D)$: D is almost equal to the set of all prime divisors of $K|k$.
- $\mathscr{P}^{(1)}_{\mathrm{v.u.}}(L|K)$: The genus of the complete normal model of $L|k$ satisfies $g_{L|k}>0$.

(2) Case $d>1$:

- $\mathscr{P}^{(d)}_{\mathrm{geom}}(D)$: With $K_0|K$ and $K_1|K$ as in Proposition 14, the set D is almost equal to the set of all prime divisors v of $K|k$ whose prolongations $v_1|v$ to $K_1|K$ satisfy $\mathscr{P}^{(d-1)}_{\mathrm{v.r}}(K_1v_1|Kv)$.
- $\mathscr{P}^{(d)}_{\mathrm{v.u.}}(L|K)$: There exists a set D of prime divisors of $K|k$ such that $\mathscr{P}^{(d)}_{\mathrm{geom}}(D)$ holds, and for almost all $v\in D$, the prolongations $w|v$ of v to $L|K$ satisfy $\mathscr{P}^{(d-1)}_{\mathrm{v.u.}}(Lw\,|Kv)$.

Remarks 16 (1) As mentioned in the introduction, if $\mathfrak{v}$ is a generalized prime divisor of $K|k$, then via the canonical projection $pr_{\mathfrak{v}}: Z_{\mathfrak{v}}\to \Pi_{K\mathfrak{v}}$, one can recover the total decomposition graph of $K\mathfrak{v}\,|\,k$ as follows: The generalized prime divisors of $K\mathfrak{v}\,|\,k$ are precisely the valuations of the from $\mathfrak{w}/\mathfrak{v}$ with $\mathfrak{w}$ a generalized prime divisor satisfying $\mathfrak{v}\leq\mathfrak{w}$. In turn, these are exactly the generalized prime divisors $\mathfrak{w}$ such that $T_{\mathfrak{v}}\subseteq T_{\mathfrak{w}}$, or equivalently $Z_{\mathfrak{w}}\subseteq Z_{\mathfrak{v}}$. If so, then $T_{\mathfrak{w}/\mathfrak{v}}\subseteq Z_{\mathfrak{w}/\mathfrak{v}}$ are the images of $T_{\mathfrak{w}}\subseteq Z_{\mathfrak{w}}$; thus the total decomposition graph of $K\mathfrak{v}\,|\,k$ can be recovered from the total decomposition graph of $K|k$ via $pr_{\mathfrak{v}}$.

(2) The finite subextensions $L|K$ of $K'|K$ are in bijection with all the open subgroups $\Delta\subseteq\Pi_K$. And if $\mathfrak{v}$ is a generalized prime divisor of $K|k$, and $\mathfrak{w}$ is a prolongation of $\mathfrak{v}$ to L, then under the canonical projection $\mathrm{pr}_{\mathfrak{v}}: Z_{\mathfrak{v}}\to\Pi_{K\mathfrak{v}}$ we have that if $L|K$ corresponds to $\Delta\subseteq\Pi_K$, then the finite residual subextension $L\mathfrak{w}\,|K\mathfrak{v}$ of $K\mathfrak{v}'|K\mathfrak{v}$ corresponds to the open subgroup $\Delta_v := \mathrm{pr}_{\mathfrak{v}}(Z_{\mathfrak{v}}\cap\Delta)$ of $\Pi_{K\mathfrak{v}}$.

(3) Let $\mathscr{G} \subset \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ be a connected full subgraph containing the origin Π_K of $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ and having all maximal oriented branches of length $d = \mathrm{td}(K|k)$. (Here "full" means that for all vertices $\Pi_{K\mathfrak{v}}$ and $\Pi_{K\mathfrak{w}}$ of $\mathscr{G}$ one has that if the edge $\mathfrak{w}/\mathfrak{v}$ from $K\mathfrak{v}$ to $K\mathfrak{w}$ exists, then this edge endowed with $T_{\mathfrak{w}/\mathfrak{v}} \subset Z_{\mathfrak{w}/\mathfrak{v}}$ is contained in $\mathscr{G}$.) In particular, the following hold:

(a) For for every vertex $\Pi_{K\mathfrak{v}}$ of $\mathscr{G}$, the trivial edge from $\Pi_{K\mathfrak{v}}$ to itself endowed with the inertia/decomposition group of the trivial valuation $\{1\} \subset \Pi_{K\mathfrak{v}}$ belongs to $\mathscr{G}$.

(b) If $\Pi_{K\mathfrak{v}}$ and $\Pi_{K\mathfrak{w}}$ belong to $\mathscr{G}$, and $\mathfrak{w}/\mathfrak{v}$ is a prime divisor of $K\mathfrak{v}$, then the edge $\mathfrak{w}/\mathfrak{v}$ endowed with $T_{\mathfrak{w}/\mathfrak{v}} \subseteq Z_{\mathfrak{w}/\mathfrak{v}}$ belongs to $\mathscr{G}$.

(c) All maximal branches of non-trivial edges have length $d := \mathrm{td}(K|k)$.

(4) Let $\mathscr{D} \subset \mathscr{D}_K^{\mathrm{tot}}$ be the (connected full) subgraph defined by $\mathscr{G}$. For every vertex $K\mathfrak{v}$ of $\mathscr{D}$, or equivalently a vertex $\Pi_{K\mathfrak{v}}$ of $\mathscr{G}$, let $D_\mathfrak{v}$ be the set of prime divisors v of $K\mathfrak{v}\,|\,k$ which are the non-trivial edges of $\mathscr{D}$ originating from $K\mathfrak{v}$. Then by the definitions one has the following:

$\mathscr{G}$ is a geometric decomposition graph iff $D_\mathfrak{v}$ is a geometric set of prime divisors of $K\mathfrak{v}\,|\,k$ for every vertex $K\mathfrak{v}$ of $\mathscr{D}$, and all maximal oriented branches of $\mathscr{G}$ have length $\mathrm{td}(K|k)$.

(5) We conclude that recovering/describing the geometric decomposition graphs inside $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ is equivalent to recovering/describing the geometric sets of prime divisors of the function fields $K\mathfrak{v}\,|\,k$ for all generalized prime divisors $\mathfrak{v}$.

We do this by showing that the geometric Criterion 15 can be recovered from, respectively interpreted in, the group-theoretical information encoded in $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$.

Gal-Criterion 17

By induction on $d_\mathfrak{v} := \mathrm{td}(K\mathfrak{v}\,|\,k)$, we consider criteria $\mathrm{Gal}\mathscr{P}^{(d)}_{\mathrm{geom}}(D)$ for sets of prime divisors D of $K\mathfrak{v}|k$ to be geometric sets of prime divisors, respectively $\mathrm{Gal}\mathscr{P}^{(d)}_{\mathrm{v.u.}}(L|K\mathfrak{v})$ for $L|K\mathfrak{v}$ finite subextensions of $(K\mathfrak{v})'\,|\,K\mathfrak{v}$ to be very unruly, as follows:

Case $d_\mathfrak{v} = 1$:

Then $K\mathfrak{v}\,|\,k$ is the function field of a complete smooth curve $X_\mathfrak{v} \to k$ with function field $\kappa(X_\mathfrak{v}) = K\mathfrak{v}$. And the set of all the non-trivial generalized prime divisors equals the set of prime divisors of $K\mathfrak{v}\,|\,k$, which is $D_{X_\mathfrak{v}}$. Let $(T_v)_v$ be the system of all the divisorial inertia groups in $\Pi_{K\mathfrak{v}}$ (which is part of the hypothesis, as $\Pi_{K\mathfrak{v}}$ comes endowed with the total decomposition graph of $K\mathfrak{v}\,|\,k$, hence encodes the set of all the T_v, $v \in D_{X_\mathfrak{v}}$), and let $T_{K\mathfrak{v}}$ be the closed subgroup of $\Pi_{K\mathfrak{v}}$ generated by all the T_v. Then $\Pi_{K\mathfrak{v}}/T_{K\mathfrak{v}} = \pi_1^{\ell,\mathrm{ab}}(X_\mathfrak{v})$ is the pro-ℓ abelian fundamental group of $X_\mathfrak{v}$. Since $\mathrm{char}(k) \neq \ell$, it follows that $\pi_1^{\ell,\mathrm{ab}}(X_\mathfrak{v}) \cong \mathbb{Z}_\ell^{2g_\mathfrak{v}}$, where $g_\mathfrak{v}$ is the genus of $X_\mathfrak{v}$. For every non-empty set $D \subset D_{X_\mathfrak{v}}$ of prime divisors of $K\mathfrak{v}\,|\,k$, let T_D be the closed subgroup of $\Pi_{K\mathfrak{v}}$ generated by T_v, $v \in D$. Then $\Pi_{K\mathfrak{v}}/T_D$ is a pro-ℓ abelian free group on $2g_{\mathfrak{v}}+r-1$ generators, where $r=|D_{X_{\mathfrak{v}}}\backslash D|$. Since D is geometric iff r is finite, we get that D is geometric iff $\Pi_{K\mathfrak{v}}/T_D$ is topologically finitely generated. Hence the geometric criterion $\mathscr{P}^{(1)}_{\mathrm{geom}}(D)$ is equivalent to:

- $\mathrm{Gal}\mathscr{P}^{(1)}_{\mathrm{geom}}(D)$: *$\Pi_{K\mathfrak{v}}/T_D$ is finitely generated.*

Let $L|K\mathfrak{v}$ be a finite subextension of $K\mathfrak{v}'\,|\,K\mathfrak{v}$ corresponding to $\Delta\subseteq\Pi_{K\mathfrak{v}}$ as above, and let $\pi_L:\Pi_{K\mathfrak{v}}\to\Pi_{K\mathfrak{v}}/\Delta=\mathrm{Gal}(L|K\mathfrak{v})$ be the corresponding finite quotient. For every prime divisor v of $K\mathfrak{v}\,|\,k$ and a prolongation $w|v$ of v to L, we have that the inertia group of $w|v$ in $G=\mathrm{Gal}(L|K\mathfrak{v})$ is precisely $T_w:=\pi_L(T_v)$; hence the ramification index of $w|v$ is $e_{L,v}:=|\pi_L(T_v)|$. Further, by the fundamental equality, the number $n_{L,v}$ of prolongations of v to L can be computed as $[L:K\mathfrak{v}]=n_{L,v}\cdot e_{L,v}$. Thus applying the Hurwitz genus formula, one has $2g_L-2=[L:K\mathfrak{v}](2g_{\mathfrak{v}}-2)+\sum_v n_{L,v}\big(e_{L,v}-1\big)$. And since $n_{L,v}=[L:K\mathfrak{v}]/e_{L,v}$, we see that $g_L>0$ iff either $g_{\mathfrak{v}}>0$ or $\sum_v(1-1/e_{L,v})\geq 2$. Therefore, taking into account that $e_{L,v}=|\pi_L(T_v)|$, and that $g_{L|k}=0$ iff $\Pi_{K\mathfrak{v}}=T_{K\mathfrak{v}}$, we get that the geometric criterion $\mathscr{P}^{(1)}_{\mathrm{v.r}}(D)$ is equivalent to:

- $\mathrm{Gal}\mathscr{P}^{(1)}_{\mathrm{v.r}}(L|K\mathfrak{v})$: *Either $\Pi_{K\mathfrak{v}}\neq T_{K\mathfrak{v}}$ or otherwise $\sum_v\big(1-1/|\pi_L(T_v)|\big)\geq 2$.*

Case $d_{\mathfrak{v}}>1$:

Recall that the total decomposition graph of $K\mathfrak{v}\,|\,k$ can be recovered from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ as mentioned above. Hence without loss of generality (and in order to simplify notions), we can suppose that $K\mathfrak{v}=K$. In particular, we will denote by v the prime divisors of $K|k$, and for finite subextensions $L|K$ of $K'|K$, we denote by $w|v$ the prolongations of v to L. Note that since $L|K$ is abelian, thus Galois, all the prolongations $w|v$ are conjugated under $\mathrm{Gal}(L|K)$, thus have the same ramification indices and residue function fields $Lw\,|\,k$, which are isomorphic over $Kv\,|\,k$.

Now let D be a set of prime divisors v of $K|k$. Then the geometric criterion $\mathscr{P}^{(d)}_{\mathrm{geom}}(D)$ is equivalent to the following:

- $\mathrm{Gal}\mathscr{P}^{(d)}_{\mathrm{geom}}(D)$: *With $K_0|K$ and $K_1|K$ as in Proposition 14, the set D is almost equal to the set of all prime divisors v of $K|k$ whose prolongations $v_1|v$ to $K_1|K$ satisfy* $\mathrm{Gal}\mathscr{P}^{(d-1)}_{\mathrm{v.r}}(K_1v_1|Kv)$.

Similarly, let $L|K$ be a finite subextension of $K'|K$. Then the geometric criterion $\mathscr{P}^{(d)}_{\mathrm{v.r}}(L|K)$ is equivalent to the following:

- $\mathrm{Gal}\mathscr{P}^{(d)}_{\mathrm{v.r}}(L|K)$: *There is a set D of prime divisors of $K|k$ satisfying* $\mathrm{Gal}\mathscr{P}^{(d)}_{\mathrm{geom}}(D)$ *such that for almost all prime divisors $v\in D$, all the prolongations $w|v$ of v to $L|K$ satisfy* $\mathrm{Gal}\mathscr{P}^{(d-1)}_{\mathrm{v.r}}(Lw\,|\,Kv)$.

This concludes the proof of the claim that the geometric sets of prime divisors of each $K\mathfrak{v}\,|\,k$ can be recovered from the total decomposition graph of $K\mathfrak{v}\,|\,k$, thus from that of $K|k$.

3.3 Geometric decomposition graphs as abstract decomposition graphs

Let $K|k$ be a function field over an algebraically closed field k with $\mathrm{char}(k) \neq \ell$. Generalizing the divisor graphs of prime divisors from the Introduction, we define a level-δ geometric prime divisor graph for $K|k$ as being a (half) oriented graph $\mathscr{D}_K$ defined as follows:

(I) The vertices of $\mathscr{D}_K$ are distinct function fields $K_i|k$ over k. And $\mathscr{D}_K$ has an origin which is $K_0 := K$.

(II) For every vertex K_i, the trivial valuation v_{i0} of K_i is the only edge from K_i to itself, and we view this edge as a non-oriented one, or a trivial edge. Further, the set of all the oriented edges starting at K_{i^*} is a geometric set D_{i^*} of prime divisors v_i of K_{i^*}. We call these edges non-trivial, and if $v_i \in D_{i^*}$ is such a non-trivial edge from K_{i^*} to K_i, then $K_i = K_{i^*}v_i$. In particular, $\mathrm{td}(K_i|k) = \mathrm{td}(K_{i^*}|k) - 1$.

(III) The trivial valuations are the only cycles of $\mathscr{D}_K$, and all the maximal branches of non-trivial edges of $\mathscr{D}_K$ have length equal to δ, hence $\delta \leq \mathrm{td}(K|k)$.

As indicated above, we attach to $\mathscr{D}_K$ the corresponding subgraph $\mathscr{G}_{\mathscr{D}_K} \subset \mathscr{G}_{\mathrm{Val}_K}$. Hence by definition one has:

(I) The vertices of $\mathscr{G}_{\mathscr{D}_K}$ are in bijection with the vertices of $\mathscr{D}_K$, via the Galois correspondence, i.e., the vertices of $\mathscr{G}_{\mathscr{D}_K}$ are the pro-ℓ groups Π_{K_i} with K_i vertex of $\mathscr{D}_K$. In particular, $\Pi_{K_0} := \Pi_K$ is the origin of $\mathscr{G}_{\mathscr{D}_K}$.

(II) The edges of $\mathscr{G}_{\mathscr{D}_K}$ are in bijection with the edges of $\mathscr{D}_K$. The trivial edge v_{i0} from K_i to itself is endowed with $\{1\} =: T_{v_{i0}} \subset Z_{v_{i0}} := \Pi_{K_i}$, i.e., with $\{1\} \subset \Pi_{K_i}$. Every non-trivial edge v_i is endowed with the inertia/decomposition groups $T_{v_i} \subseteq Z_{v_i}$. In particular, if v_i is an edge from $K_{i'}$ to $K_i = K_{i'}v_i$, then $\Pi_{K_i} = Z_{v_i}/T_{v_i}$.

(III) The trivial valuations are the only cycles of $\mathscr{G}_{\mathscr{D}_K}$, and all the maximal branches originating from Π_{K_0} and consisting of non-trivial edges of $\mathscr{G}_{\mathscr{D}_K}$ have length δ.

Proposition 18 *In the above context, $\mathscr{G}_{\mathscr{D}_K}$ is a level-δ abstract decomposition graph.*

Proof. Indeed, all the axioms of an abstract decomposition graph are more or less well known facts concerning Hilbert decomposition theory for valuations. For instance, if v_i is a prime divisor of $K_{i^*}|k$, then all the prolongations v'_i of v_i to K'_{i^*} are conjugated under $\Pi_{K_{i^*}}$; therefore, their inertia, respectively decomposition, groups are equal, say equal to $T_{v_i} \subseteq Z_{v_i}$. Further, $T_{v_i} \cong \mathbb{Z}_\ell$, and the residue field $K'_{i^*}v'_i$ equals $(K_{i^*}v_i)'$, thus $(K_{i^*}v_i)' = Z_{v_i}/T_{v_i}$, etc. Moreover, for prime divisors $v_i \neq w_i$ of $K_{i^*}|k$ one has the following; see e.g., Pop [28], Introduction, and especially Proposition 2.5 (2): The decomposition groups Z_{v_i} and Z_{w_i} have trivial intersection. And finally, if $X_{i^*} \to k$ is any quasi-projective normal variety, and $D_{X_{i^*}}$ is the set of Weil prime divisors of X_{i^*}, then every open subgroup of $\Pi_{K_{i^*}}$ contains almost all inertia groups T_{v_i}. Indeed, in every finite extension of K_{i^*} only finitely many Weil prime divisors of X_{i^*} are ramified, etc. $\square$

Remarks 19 Let $\mathscr{G}_{\mathscr{D}_K}$ be a level-δ abstract decomposition graph as above, and to simplify notation a little bit, set $\widehat{\Lambda}_{\mathscr{D}_K} := \widehat{\Lambda}_{\mathscr{G}_{\mathscr{D}_K}}$, $\widehat{U}_{\mathscr{D}_K} := \widehat{U}_{\mathscr{G}_{\mathscr{D}_K}}$, $\widehat{\mathrm{Div}}_{\mathscr{D}_K} := \widehat{\mathrm{Div}}_{\mathscr{G}_{\mathscr{D}_K}}$, and $\widehat{\mathfrak{Cl}}_{\mathscr{D}_K} := \widehat{\mathfrak{Cl}}_{\mathscr{G}_{\mathscr{D}_K}}$. We next analyze/describe the abstract objects $\widehat{U}_{\mathscr{D}_K}$, $\widehat{\Lambda}_{\mathscr{D}_K}$, $\widehat{\mathrm{Div}}_{\mathscr{D}_K}$ and $\widehat{\mathfrak{Cl}}_{\mathscr{D}_K}$ and relate the abstract exact sequence

$$1 \to \widehat{U}_{\mathscr{D}_K} \to \widehat{\Lambda}_{\mathscr{D}_K} \xrightarrow{j^{\mathscr{D}_K}} \widehat{\mathrm{Div}}_{\mathscr{D}_K} \to \widehat{\mathfrak{Cl}}_{\mathscr{D}_K} \to 0$$

to the geometry of $K|k$ as reflected in the geometric information encoded in the prime divisor graph $\mathscr{D}_K$. In order to do so, let us consider some normal model $X \to k$ of $K|k$ such that $D := D_X$ is the set of 1-edges of the given $\mathscr{D}_K$. Without loss of generality, we can and will suppose that X is quasi-projective. Then by Krull's Hauptidealsatz, $\mathscr{U}_D := \Gamma(X, \mathscr{O}_X)^\times$ depends on D only, and not on X, and $\mathscr{H}_D(K) := K^\times / \mathscr{U}_D$ is isomorphic to the group of principal divisors on X. Hence since $\mathrm{Div}(D) := \mathrm{Div}(X)$ depends on D only, and not on X, it follows that $\mathfrak{Cl}(D) := \mathfrak{Cl}(X)$ depends on D only, and not on X, and one has a canonical exact sequence

$$0 \to \mathscr{H}_D(K) \xrightarrow{\mathrm{div}_D} \mathrm{Div}(D) \xrightarrow{\mathrm{pr}} \mathfrak{Cl}(D) \to 0$$

and the resulting exact sequence of ℓ-adically complete groups:

$$0 \to \mathbb{T}_{\ell, \mathfrak{Cl}(D)} \longrightarrow \widehat{\mathscr{H}_D}(K) \xrightarrow{\mathrm{div}_D} \widehat{\mathrm{Div}}(D) \longrightarrow \widehat{\mathfrak{Cl}}(D) \to 0,$$

where $\mathbb{T}_{\ell, \mathfrak{Cl}(D)} = \varprojlim_n {}_n\mathfrak{Cl}(D)$, with $n = \ell^e$ and $e \geq 0$, is the ℓ-adic Tate module of $\mathfrak{Cl}(D)$. Since $1 \to \mathscr{U}_D / k^\times \to K^\times / k^\times \to \mathscr{H}_D(K) \to 1$ is an exact sequence of free abelian groups, so is $1 \to \widehat{\mathscr{U}_D} \to \widehat{K} \to \widehat{\mathscr{H}_D}(K) \to 1$. Hence if $\widehat{U}_D \subset \widehat{K}$ is the preimage of $\mathbb{T}_{\ell, \mathfrak{Cl}(D)} \hookrightarrow \widehat{\mathscr{H}_D}(K)$ under $\widehat{K} \to \widehat{\mathscr{H}_D}(K)$, we finally get an exact sequence of the form $0 \to \widehat{U}_D \to \widehat{K} \to \widehat{\mathrm{Div}}(D) \to \widehat{\mathfrak{Cl}}(D) \to 0$, and therefore, $\widehat{U}_D$ fits canonically into an exact sequence $1 \to \widehat{\mathscr{U}_D} \to \widehat{U}_D \to \mathbb{T}_{\ell, \mathfrak{Cl}(D)} \to 0$.

(1) By Kummer Theory we have an identification: $\widehat{\Lambda}_{\mathscr{D}_K} := \mathrm{Hom}\big(\Pi_K, \mathbb{Z}_\ell\big) = \widehat{K}$.

(2) Concerning/describing $\widehat{U}_{\mathscr{D}_K}$: Recall that we defined $\widehat{U}_{\mathscr{D}_K} := \mathrm{Hom}\big(\Pi_{1,\mathscr{D}_K}, \mathbb{Z}_\ell\big)$, where $\Pi_{1,\mathscr{D}_K} := \Pi_K / T_{\mathscr{D}_K}$ and $T_{\mathscr{D}_K}$ is the group generated by all the inertia groups T_v with v all the 1-edges of $\mathscr{D}_K$. By the definitions, we have $T_{\mathscr{D}_K} = T_D$ and $\Pi_{1,\mathscr{D}_K} = \Pi_{1,D}$. Further, in the notations from Fact 55, it follows that $\Pi_{1,D}$ is the Pontryagin dual of Δ_∞, which fits canonically in the exact sequence

$$0 \to \mathscr{U}_D \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \to \Delta_\infty \to {}_{\ell^\infty}\mathfrak{Cl}(D) \to 0.$$

Let $\Delta_0 \subset \Delta_\infty$ be the maximal divisible subgroup. Since $\mathscr{U}_D \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell$ is divisible, it follows by Fact 54 that Δ_0 fits into an exact sequence of the form

$$0 \to \mathscr{U}_D \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \to \Delta_0 \to {}_{\ell^\infty}A_0(X) \to 0,$$

and Δ_∞ / Δ_0 finite. Hence $\Pi_{1,D}$ has finite torsion, $\widehat{U}_{\mathscr{D}_K} := \mathrm{Hom}\big(\Pi_{1,\mathscr{D}_K}, \mathbb{Z}_\ell\big)$ is the Pontryagin dual of Δ_0, and we get an exact sequence $1 \to \widehat{\mathscr{U}_D} \to \widehat{U}_{\mathscr{D}_K} \to \mathbb{T}_{\ell, \mathfrak{Cl}(D)} \to 0$. Finally $\widehat{U}_{\mathscr{D}_K} = \widehat{U}_D$, and this gives the precise description of $\widehat{U}_{\mathscr{D}_K}$ in geometric terms.

(3) Concerning $\widehat{\mathrm{Div}}_{\mathscr{D}_K}$: For every prime divisor v one has a commutative diagram

$$(*)_v \qquad \begin{array}{ccc} \widehat{K} & \xrightarrow{v} & \widehat{vK} \\ \downarrow & & \downarrow \theta^v \\ \mathrm{Hom}\big(\Pi_K,\mathbb{Z}_\ell\big) & \xrightarrow{j^v} & \mathrm{Hom}\big(T_v,\mathbb{Z}_\ell\big) \end{array}$$

The diagrams $(*)_v$ with $v \in D$ give rise canonically to a commutative diagram

$$\begin{array}{ccccccc} \widehat{K} & \to & \widehat{\mathrm{Div}}(D) = \widehat{\oplus_v} vK & \to & \widehat{\mathfrak{Cl}}(D) & \to & 0 \\ \downarrow & & \downarrow \widehat{\oplus}\theta^v & & \downarrow & & \\ \mathrm{Hom}\big(\Pi_K,\mathbb{Z}_\ell\big) & \xrightarrow{j^{\mathscr{G}}} & \widehat{\oplus_v}\,\mathrm{Hom}\big(T_v,\mathbb{Z}_\ell\big) & \xrightarrow{\mathrm{can}} & \widehat{\mathscr{P}}_{\mathscr{D}_K} & \to & 0 \end{array}$$

where the vertical maps are isomorphisms, and $\widehat{\mathscr{P}}_{\mathscr{D}_K}$ is simply the quotient of the middle group by the first one. Thus the identification $\widehat{K} \to \mathrm{Hom}\big(\Pi_K,\mathbb{Z}_\ell\big) =: \widehat{\Lambda}_{\mathscr{D}_K}$ gives rise a canonical isomorphism $\widehat{\mathrm{Div}}(D) \to \widehat{\oplus_v}\,\mathrm{Hom}\big(T_v,\mathbb{Z}_\ell\big) =: \widehat{\mathrm{Div}}_{\mathscr{D}_K}$.

4) Finally, the above identifications $\widehat{K} \to \mathrm{Hom}\big(\Pi_K,\mathbb{Z}_\ell\big)$ and $\widehat{\mathrm{Div}}(D) \to \widehat{\mathrm{Div}}_{\mathscr{D}_K}$ give rise to an identification $\widehat{\mathfrak{Cl}}(D) \to \widehat{\mathscr{P}}_{\mathscr{D}_K} =: \widehat{\mathfrak{Cl}}_{\mathscr{D}_K}$. Hence by the structure of $\mathfrak{Cl}(D) := \mathfrak{Cl}(X)$ given in Fact 54, it follows that $\widehat{\mathfrak{Cl}}_{\mathscr{D}_K} \cong \widehat{\mathfrak{Cl}}(D) = \widehat{A}_1(X)$, and is thus a finite $\mathbb{Z}_\ell$-module.

Fact 20 *With the above notation, let* $\mathrm{Div}'(D)$ *be the preimage in* $\mathrm{Div}(D) := \mathrm{Div}(X)$ *of the maximal* ℓ*-divisible subgroup* $\mathfrak{Cl}'(D)$ *of* $\mathfrak{Cl}(D) := \mathfrak{Cl}(X)$*. Then one has:*

(1) $\mathrm{Div}'(D) \hookrightarrow \mathrm{Div}(D)$ *gives rise to an embedding* $\widehat{\mathrm{Div}'}(D) \hookrightarrow \widehat{\mathrm{Div}}(D)$.

(2) $\widehat{\mathrm{Div}'}(D) = \ker\big(\widehat{\mathrm{Div}}(D) \to \widehat{\mathfrak{Cl}}(D)\big) = \mathrm{div}_D(\widehat{K})$, *and* $\mathrm{Div}'(D) = \widehat{\mathrm{Div}'}(D) \cap \mathrm{Div}(D)$.

(3) *Let* $\tilde{D} \supseteq D$ *be geometric sets such that* $\Pi_{1,D} = \Pi_{1,\tilde{D}}$*. Then* $\mathrm{Div}'(\tilde{D}) \subseteq \mathrm{Div}'(D)$ *has finite bounded index. Finally, for every D large enough,* $\mathrm{Div}'(D) \subset \mathrm{Div}(D)$ *depends on* $K|k$ *only, and* $\widehat{\mathfrak{Cl}}(\tilde{D}) \cong \widehat{\mathfrak{Cl}}(D) \oplus \mathbb{Z}_\ell^{|\tilde{D}\setminus D|}$.

Proof. To (1) and (2): We get a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{H}_D(K) & \to & \mathrm{Div}'(D) & \to & \mathfrak{Cl}'(D) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{H}_D(K) & \to & \mathrm{Div}(D) & \to & \mathfrak{Cl}(D) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathscr{D} & \to & \mathscr{C} & & \end{array}$$

where $\mathscr{D} = \mathrm{Div}(D)/\mathrm{Div}'(D)$, $\mathscr{C} = \mathfrak{Cl}(D)/\mathfrak{Cl}'(D)$, and $\mathscr{D} \to \mathscr{C}$ is an isomorphism. By the structure of $\mathfrak{Cl}(D) = \mathfrak{Cl}(X)$ as given in Fact 54, $\mathscr{C} := \mathfrak{Cl}(X)/\mathfrak{Cl}'(X)$ is of the form $\mathscr{C} = C_1 + C_2$ with C_1 a finite abelian ℓ-group, and C_2 a finitely generated free abelian group. Hence we get embeddings of the ℓ-adic completions $\mathscr{C} \hookrightarrow \widehat{\mathscr{C}}$ and $\mathscr{D} \hookrightarrow \widehat{\mathscr{D}}$. Further, since $\mathfrak{Cl}(D)'$ is ℓ-divisible, its ℓ-adic completion is trivial, hence $\widehat{\mathfrak{Cl}}(D) \to \widehat{\mathscr{C}}$ is an isomorphism. And finally, the middle (exact) column defines an exact sequence $\widehat{\mathrm{Div}'}(X) \hookrightarrow \widehat{\mathrm{Div}}(X) \to \widehat{\mathscr{D}} \to 0$. The remaining assertions follow easily in the same way, by chasing in the commutative diagram above.

To (3): Let $D = D_X$ and $\tilde{D} = D_Y$ for some normal models $X \to k$ and $Y \to k$ of $K|k$. By Fact 55, especially assertion (4), it follows that $\mathscr{U}_X = \mathscr{U}_Y$, and with $Y \to k$ as in Fact 55 we have that $A_{\mathrm{tors}}(Y) \subseteq A_0(Y) \subseteq A_\tau(Y)$ are birational invariants of $K|k$, etc. □

Definition/Remark 21 Consider notation as above.

(1) A geometric set $D := D_X$ of prime divisors for $K|k$ is called complete regular-like, if $\Pi_{1,D} = \Pi_{1,K}$ and $\widehat{\mathfrak{Cl}}(D)$ has positive rational rank and for every geometric set of prime divisors $\tilde{D} \supseteq D$ one has $\widehat{\mathfrak{Cl}}(\tilde{D}) \cong \widehat{\mathfrak{Cl}}(D) \oplus \mathbb{Z}_\ell^{|\tilde{D}\backslash D|}$.

• Note that if $X \to k$ is a complete regular variety, then D_X is complete regular-like, but the converse is not true in general. Nevertheless, if X is a curve, then D_X is complete regular-like iff X is a complete regular curve.

(2) Let $\mathscr{D}_K$ be a level-δ geometric graph of prime divisors for $K|k$. For each vertex $\mathfrak{v}$ of $\mathscr{D}_K$, let $D_\mathfrak{v}$ is the set of non-trivial 1-edges of $\mathscr{D}$ with origin $K\mathfrak{v}$. We say that $\mathscr{D}_K$ is complete regular-like, if $D_\mathfrak{v}$ is complete regular-like for all $\mathfrak{v}$ with $\mathrm{td}(K\mathfrak{v}|k) > 0$.

(3) The complete regular-like prime divisor graphs for $K|k$ are abundant. Moreover, for every geometric prime divisor graph $\mathscr{D}'_K \subset \mathscr{D}_K^{\mathrm{tot}}$ there exist complete regular-like decomposition graphs $\mathscr{D}_K$ with $\mathscr{D}'_K \subseteq \mathscr{D}_K$. Indeed, we proceed by induction on the transcendence degree as follows: Let D' be the set of 1-indices of $\mathscr{D}'_K$. Let $D \supseteq D'$ be any complete regular-like set of prime divisors of K. Then $\mathrm{td}(Kv|k) < \mathrm{td}(K|k)$ for all $v \in D$, hence for every $v \in D$ by induction one has the following: There exists complete regular-like prime divisor graphs $\mathscr{D}_{Kv}$ for $Kv|k$. Moreover, if $v \in D'$, then there exists a complete regular-like prime divisor graph $\mathscr{D}_{Kv}$ which contains the residual prime divisor graph $\mathscr{D}'_{Kv}$ of $\mathscr{D}'_K$. Then the resulting prime divisor graph $\mathscr{D}_K$ having D as set of 1-indices and $\mathscr{D}_{Kv}$ as residual prime divisor graph at each $v \in D$ is by definition complete regular-like.

Combining the above discussion with the one in the previous subsection, we get:

Proposition 22 *In the above notations and context, the following hold:*

(1) *The geometric decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ for $K|k$ can be recovered by a group-theoretical recipe from the group-theoretical information encoded in $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$.*

(2) *Moreover, given a geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$, the fact that $\mathscr{G}_{\mathscr{D}_K}$ is complete regular-like can be recovered from the total decomposition graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ endowed with $\mathscr{G}_{\mathscr{D}_K}$.*

(3) *In particular, the complete regular-like decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ for $K|k$ can be recovered from the group-theoretical information encoded in $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$.*

Proof. To (1): This is more or less the Gal-Criterion 17 combined with Remarks 16.

To (2): Let $\mathscr{D}_K$ be a geometric graph of prime divisors for $K|k$, and $\mathscr{G}_{\mathscr{D}_K}$ the corresponding geometric decomposition graph for $K|k$. For every vertex $K\mathfrak{v}$ of $\mathscr{D}_K$, let $X_\mathfrak{v} \to k$ be a normal model of $K\mathfrak{v}\,|\,k$ such that $D_\mathfrak{v} = D_{X_\mathfrak{v}}$. Let $T_{D_\mathfrak{v}} \subseteq \Pi_{K\mathfrak{v}}$ be the closed subgroup generated by all the inertia groups T_v, $v \in D_\mathfrak{v}$. Further, by Remark 16 (1), the total decomposition graph of $K\mathfrak{v}\,|\,k$ can be recovered from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$. In particular, the set of the inertia groups T_w of all the prime divisors w of $K\mathfrak{v}\,|\,k$, hence the closed subgroup $T_{K\mathfrak{v}}$ generated by all these inertia groups, can be recovered from $\mathscr{G}_{\mathscr{D}_K^{\rm tot}}$ endowed with $\mathscr{G}_{\mathscr{D}_K}$. Therefore, from $\mathscr{G}_{\mathscr{D}_K^{\rm tot}}$ endowed with $\mathscr{G}_{\mathscr{D}_K}$ one can check whether the following hold:

(a) $\{T_v \mid v \in D_\mathfrak{v}\}$ equals the set $\{T_w \mid w$ all the prime divisors of $K\mathfrak{v}\,|\,k\}$.

(b) $T_{K\mathfrak{v}} = T_{D_\mathfrak{v}}$.

(c) $\widehat{\mathfrak{Cl}}(D_\mathfrak{v})$ is not finite.

(d) $\widehat{\mathfrak{Cl}}(\tilde{D}_\mathfrak{v}) \cong \widehat{\mathfrak{Cl}}(D_\mathfrak{v}) \oplus \mathbb{Z}_\ell^{|\tilde{D}_\mathfrak{v}\setminus D_\mathfrak{v}|}$ for every geometric set $\tilde{D}_\mathfrak{v} \supseteq D_\mathfrak{v}$ of prime divisors of $K\mathfrak{v}|k$.

Note that (a) holds iff $X_\mathfrak{v}$ is a complete normal curve. Indeed, if $\dim(X_\mathfrak{v}) > 1$, then for every given normal model $Y_\mathfrak{v} \to k$ of $K\mathfrak{v}\,|\,k$ there exist infinitely many prime divisors of $K\mathfrak{v}\,|\,k$ which are not Weil prime divisors of $Y_\mathfrak{v}$. And if $X_\mathfrak{v}$ is a normal curve over k, then the set of prime divisors of $K\mathfrak{v}\,|\,k$ is in bijection with the set of all the points in the normal completion of $X_\mathfrak{v}$, etc. Further, the conditions (b), (c), (d), are by definition the necessary conditions for $D_\mathfrak{v}$ to be complete regular-like. This concludes the proof of assertion (2).

To (3): This follows immediately by combining assertions (1) and (2) above. □

Proposition 23 *With the above notation, let $\mathscr{G}_{\mathscr{D}_K}$ be a complete regular-like decomposition graph for $K|k$. Then the following hold:*

(1) *$\mathscr{G}_{\mathscr{D}_K}$ viewed as an abstract decomposition graph of level $\delta := \mathrm{td}(K|k)$ is δ complete curve like and ample up to level δ. Thus $\mathscr{G}_{\mathscr{D}_K}$ is divisorial.*

(2) *Let D be the set of 1-edges of $\mathscr{G}_{\mathscr{D}_K}$. Then the following hold:*

- (a) *$\widehat{U}_{\mathscr{D}_K}$ is the ℓ-adic dual of $\Pi_{1,K} = \Pi_{1,D}$, and thus it depends on $K|k$ only, and not on $\mathscr{G}_{\mathscr{D}_K}$. Therefore we denote $\widehat{U}_K := \widehat{U}_{\mathscr{D}_K}$.*
- (b) *$\mathrm{Div}(D)_{(\ell)} := \mathrm{Div}(D) \otimes \mathbb{Z}_{(\ell)}$ is an abstract divisor group of $\mathscr{G}_{\mathscr{D}_K}$, which we call the canonical abstract divisor group of $\mathscr{G}_{\mathscr{D}_K}$.*
- (c) *The preimage $\Lambda_{\mathscr{D}_K}$ of $\mathrm{Div}(D)_{(\ell)}$ in $\widehat{K}$ will be called the canonical divisorial $\widehat{U}_{\mathscr{D}_K}$-lattice of $\mathscr{G}_{\mathscr{D}_K}$, and it has the following description:*
- *Let $\mathrm{Div}'(D) \subseteq \mathrm{Div}(D)$ be the preimage of the ℓ-divisible subgroup $\mathfrak{Cl}'(D)$ of $\mathfrak{Cl}(D)$. Then $\mathrm{Div}'(D)_{(\ell)} = \mathrm{div}_D(\Lambda_{\mathscr{D}_K})$ and $\mathrm{div}_D(\Lambda_{\mathscr{D}_K})/\mathscr{H}_D(K)_{(\ell)} = \mathfrak{Cl}'(D)_{(\ell)}$.*

(3) *Up to multiplication by ℓ-adic units, the canonical $\widehat{U}_K$-lattice $\Lambda_K := \Lambda_{\mathscr{D}_K}$ depends only on $K|k$, and not on $\mathscr{D}_K$.*

Proof. To (1): First, the fact that $\mathscr{G}_{\mathscr{D}_K}$ is complete curve-like follows from the fact that all the $(\delta - 1)$ residual function fields $K\mathfrak{v}|k$ have $\mathrm{td}(K\mathfrak{v}|k) = 1$, and the facts that $\mathscr{G}_{\mathscr{D}_K}$ is complete regular-like. In order to show that $\mathscr{G}_{\mathscr{D}_K}$ is ample up to level $\delta = \mathrm{td}(K|k)$, we have to show that conditions (i), (ii), from Definition/Remark 3 (7) are satisfied. First, condition (i) follows immediately from the weak Approximation Lemma. To check condition (ii) is a little bit more technical though. Since $\mathscr{G}_{\mathscr{D}_K}$ is complete regular-like, for every multi-index $\mathfrak{v}$ of $\mathscr{G}_{\mathscr{D}_K}$, there exists a complete regular-like model $X_{\mathfrak{v}} \to k$ of $K\mathfrak{v}|k$ such that $D_{X_{\mathfrak{v}}}$ is the set of 1-vertices of $\mathscr{G}_{\mathscr{D}_{K\mathfrak{v}}}$; hence in particular, one has $\Pi_{1,K\mathfrak{v}} = \Pi_{1,D_{X_{\mathfrak{v}}}}$. And we will denote by $X \to k$ the corresponding model of $K|k$; thus $\Pi_{1,K} = \Pi_{1,D_X}$.

We carry out induction on $d = \mathrm{td}(K|k) > 1$ as follows:

Let $\Delta \subseteq \widehat{K}$ be an ℓ-adically closed submodule. Then Δ gives rise functorially to a subextension $K_\Delta|K$ of $K'|K$ by setting $K_\Delta := \cup_n K_n$, where $K_n := K[\sqrt[n]{\Delta_n}]$, and $\Delta_n \subseteq K^\times/n$ is the image of Δ in $K^\times/n = \widehat{K}/n$ for all $n = \ell^e$. (Note that since $\Delta \subseteq \widehat{K}$ is closed, Δ is the projective limit of the Δ_n's inside $\widehat{K}$.) We notice that $K_n|K$ is $\mathbb{Z}/n$ elementary abelian with Galois group equal to $\mathrm{Hom}(\Delta_n, \mu_n)$; thus $K_\Delta|K$ has Galois group $\mathrm{Hom}_{\mathrm{cont}}(\Delta, \mathbb{T}_\ell)$, where $\mathbb{T}_\ell$ is the Tate module of the μ_{ℓ^∞} roots of unity, which we have identified with $\mathbb{Z}_\ell$.

Further, let v be an *arbitrary* valuation of K, and v_Δ a prolongation to K_Δ. Then under the above correspondence one has the following: The decomposition field of $v_\Delta|v$ is K_{Δ_0}, where $\Delta_0 := \Delta \cap \ker(j_v)$, and the inertia field of $v_\Delta|v$ is K_{Δ_1}, where $\Delta_1 := \Delta \cap \widehat{U}_v$. In particular, v is unramified in $K_\Delta|K$ iff $\Delta \subset \widehat{U}_v$.

Checking condition (ii) (a) from Definition/Remark 3 (7): The main technical tool for the proof is Theorem B from Pop [29], which implies the following: Since $\Delta := \widehat{U}_{\mathscr{D}_K}$ is the ℓ-adic dual of $\Pi_{1,K}$, it follows by the definition of $\Pi_{1,K}$ that the corresponding subextension $K_\Delta|K$ is the maximal subextension of $K'|K$ in which all prime divisors v of $K|k$ are unramified. But then by [29] it follows that all the k-valuations of $K|k$ are unramified in $K_\Delta|K$. And correspondingly, the same is true for all the residue function fields $K\mathfrak{v}|k$ of $\mathscr{D}_K$. Now for v a fixed prime divisor of $K|k$, let $\mathscr{V}$ be the set of the k-valuations $\mathfrak{v} = w \circ v$ of $K|k$, with $w \in D_{X_v}$ the set of prime divisors defined by the complete regular-like model $X_v \to k$ mentioned at the beginning of the proof. Since $\widehat{U}_{\mathfrak{v}} \subseteq \widehat{U}_v$ and $j_v(\widehat{U}_{\mathfrak{v}}) = \widehat{U}_w$ for $\mathfrak{v} = w \circ v$, it follows that setting $\Delta_v := \cap_{\mathfrak{v} \in \mathscr{V}} \widehat{U}_{\mathfrak{v}}$, we have

$$\Delta_v \subseteq \widehat{U}_v \ \text{ and } \ j_v(\Delta_v) = j_v\big(\cap_{\mathfrak{v}} \widehat{U}_{\mathfrak{v}}\big) \subseteq \cap_{\mathfrak{v}}\, j_v(\widehat{U}_{\mathfrak{v}}) = \cap_w \widehat{U}_w = \widehat{U}_{\mathscr{D}_{Kv}}.$$

On the other hand, by the discussion above, all the k-valuations of $K|k$ are unramified in $K_\Delta|K$. Hence in particular so are all the $\mathfrak{v} \in \mathscr{V}$; thus $\Delta \subseteq \widehat{U}_{\mathfrak{v}}$ for all $\mathfrak{v} \in \mathscr{V}$. We conclude that $\Delta \subseteq \Delta_v$. Therefore, j_v maps $\widehat{U}_{\mathscr{D}_K} =: \Delta$ into $j_v(\Delta_v) \subseteq \widehat{U}_{\mathscr{D}_{Kv}}$, as claimed.

For the second assertion of condition (ii) (a) from Definition/Remark 3 (7), let $\widehat{K}_{\mathrm{fin}} \subset \widehat{K}$ be the union of all the finite-corank submodules of $\widehat{K}$, and define $\widehat{Kv}_{\mathrm{fin}} \subset \widehat{Kv}$ correspondingly. We then have to show that $\widehat{U}_{\mathscr{D}_{Kv}} \cdot j_v(\widehat{K}_{\mathrm{fin}} \cap \widehat{U}_v) = \widehat{Kv}_{\mathrm{fin}}$. Let $\Delta \subset \widehat{K}_{\mathrm{fin}}$ be a finite-corank $\mathbb{Z}_\ell$-submodule. Since Δ has finite corank, there exits a cofinite subset $D' \subset D_X$ such that $v'(\Delta) = 0$ for every $v' \in D$. Therefore, if x_v is the center of v on X, for every sufficiently small open neighborhood $X' \subset X$, we have $v'(\Delta) = 0$ for all $v \in D_{X'}$, $v' \neq v$. In particular, since X is normal, thus smooth at x_v, we can choose $X' \subset X$ to be smooth such that $x_v \in X'$ and $w(\Delta) = 0$ for all $w \in D_{X'}$, $w \neq v$. Since $\Delta \cap \widehat{U}_v$ is contained in $\widehat{U}_{D_{X'}}$, it is sufficient to show that $j_v(\widehat{U}_{D_{X'}})$ is contained in $\widehat{Kv}_{\mathrm{fin}}$, hence mutatis mutandis, we can suppose that $\Delta := \widehat{U}_{D_{X'}}$. If so, $K_\Delta|K$ is the maximal subextension of $K'|K$ in which all $v' \in D_{X'}$ are unramified. Let $\tilde{X} \to k$ be a projective normal completion of X (note that $\tilde{X} \to k$ exists, because $X' \subset X$, and X is normal quasi-projective), and set $\tilde{S} := \tilde{X} \backslash X'$; hence S is a proper closed subset of X which does not contain x_v. Let further $X_{x_v} \subset \tilde{X}$ be the closure of x_v in $\tilde{X}$, and set $S := \tilde{S} \cap X_{x_v}$, thus $S \subset X_{x_v}$ is a proper closed subset. Further, we view $X_{x_v} \to k$ as a projective, thus proper (not necessarily normal) model of $Kv|k$. For every k-valuation w of $Kv|k$ we claim the following:

Claim. Suppose that $w\big(j_v(\varDelta)\big) \neq 0$. Then the center x_w of w on X_{x_v} lies in S.

Indeed, since $w\big(j_v(\varDelta)\big) \neq 0$, it follows that setting $\mathfrak{v} = w \circ v$ as a valuation of $K|k$, we have $\mathfrak{v}(\varDelta) \neq 0$. Therefore, by the introductory discussion above, it follows that $\mathfrak{v}$ is ramified in $K_\varDelta|K$. We claim that the center $x_\mathfrak{v}$ of $\mathfrak{v}$ on $\tilde{X}$ lies in $\tilde{S}$. By contradiction, let $x_\mathfrak{v} \in X'$. Since X' is smooth, hence regular, by the purity of the branch locus, one has $\varPi_{1,D_{X'}} = \varPi_1(X)$. Hence every finite cover $Y' \to X'$ defined by some open subgroup of $\varPi_{1,X'}$ is étale. Hence the cover $Y' \times_X \operatorname{Spec} \mathscr{O}_\mathfrak{v} \to \operatorname{Spec} \mathscr{O}_\mathfrak{v}$ is étale, thus unramified. Therefore, $\mathfrak{v}$ is unramified in $K_\varDelta|K$, contradiction! Since $\mathscr{O}_\mathfrak{v} \subset \mathscr{O}_v$, by the valuative criterion for properness, we have $\mathscr{O}_{\tilde{X},x_\mathfrak{v}} \subset \mathscr{O}_{\tilde{X},x_v}$, and $x_\mathfrak{v}$ lies in the closure of $\{x_v\}$ in $\tilde{X}$, hence in X_{x_v}. In particular, since $\mathfrak{v}$ has no center on X', it follows that $x_\mathfrak{v} \in S = X_{x_v} \cap \tilde{S}$. Finally, using the valuative criterion for properness again, it follows that viewing X_{x_v} as a projective (not necessarily normal) model of $Kv|k$, the center of w on X_{x_v} is precisely $x_\mathfrak{v}$. This concludes the proof of the Claim.

Using the claim above, we finish the proof of (ii) (a) from Definition/Remark 3 (7) as follows: For every geometric set of prime divisors D_v of $Kv|k$, only finitely many $w \in D_v$ have center in S. Hence by the claim, only finitely many $w \in D_v$ satisfy $w\big(j_v(\varDelta)\big) \neq 0$. From this we conclude that $j_v(\varDelta)$ has finite corank.

Checking (ii) (b) from Definition/Remark 3 (7): Let $\varDelta \subset \widehat{K}_{\mathrm{fin}}$ be a finite corank $\mathbb{Z}_\ell$-module. Further let $X' \subset X$ be a smooth open subvariety such that $\varDelta \subset \widehat{U}_{D_{X'}}$. As in the proof of (ii) (a), mutatis mutandis, it sufficient to check (ii) (b) for the $\mathbb{Z}_\ell$-submodule of finite-corank $\varDelta := \cap_{v \in D_{X'}} \widehat{U}_v$.

In order to do so, let $X' \subset \tilde{X}$ be a normal projective completion of X', and $\tilde{X} \hookrightarrow \mathbb{P}^N_k$ some projective embedding. Then if H is a general hyperplane, and $Z := \tilde{X} \cap H$ and $Z' := X' \cap H$ are the corresponding hyperplane sections, it follows that $Z' \hookrightarrow X'$ is a prime Weil divisor such that $Z' \to k$ is smooth, because $X' \to k$ was so. Further, $Z' \hookrightarrow X'$ gives rise to a surjective group homomorphism $\varPi_1(Z') \to \varPi_1(X')$, which is an isomorphism if $\dim(X') > 2$. Hence since X' and Z' are smooth, thus regular, by the purity of the branch locus we have $\varPi_{1,D_{X'}} = \varPi_1(X')$ and $\varPi_{1,D_{Z'}} = \varPi_1(Z')$; thus we get a surjective projection $\varPi_{1,D_{Z'}} \to \varPi_{1,D_{X'}}$. Let $v := v_{Z'}$ be the prime divisor of $K|k$ defined by the Weil prime divisor Z' of X'. Then taking ℓ-adic duals, it follows as in the proof of (ii) (a) above that the surjectivity of the projection $\varPi_{1,D_{Z'}} \to \varPi_{1,D_{X'}}$ implies that $j_v : \widehat{U}_{X'} \to \widehat{U}_{Z'}$ is injective, as claimed.

To (3): It follows immediately from (the proof of) assertion (1) above, together with Remarks 19 (2), (3), and (4) and Fact 20 and Fact 55 (4). $\square$

4 Morphisms and rational quotients of abstract decomposition graphs

4.1 Morphisms

Let $\mathscr{G}$ and $\mathscr{H}$ be given abstract decomposition graphs of levels $\delta_{\mathscr{G}}$ and $\delta_{\mathscr{H}}$, based on $G = G_0$, respectively $H = H_0$. We denote as usual by $T_v \subset Z_v$ and $G_v = Z_v/T_v$ the 1-edges, respectively the 1-vertices, of $\mathscr{G}$, and correspondingly by $T_w \subset Z_w$ and $G_w = Z_w/T_w$ those for $\mathscr{H}$. Further, $\mathscr{G}_v$ and $\mathscr{H}_w$ are the corresponding 1-residual abstract decomposition graphs, which have then level $\delta_{\mathscr{G}} - 1$, respectively $\delta_{\mathscr{H}} - 1$. We also recall that v_0 and w_0 are the trivial valuations of G, respectively H, and that their inertia groups are trivial by definition.

Definition/Remark 24 In the above context we define:

(1) Let $\Phi : G_0 \to H_0$ be a (continuous) group homomorphism. Let $\mathfrak{v}$ and $\mathfrak{w}$ be multi-indices for $\mathscr{G}$ and $\mathscr{H}$. We define inductively on the length of $\mathfrak{v}$ the fact that $\mathfrak{w}$ corresponds to $\mathfrak{v}$ via Φ as follows; see Definition/Remark 2, especially points (3) and (4), to recall notation:

(i) The trivial multi-index $\mathfrak{w} = w_0$ corresponds to $\mathfrak{v}$ if and only if $\Phi(T_{\mathfrak{v}}) = 1$ and $\Phi(Z_{\mathfrak{v}})$ is open in H_0. And the only $\mathfrak{w}$ which corresponds to the trivial multi-index $\mathfrak{v} = v_0$ is the trivial multi-index $\mathfrak{w} = w_0$.

(ii) Suppose that $\mathfrak{w} = (w_s, \dots, w_1)$ and $\mathfrak{v} = (v_r, \dots, v_1)$ are both non-trivial, and let us set $\mathfrak{v} = (\mathfrak{v}_1, v_1)$ and $\mathfrak{w} = (\mathfrak{w}_1, w_1)$ with $\mathfrak{v}_1$ and $\mathfrak{w}_1$ the corresponding multi-indices for the residual abstract decomposition graphs $\mathscr{G}_{v_1}$, respectively $\mathscr{H}_{w_1}$. (Note that $\mathfrak{v}_1$ and/or $\mathfrak{w}_1$ might be trivial.) Then we say that $\mathfrak{w}$ corresponds to $\mathfrak{v}$ if and only if one of the following hold:

(a) If $\Phi(T_{v_1}) = 1$, then under $\Phi_{v_1} : G_{v_1} = Z_{v_1}/T_{v_1} \to H_0$, inductively one has that $\mathfrak{w}$ corresponds to $\mathfrak{v}_1$.

(b) If $\Phi(T_{v_1}) \neq 1$, then $\Phi(T_{v_1}) \subseteq T_{w_1}$ and $\Phi(Z_{v_1}) \subseteq T_{w_1}$ are open subgroups, and under $\Phi_{v_1} : G_{v_1} = Z_{v_1}/T_{v_1} \to Z_{w_1}/T_{w_1} = H_{w_1}$, inductively one has that $\mathfrak{w}_1$ corresponds to $\mathfrak{v}_1$.

(2) Let $\mathfrak{w}$ correspond to some $\mathfrak{v}$. Then for every $\mathfrak{w}' \leq \mathfrak{w}$, there exists $\mathfrak{v}' \leq \mathfrak{v}$ such that $\mathfrak{w}'$ corresponds to $\mathfrak{v}'$. The proof of this assertion follows easily by induction on the length of $\mathfrak{v}$, and we will omit it.

(3) Finally, let $\mathscr{V}_{\mathscr{G}}$ and $\mathscr{V}_{\mathscr{H}}$ be the sets of the multi-indices $\mathfrak{v}$ of $\mathscr{G}$, respectively $\mathfrak{w}$ of $\mathscr{H}$, and let $\mathscr{V}_{\mathscr{G},\Phi} \subseteq \mathscr{V}_{\mathscr{G}}$ be the set of all $\mathfrak{v} \in \mathscr{V}_{\mathscr{G}}$ such that there exists some $\mathfrak{w}_{\mathfrak{v}} \in \mathscr{V}_{\mathscr{H}}$ which corresponds to $\mathfrak{v}$. Then the correspondence defined at (1) above gives rise to a well-defined map $\varphi_\Phi : \mathscr{V}_{\mathscr{G},\Phi} \to \mathscr{V}_{\mathscr{H}}, \ \ \mathfrak{v} \mapsto \varphi_\Phi(\mathfrak{v}) = \mathfrak{w} := \mathfrak{w}_{\mathfrak{v}}$.

(4) If $\varphi_\Phi(\mathfrak{v}) = \mathfrak{w}$, we say that Φ maps $\mathfrak{v}$ to $\mathfrak{w}$, or that $\mathfrak{w}$ is the image of $\mathfrak{v}$ under Φ.

Definition 25 With the above notation, let δ be a non-negative integer which satisfies $\delta \leq \delta_{\mathscr{G}}, \delta_{\mathscr{H}}$. We define a level-$\delta$ morphism $\Phi : \mathscr{G} \to \mathscr{H}$ inductively on δ and $\delta_{\mathscr{G}}$ as follows:

(1) A level-zero morphism $\Phi : \mathscr{G} \to \mathscr{H}$ is any group homomorphism $\Phi : G \to H$ under which w_0 corresponds to v_0. Equivalently, Φ is open.

(2) A level-δ morphism $\Phi : \mathscr{G} \to \mathscr{H}$ is any level-zero morphism $\Phi : G \to H$ which inductively on $\delta_{\mathscr{G}}$ and on $\delta > 0$ satisfies the following:

(i) Almost all 1-vertices of $\mathscr{H}$ correspond to some 1-vertices of $\mathscr{G}$, and every 1-vertex of $\mathscr{H}$ corresponds to only finitely many (maybe to none) of the 1-vertices of $\mathscr{G}$.

(ii) If the trivial valuation w_0 corresponds to a 1-edge v, then $\delta_{\mathscr{G}_v} = \delta_{\mathscr{G}} - 1 \geq \delta$, and the canonical group homomorphism $\Phi_v : G_v = Z_v/T_v \to H_0$ defines a level-δ morphism of the corresponding residual abstract decomposition graphs $\mathscr{G}_v$ and $\mathscr{H}$.

(iii) If w is a 1-edge corresponding to the 1-edge v, then the group homomorphism $\Phi_v : G_v = Z_v/T_v \to Z_w/T_w = H_w$ defines a level-$(\delta - 1)$ morphism of the corresponding residual abstract decomposition graphs $\mathscr{G}_v$ and $\mathscr{H}_w$.

Remarks 26 In the above context, let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-δ morphism of abstract decomposition graphs.

(1) The morphism Φ gives rise to a Kummer homomorphism

$$\widehat{\Lambda}_{\mathscr{H}} := \mathrm{Hom}\big(H, \mathbb{Z}_\ell\big) \xrightarrow{\hat{\phi}} \mathrm{Hom}\big(G, \mathbb{Z}_\ell\big) =: \widehat{\Lambda}_{\mathscr{G}}, \quad \varphi \mapsto \varphi \circ \Phi.$$

Since Φ has an open image, and $\widehat{\Lambda}_{\mathscr{G}}$ and $\widehat{\Lambda}_{\mathscr{H}}$ are torsion-free, $\hat{\phi}$ is *injective.*

From now on *suppose that* $\delta > 0$, and that $\mathfrak{v}$ and $\mathfrak{w}$ are the multi-indices of $\mathscr{G}$, respectively $\mathscr{H}$, which correspond to each other. Let $\delta_{\mathfrak{v}}$ and $\delta_{\mathfrak{w}}$ be their lengths, and suppose that $\delta_{\mathfrak{w}} < \delta$.

(2) $\Phi_{\mathfrak{v}} : \mathscr{G}_{\mathfrak{v}} \to \mathscr{H}_{\mathfrak{w}}$ has level $(\delta - \delta_w)$ and the resulting residual Kummer homomorphism $\hat{\phi}_{\mathfrak{v}} : \widehat{\Lambda}_{\mathscr{H}_{\mathfrak{w}}} \to \widehat{\Lambda}_{\mathscr{G}_{\mathfrak{v}}}$ is injective by the remark above applied to $\Phi_{\mathfrak{v}}$.

(3) To simplify notation, let us set $\widehat{\Lambda}_{Z_{\mathfrak{v}}} = \mathrm{Hom}\big(Z_{\mathfrak{v}}, \mathbb{Z}_\ell\big)$ and $\widehat{\Lambda}_{T_{\mathfrak{v}}} = \mathrm{Hom}\big(T_{\mathfrak{v}}, \mathbb{Z}_\ell\big)$, thus in particular, $\widehat{\Lambda}_{T_{\mathfrak{v}}} \cong \mathbb{Z}_\ell^{\delta_{\mathfrak{v}}}$. The inclusions $T_{\mathfrak{v}} \hookrightarrow Z_{\mathfrak{v}} \hookrightarrow G$ and the canonical exact sequence $1 \to T_{\mathfrak{v}} \to Z_{\mathfrak{v}} \to G_{\mathfrak{v}} \to 1$ give rise in the same way as at Definition/Remark 3, points (3) and (6), to morphisms of ℓ-adically complete $\mathbb{Z}_\ell$-modules as follows:

$$j^{\mathfrak{v}} : \widehat{\Lambda}_{\mathscr{G}} \xrightarrow{\mathrm{res}_Z} \widehat{\Lambda}_{Z_{\mathfrak{v}}} \xrightarrow{\mathrm{res}_T} \widehat{\Lambda}_{T_{\mathfrak{v}}} \quad \text{and} \quad 0 \to \widehat{\Lambda}_{\mathscr{G}_{\mathfrak{v}}} \xrightarrow{\mathrm{inf}} \widehat{\Lambda}_{Z_{\mathfrak{v}}} \xrightarrow{\mathrm{res}_T} \widehat{\Lambda}_{T_{\mathfrak{v}}} \to 0.$$

In particular, setting $\widehat{U}^1_{\mathfrak{v}} := \ker(\mathrm{res}_Z)$ and $\widehat{U}_{\mathfrak{v}} = \ker(j^{\mathfrak{v}})$, we get exact sequences

$$0 \to \widehat{U}_{\mathfrak{v}} \longrightarrow \widehat{\Lambda}_{\mathscr{G}} \xrightarrow{j^{\mathfrak{v}}} \widehat{\Lambda}_{T_{\mathfrak{v}}} \to 0 \quad \text{and} \quad 0 \to \widehat{U}^1_{\mathfrak{v}} \hookrightarrow \widehat{U}_{\mathfrak{v}} \xrightarrow{j_{\mathfrak{v}}} \widehat{\Lambda}_{\mathscr{G}_{\mathfrak{v}}} \to 0.$$

The surjective morphism $j_{\mathfrak{v}} : \widehat{U}_{\mathfrak{v}} \to \widehat{\Lambda}_{\mathscr{G}_{\mathfrak{v}}}$ is called the canonical $\mathfrak{v}$-reduction homomorphism.

(4) By induction on $\delta_{\mathfrak{v}}$ and $\delta_{\mathfrak{w}}$, one gets the following: $\Phi(Z_{\mathfrak{v}}) \subseteq Z_{\mathfrak{w}}$ and $\Phi(Z_{\mathfrak{v}})$ is open in $Z_{\mathfrak{w}}$, and $\Phi(T_{\mathfrak{v}}) \subseteq T_{\mathfrak{w}}$ and $\Phi(T_{\mathfrak{v}})$ is open in $T_{\mathfrak{w}}$. Hence since Φ is open and restricts to an open homomorphism $Z_{\mathfrak{v}} \to Z_{\mathfrak{w}}$ and $T_{\mathfrak{v}} \to T_{\mathfrak{w}}$, by taking ℓ-adic duals we get commutative diagrams with *injective columns and exact rows* as follows:

$$\begin{array}{ccccc} \widehat{U}_{\mathfrak{w}} & \longrightarrow & \widehat{\Lambda}_{Z_{\mathfrak{w}}} & \longrightarrow & \widehat{\Lambda}_{T_{\mathfrak{w}}} \\ \big\downarrow \hat{\phi} & & \big\downarrow \hat{\phi} & & \big\downarrow \hat{\phi}^{\mathfrak{v}} \\ \widehat{U}_{\mathfrak{v}} & \longrightarrow & \widehat{\Lambda}_{Z_{\mathfrak{v}}} & \longrightarrow & \widehat{\Lambda}_{T_{\mathfrak{v}}} \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \widehat{U}^1_{\mathfrak{w}} & \hookrightarrow & \widehat{U}_{\mathfrak{w}} & \xrightarrow{j_{\mathfrak{w}}} & \widehat{\Lambda}_{\mathscr{H}_{\mathfrak{w}}} \\ \big\downarrow \hat{\phi} & & \big\downarrow \hat{\phi} & & \big\downarrow \hat{\phi}_{\mathfrak{v}} \\ \widehat{U}^1_{\mathfrak{v}} & \hookrightarrow & \widehat{U}_{\mathfrak{v}} & \xrightarrow{j_{\mathfrak{v}}} & \widehat{\Lambda}_{\mathscr{H}_{\mathfrak{v}}} \end{array}$$

(5) A special case of the above discussion is that $\mathfrak{v} = v$ and $\mathfrak{w} = w$ are 1-vertices. If τ_v and τ_w are inertia generators at v, respectively w, there exists a unique $a_{vw} \in \mathbb{Z}_\ell$ such that $\Phi(\tau_v) = \tau_w^{a_{vw}}$. And we have commutative diagrams dual to each other:

$$\begin{array}{ccc} T_v & \longrightarrow & G \\ \big\downarrow \Phi & & \big\downarrow \Phi \\ T_w & \longrightarrow & H \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widehat{\Lambda}_{\mathscr{H}} & \xrightarrow{j^w} & \mathbb{Z}_\ell \varphi_w \\ \big\downarrow \hat{\phi} & & \big\downarrow a_{vw} \\ \widehat{\Lambda}_{\mathscr{G}} & \xrightarrow{j^v} & \mathbb{Z}_\ell \varphi_v \end{array}$$

where φ_w and φ_v are as in Construction 5. Further, the horizontal maps in the first diagram are the inclusions, and the last vertical map in the second diagram denotes the $\mathbb{Z}_\ell$-morphism defined by $\varphi_w \mapsto a_{vw}\varphi_v$.

Definition/Remark 27 Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-δ morphism of abstract decomposition graphs. We will say that:

(1) Φ is proper if first each $\mathfrak{w}$ corresponds to some $\mathfrak{v}$ and every $\mathfrak{v}$ has an image $\mathfrak{w}$, and second, inductively on $\delta_{\mathscr{G}}$, for every 1-edge v of $\mathscr{G}$ and the corresponding edge w of $\mathscr{H}$ (which could be the trivial edge), the residual morphism $\Phi_v : \mathscr{G}_v \to \mathscr{G}_w$ is a proper one.

(2) Φ defines $\mathscr{H}$ as a level-δ quotient of $\mathscr{G}$, or that $\mathscr{H}$ is a level-δ quotient of $\mathscr{G}$ via Φ, if Φ is proper, and we have $\Phi(G) = H$.

(3) We notice that a level δ morphism $\Phi : \mathscr{G} \to \mathscr{H}$ is a proper morphism iff in the notations from Definition/Remark 24 (3), for every $\mathfrak{w}$ and $\mathfrak{v}$ which correspond to each other one has $\mathscr{V}_{\mathscr{G}_{\mathfrak{v}},\Phi_{\mathfrak{v}}} = \mathscr{V}_{\mathscr{G}_{\mathfrak{v}}}$ and the residual map $\varphi_{\Phi_{\mathfrak{v}}} : \mathscr{V}_{\mathscr{G}_{\mathfrak{v}}} \to \mathscr{V}_{\mathscr{H}_{\mathfrak{w}}}$ is onto.

(4) If $\Phi : \mathscr{G} \to \mathscr{H}$ is a proper morphism, and $\mathfrak{w}$ and $\mathfrak{v}$ correspond to each other, then the corresponding residual morphism $\Phi_{\mathfrak{v}} : \mathscr{G}_{\mathfrak{v}} \to \mathscr{G}_{\mathfrak{w}}$ is proper.

Remarks 28 Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level $\delta > 0$ proper morphism.

(1) Let $\mathfrak{v}$ be a multi-index of $\mathscr{G}$, and let $T_{\mathfrak{v}} \cong \mathbb{Z}_\ell^{\delta_{\mathfrak{v}}}$ be the inertia group at $\mathfrak{v}$, as defined in Remark/Definition 2 (3). Then $\Phi(T_{\mathfrak{v}})$ is a free $\mathbb{Z}_\ell$-module of rank $\delta' \leq \delta_{\mathfrak{v}}$. Suppose that $\delta' \leq \delta$. Then using the *properness* of Φ, one checks by induction on $\delta_{\mathfrak{v}}$ that there exists a unique multi-index $\mathfrak{w}$ such that the following hold: $\Phi(Z_{\mathfrak{v}}) \subseteq Z_{\mathfrak{w}}$ and $\Phi(T_{\mathfrak{v}}) \subseteq T_{\mathfrak{w}}$ are open subgroups. Thus in particular, $\mathfrak{w}$ corresponds to $\mathfrak{v}$.

(2) Denote by $T_{\mathscr{G}}$ the subgroup of G generated by all the inertia elements of G, and define $T_{\mathscr{H}} \subseteq H$ correspondingly. Then in the notation from Definition/Remark 3 (2), Φ gives rise to a commutative diagram as follows:

$$\begin{array}{ccccccccc} 1 \to & T_{\mathscr{G}} & \longrightarrow & G & \longrightarrow & \Pi_{1,\mathscr{G}} & \to 1 \\ & \big\downarrow \Phi & & \big\downarrow \Phi & & \big\downarrow & \\ 1 \to & T_{\mathscr{H}} & \longrightarrow & H & \longrightarrow & \Pi_{1,\mathscr{H}} & \to 1 \end{array}$$

Next suppose that $\mathscr{G}$ and $\mathscr{H}$ are divisorial, and let $\mathfrak{T}_{\mathscr{G}} = (\tau_v)_v$ and $\mathfrak{T}_{\mathscr{H}} = (\tau_w)_w$ be distinguished systems of generators for $\mathscr{G}$, respectively $\mathscr{H}$, which give rise to abstract divisor groups $\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}}$ and $\mathrm{Div}_{\mathfrak{T}_{\mathscr{H}}}$ for $\mathscr{G}$, respectively $\mathscr{H}$, and abstract divisorial lattices $\Lambda_{\mathscr{G}}$ and $\Lambda_{\mathscr{H}}$.

(3) For every w, denote by X_w the set of all the v to which w corresponds. Then X_w is finite non-empty (by the fact that Φ is proper). For every w and $v \in X_w$, there exists a unique $a_{vw} \in \mathbb{Z}_\ell$ such that $\Phi(\tau_v) = \tau_w^{a_{vw}}$. Equivalently, denoting $\mathfrak{B}_{\mathscr{G}} = (\varphi_v)_v$ and $\mathfrak{B}_{\mathscr{H}} = (\varphi_w)_w$ the dual bases to $\mathfrak{T}_{\mathscr{G}} = (\tau_v)_v$ and $\mathfrak{T}_{\mathscr{H}} = (\tau_w)_w$ as defined/introduced at Construction 5, by Remark 26 (4), above, via $\hat{\phi}$ we have

$$\varphi_w \mapsto \textstyle\sum_{v \in X_w} a_{vw} \varphi_v,$$

and therefore $\hat{\phi}$ gives rise to a morphism

$$\mathrm{div}_\Phi : \widehat{\mathrm{Div}}_{\mathscr{H}} \to \widehat{\mathrm{Div}}_{\mathscr{G}}$$

which maps $\mathrm{Div}_{\mathfrak{T}_{\mathscr{H}}} \otimes \mathbb{Z}_\ell$ into $\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}} \otimes \mathbb{Z}_\ell$ and fits into the following commutative diagram:

$$(*) \qquad \begin{array}{ccccccccc} 0 \to & \widehat{U}_{\mathscr{H}} & \longrightarrow & \widehat{\Lambda}_{\mathscr{H}} & \xrightarrow{j^{\mathscr{H}}} & \widehat{\mathrm{Div}}_{\mathscr{H}} & \longrightarrow & \widehat{\mathfrak{Cl}}_{\mathscr{H}} & \to 0 \\ & \big\downarrow \hat{\phi} & & \big\downarrow \hat{\phi} & & \big\downarrow \mathrm{div}_\Phi & & \big\downarrow \mathrm{can} & \\ 0 \to & \widehat{U}_{\mathscr{G}} & \longrightarrow & \widehat{\Lambda}_{\mathscr{G}} & \xrightarrow{j^{\mathscr{G}}} & \widehat{\mathrm{Div}}_{\mathscr{G}} & \longrightarrow & \widehat{\mathfrak{Cl}}_{\mathscr{G}} & \to 0 \end{array}$$

(4) Recall that for every divisorial $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathfrak{T}_{\mathscr{G}}}$ in $\widehat{\Lambda}_{\mathscr{G}}$ one has the following: $\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}} = \Lambda_{\mathfrak{T}_{\mathscr{G}}} \otimes \mathbb{Z}_\ell$, and therefore $\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$ is exactly the preimage of $\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}} \otimes \mathbb{Z}_\ell$ under $j^{\mathscr{G}}$. Hence from the commutative diagram $(*)$ above it follows that

$$(**)\ \widehat{\Lambda}_{\mathscr{H},\mathrm{fin}} = \hat{\phi}^{-1}(\widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}),\ \hat{\phi}(\widehat{\Lambda}_{\mathscr{H},\mathrm{fin}}) \cap \widehat{U}_{\mathscr{G}} = \hat{\phi}(\widehat{U}_{\mathscr{H}}),\ \hat{\phi}(\widehat{\Lambda}_{\mathscr{H},\mathrm{fin}}) = \hat{\phi}(\widehat{\Lambda}_{\mathscr{H}}) \cap \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}.$$

In particular, $\hat{\phi}$ maps finite-corank submodules into such, and preimages of finite-corank submodules under $\hat{\phi}$ are again such.

(5) With the above notation, the following are equivalent:

(a) There exist $\mathfrak{T}_{\mathscr{G}} = (\tau_v)_v$, $\mathfrak{T}_{\mathscr{H}} = (\tau_w)_w$ such that $a_{vw} \in \mathbb{Z}_{(\ell)}$ for all w, $v \in X_w$.

(b) $\mathrm{div}_\Phi(\mathrm{Div}_{\mathfrak{T}_{\mathscr{H}}}) \subseteq \mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}}$.

(c) $\hat{\phi}(\Lambda_{\mathscr{H}}) \subseteq \Lambda_{\mathscr{G}}$.

And if the above equivalent conditions are satisfied, one has equalities as follows:

$$(***)\ \ \mathrm{Div}_{\mathfrak{T}_{\mathscr{H}}} = \mathrm{div}_\Phi^{-1}(\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}}),\ \Lambda_{\mathfrak{T}_{\mathscr{H}}} = \hat{\phi}^{-1}(\Lambda_{\mathfrak{T}_{\mathscr{G}}}),\ \hat{\phi}(\Lambda_{\mathfrak{T}_{\mathscr{H}}}) = \hat{\phi}(\widehat{\Lambda}_{\mathscr{H}}) \cap \Lambda_{\mathfrak{T}_{\mathscr{G}}}.$$

Thus in particular, $\Lambda_{\mathfrak{T}_{\mathscr{H}}}$ can be recovered from $\Lambda_{\mathfrak{T}_{\mathscr{G}}}$ via $\hat{\phi}$.

Proof of (5): The implication (a) $\Rightarrow$ (b) follows immediately from the definition of div_{Φ}, and the implication (b) $\Rightarrow$ (c) follows from the definition of $\Lambda_{\mathscr{H}}$ and $\Lambda_{\mathscr{G}}$. In order to prove (c) $\Rightarrow$ (a), let $v \in X_w$ be given. Then combining Remark 10 (1), with the second diagram from Remark 26 (4), we get a commutative diagram of the form

$$\begin{array}{ccc} \Lambda_{\mathfrak{T}_{\mathscr{H}}} & \xrightarrow{j^w} & \mathbb{Z}_{(\ell)}\varphi_w \\ \downarrow\hat{\phi} & & \downarrow a_{vw} \\ \Lambda_{\mathfrak{T}_{\mathscr{G}}} & \xrightarrow{j^v} & \mathbb{Z}_{(\ell)}\varphi_v \end{array}$$

hence it follows that $a_{vw} \in \mathbb{Z}_{(\ell)}$, as claimed. Finally, let us show that in case the equivalent conditions (a), (b), (c), are satisfied, the equalities $(**)$ hold. First observe that since $\hat{\phi}$ and div_{Φ} are injective, all the above equalities are equivalent. Thus it is enough to prove one of them, say the first one: Recall that by point (3) above, $\mathrm{Div}_{\mathfrak{T}_{\mathscr{H}}} \otimes \mathbb{Z}_\ell$ and $\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}} \otimes \mathbb{Z}_\ell$ are free $\mathbb{Z}_\ell$-modules on the bases $\mathfrak{B}_{\mathscr{H}} = (\varphi_w)_w$, respectively $\mathfrak{B}_{\mathscr{G}} = (\varphi_v)_v$, and that div_{Φ} maps the former $\mathbb{Z}_\ell$-module into the latter one. Hence $\mathrm{div}_{\Phi}^{-1}(\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}}) \subseteq \mathrm{Div}_{\mathfrak{T}_{\mathscr{H}}} \otimes \mathbb{Z}_\ell$. Now let $x = \sum_w b_w \varphi_w$ with $b_w \in \mathbb{Z}_\ell$ be an element of $\mathrm{div}_{\Phi}^{-1}(\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}})$. Then $\mathrm{div}_{\Phi}(x) = \sum_w \sum_{v \in X_w} b_w a_{vw} \varphi_v$ lies in $\mathrm{Div}_{\mathfrak{T}_{\mathscr{G}}}$; hence $b_w a_{vw} \in \mathbb{Z}_{(\ell)}$ for all w and $v \in X_w$. On the other hand, since $a_{vw} \in \mathbb{Z}_{(\ell)}$, it follows that b_w are rational numbers. Since they lie in $\mathbb{Z}_\ell$ too, it follows that $b_w \in \mathbb{Z}_{(\ell)}$. But then we finally get that $x = \sum_w b_w \varphi_w$ lies in $\Lambda_{\mathfrak{T}_{\mathscr{H}}}$ as claimed.

Definition/Remark 29 Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-δ proper morphism of divisorial abstract decomposition graphs with $\delta > 0$.

(1) We say that Φ is divisorial, if all residual morphisms $\Phi_{\mathfrak{v}} : \mathscr{G}_{\mathfrak{v}} \to \mathscr{H}_{\mathfrak{w}}$ with $\mathfrak{w}$ of length $< \delta$ satisfy the equivalent conditions (a), (b), (c) from (5) above.

(2) If $\Phi : \mathscr{G} \to \mathscr{H}$ is divisorial, the residual Kummer morphism $\hat{\phi}_{\mathfrak{v}}$ maps divisorial lattices for $\mathscr{H}_{\mathfrak{w}}$ into divisorial lattices for $\mathscr{G}_{\mathfrak{v}}$. And the commutative diagram $(*)$ from Remarks 28 above gives rise to a commutative sub-diagram:

$$(**) \qquad \begin{array}{ccccccccc} 0 \to & \widehat{U}_{\mathscr{H}} & \longrightarrow & \Lambda_{\mathscr{H}} & \xrightarrow{j^{\mathscr{H}}} & \mathrm{Div}_{\mathscr{H}} & \longrightarrow & \mathfrak{Cl}_{\mathscr{H}} & \to 0 \\ & \downarrow\hat{\phi} & & \downarrow\hat{\phi} & & \downarrow\mathrm{div}_{\Phi} & & \downarrow\mathrm{can} & \\ 0 \to & \widehat{U}_{\mathscr{G}} & \longrightarrow & \Lambda_{\mathscr{G}} & \xrightarrow{j^{\mathscr{G}}} & \mathrm{Div}_{\mathscr{G}} & \longrightarrow & \mathfrak{Cl}_{\mathscr{G}} & \to 0 \end{array}$$

(3) It is not too difficult to give examples of proper morphisms Φ of divisorial abstract decomposition graphs such that Φ are not divisorial. Indeed, one can give such examples even in the case that both $\mathscr{G}$ and $\mathscr{H}$ are complete curve like, and Φ is a proper morphism of level $\delta = 1$. The next proposition shows that actually the case $\delta = 1$ is the "generic" source for proper non-divisorial morphisms.

Proposition 30 *Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-δ proper morphism of divisorial abstract decomposition graphs, where $\delta = \delta_{\mathscr{H}} > 0$. Then the following hold:*

(1) *Let $\mathfrak{v}$, $\mathfrak{w}$ be all pairs of multi-indices of $\mathscr{G}$, respectively of $\mathscr{H}$, such that $\mathfrak{w}$ has length $\delta_{\mathscr{H}} - 1$ and corresponds to $\mathfrak{v}$. Suppose that* for all such pairs $\mathfrak{v}, \mathfrak{w}$ *the residual morphism $\Phi_{\mathfrak{v}} : \mathscr{G}_{\mathfrak{v}} \to \mathscr{H}_{\mathfrak{w}}$ is divisorial. Then $\Phi : \mathscr{G} \to \mathscr{H}$ is divisorial.*

(2) *If Φ is an isomorphism, then $\hat{\phi}$ is an isomorphism too, and Φ is a divisorial morphism of abstract decomposition graphs. Hence for any abstract divisor groups* $\mathrm{Div}_{\mathscr{G}}$ *and* $\mathrm{Div}_{\mathscr{H}}$ *of $\mathscr{G}$, respectively $\mathscr{H}$, there exists $\varepsilon \in \mathbb{Z}_\ell^\times$ such that the diagram below is commutative:*

$$(***)\qquad \begin{array}{ccccccccc} 0 \to & \widehat{U}_{\mathscr{H}} & \longrightarrow & \Lambda_{\mathscr{H}} & \xrightarrow{j^{\mathscr{H}}} & \mathrm{Div}_{\mathscr{H}} & \longrightarrow & \mathfrak{Cl}_{\mathscr{H}} & \to 0 \\ & \downarrow{\scriptstyle \varepsilon\cdot\hat{\phi}} & & \downarrow{\scriptstyle \varepsilon\cdot\hat{\phi}} & & \downarrow{\scriptstyle \varepsilon\cdot \mathrm{div}_{\Phi}} & & \downarrow{\scriptstyle \varepsilon\cdot \mathrm{can}} & \\ 0 \to & \widehat{U}_{\mathscr{G}} & \longrightarrow & \Lambda_{\mathscr{G}} & \xrightarrow{j^{\mathscr{G}}} & \mathrm{Div}_{\mathscr{G}} & \longrightarrow & \mathfrak{Cl}_{\mathscr{G}} & \to 0 \end{array}$$

Proof. To (1): One carries out induction on $\delta_{\mathscr{G}}$.

Case (1) $\delta_{\mathscr{G}} = 1$. Then $1 = \delta_{\mathscr{G}} \geq \delta = \delta_{\mathscr{H}} > 0$; hence all these numbers equal 1, and the assertion follows from/by the definitions and the hypothesis of the proposition.

Case (2) $\delta > 1$ arbitrary. Let v be some 1-index of $\mathscr{G}$, and w the image of v under Φ. Note that w is either the trivial valuation w_0, or otherwise w is a 1-index of $\mathscr{H}$. We show that the resulting residual morphism $\Phi_v : \mathscr{G}_v \to \mathscr{H}_w$ satisfies the hypothesis of the proposition: First $\Phi_v : \mathscr{G}_v \to \mathscr{H}_w$ is a proper morphism, as $\Phi : \mathscr{G} \to \mathscr{H}$ was so by hypothesis. Second, let $\mathfrak{w}_w$ be a multi-index of $\mathscr{H}_w$ of length $\delta_{\mathscr{H}_w} - 1$, and $\mathfrak{v}_w$ a multi-index of $\mathscr{G}_v$ such that $\mathfrak{w}_w$ corresponds to $\mathfrak{v}_w$ under Φ_v.

Claim 1. $\Phi_{\mathfrak{v}_w} : \mathscr{G}_{\mathfrak{v}_w} \to \mathscr{H}_{\mathfrak{w}_w}$ is divisorial.

Indeed, let us first suppose that $w = w_0$ is the trivial valuation. Then $\mathscr{H}_w = \mathscr{H}$, and $\mathfrak{w} := \mathfrak{w}_w$ is a multi-index of $\mathscr{H}$ of length $\delta - 1$. Further, $(\mathfrak{v}_w, v)$ is a multi-index of $\mathscr{G}$ which corresponds to $\mathfrak{w}$. And we have that $\mathscr{G}_{\mathfrak{v}_w} = \mathscr{G}_{(\mathfrak{v}_w,v)}$, $\mathscr{H}_{\mathfrak{w}_w} = \mathscr{H}_{\mathfrak{w}}$, and $\Phi_{\mathfrak{v}_w} : \mathscr{G}_{\mathfrak{v}_w} \to \mathscr{H}_{\mathfrak{w}_w}$ is actually the same as $\Phi_{(\mathfrak{v}_w,v)} : \mathscr{G}_{(\mathfrak{v}_w,v)} \to \mathscr{H}_{\mathfrak{w}}$. But then the claim follows from the hypothesis of the Proposition. Next suppose that $w \neq w_0$ is a 1-index of $\mathscr{H}$. Then $\Phi_{\mathfrak{v}}$ has level $\delta - 1 = \delta_{\mathscr{H}} - 1 = \delta_{\mathscr{H}_w}$. Moreover, if $\mathfrak{w}_w$ is a multi-index of $\mathscr{H}_w$ of length $\delta_{\mathscr{H}_w} - 1$, then $(\mathfrak{w}_w, w)$ is a multi-index of $\mathscr{H}$ of length

$$(\delta_{\mathscr{H}_w} - 1) + 1 = \delta_{\mathscr{H}_w} = \delta_{\mathscr{H}} - 1.$$

And since $\mathfrak{w}_w$ corresponds to $\mathfrak{v}_w$, and w to v, it follows that $(\mathfrak{w}_w, w)$ corresponds to $(\mathfrak{v}_w, v)$. But then by the hypothesis of the proposition, the residual morphism

$$\Phi_{(\mathfrak{v}_w,v)} : \mathscr{G}_{(\mathfrak{v}_w,v)} \to \mathscr{H}_{(\mathfrak{w}_w,w)}$$

is divisorial. On the other hand, by definitions we have identifications $\mathscr{G}_{(\mathfrak{v}_w,v)} = \mathscr{G}_{\mathfrak{v}_w}$ and $\mathscr{H}_{(\mathfrak{w}_w,w)} = \mathscr{H}_{\mathfrak{w}_w}$, and $\Phi_{(\mathfrak{v}_w,v)} = \Phi_{\mathfrak{v}_w}$. This completes the proof of the Claim 1.

Coming back to the proof of assertion 1 of the proposition, let $\Lambda_{\mathscr{H}}$ and $\Lambda_{\mathscr{G}}$ be divisorial lattices in $\widehat{\Lambda}_{\mathscr{H}}$, respectively $\widehat{\Lambda}_{\mathscr{G}}$. For $\Gamma \subset \widehat{\Lambda}_{\mathscr{H}}$ of finite corank and satisfying $\Gamma \cap \widehat{U}_{\mathscr{H}} = (0)$, set $\Delta := \hat{\phi}(\Gamma)$.

Claim 2. $\hat{\phi} : \widehat{\Lambda}_{\mathscr{H}} \to \widehat{\Lambda}_{\mathscr{G}}$ maps Γ isomorphically onto its image Δ, and further one has that $\Delta \cap \widehat{U}_{\mathscr{G}} = (0)$, and Δ is a finite-corank $\mathbb{Z}_\ell$-submodule of $\widehat{\Lambda}_{\mathscr{G}}$.

Indeed, by the diagram $(*)$ from Remark 28 (3), and in the notation from there, we have that $j^{\mathscr{H}}$ is injective on Γ, since $\Gamma \cap \widehat{U}_{\mathscr{H}} = (0)$. Since $\widehat{\mathrm{div}}_{\Phi}$ is injective, it finally follows that $j^{\mathscr{H}}(\Gamma)$ is mapped injectively into $\widehat{\mathrm{Div}}_{\mathscr{G}}$. Therefore, $\hat{\phi}$ maps Γ

injectively into $\widehat{\Lambda}_{\mathscr{G}}$, and $\Delta := \hat{\phi}(\Gamma)$ has trivial intersection with $\widehat{U}_{\mathscr{G}}$. Now let us check that Δ has finite corank in $\widehat{\Lambda}_{\mathscr{G}}$: First let v be a 1-edge of $\mathscr{G}$ such that $j^v(\Gamma) \neq (0)$. Equivalently, $\Delta = \hat{\phi}(\Gamma)$ has a non-trivial image under

$$j^v \circ \hat{\phi} : \widehat{\Lambda}_{\mathscr{H}} \xrightarrow{\hat{\phi}} \widehat{\Lambda}_{\mathscr{G}} \xrightarrow{j^v} \mathbb{Z}_\ell \varphi_v.$$

Hence $j^v \circ \hat{\phi}(\Delta)$ is non-trivial. Since the above sequence is ℓ-adically dual to $T_v \hookrightarrow G \xrightarrow{\Phi} H$, it follows that $\Phi(T_v) \neq 1$ in H. Since Φ is proper by hypothesis, it follows that there exists w such that $\Phi(T_v) \subseteq T_w$, and $\Phi(T_v)$ is open in T_w. Hence finally w corresponds to v. Therefore, if $j^v(\Delta) \neq (0)$, then there exists some $w \neq w_0$ corresponding to v.

Next, by the commutativity of the second diagram in Remark 26 (4), it follows that $j^v(\Delta) \neq (0)$ if and only if $j^w(\Gamma) \neq (0)$. Now since Γ has finite corank, there exist only finitely many valuations w of H such that $j^w(\Gamma) \neq (0)$. Finally, for each such w there exist only finitely many v's such that w corresponds to one of the v's. Thus finally there are only finitely many valuations v of G such that $j^v(\Delta) \neq (0)$. This completes the proof of Claim 2.

Now suppose that Γ is non-trivial. Then we have the following situation: Γ and its isomorphic image Δ are non-trivial finite-corank submodules of $\widehat{\Lambda}_{\mathscr{H}}$, respectively $\widehat{\Lambda}_{\mathscr{G}}$. Since $\Delta \cap \widehat{U}_{\mathscr{G}} = (0)$, it follows that $\Delta \cap \Lambda_{\mathscr{G}}$ is a lattice in Δ which completely determines the divisorial lattice $\Lambda_{\mathscr{G}}$ in the ℓ-adic equivalence class of all the divisorial lattices of $\mathscr{G}$. Correspondingly, the same is true for $\Gamma \cap \widehat{U}_{\mathscr{H}}$ and $\Lambda_{\mathscr{H}}$, etc. On the other hand, since Δ has finite corank, by the ampleness of $\mathscr{G}$, there exist valuations v of G such that the following are satisfied:

(j) $\Delta \subset \widehat{U}_v$ and j_v maps Δ injectively into $\widehat{\Lambda}_{\mathscr{G}_v}$, and for such v set $\Delta_v := j_v(\Delta)$.

(jj) $\Delta_v \cap \widehat{U}_{\mathscr{G}_v} = (0)$, because $\Delta \cap \widehat{U}_{\mathscr{G}} = (0)$ by the discussion above.

For such a valuation v, the lattice $\Delta \cap \Lambda_{\mathscr{G}}$ is mapped by j_v isomorphically onto a lattice in Δ_v. Hence by the properties (i), (ii), from Fact 8, we get that there exists a *unique* divisorial $\widehat{U}_{\mathscr{G}_v}$-lattice $\Lambda_{\mathscr{G}_v}$ of $\mathscr{G}_v$ such that $j_v(\Delta \cap \Lambda_{\mathscr{G}}) = \Delta_v \cap \Lambda_{\mathscr{G}_v}$. For v as above we analyze the following cases:

Case (a): $\Phi(T_v) = 1$. Then the trivial valuation w_0 corresponds to v, and for the residual morphism $\Phi_v : \mathscr{G}_v \to \mathscr{H}$ we have that Φ_v is divisorial by Claim 1. Hence by Remark 28 (5), there exists a unique divisorial $\widehat{U}_{\mathscr{H}}$-lattice $\Lambda_{\mathscr{H}}$ of $\mathscr{H}$ such that the Kummer homomorphism $\hat{\phi}_v : \widehat{\Lambda}_{\mathscr{H}} \to \widehat{\Lambda}_{\mathscr{G}_v}$ maps $\Lambda_{\mathscr{H}}$ into $\Lambda_{\mathscr{G}_v}$.

Case (b): $\Phi(T_v) \neq 1$. Then there is a non-trivial valuation w corresponding to v, and for the corresponding residual morphism $\Phi_v : \mathscr{G}_v \to \mathscr{H}_w$ we have that Φ_v is divisorial by Claim 1. Hence by Remark 28 (5), there exists a unique divisorial $\widehat{U}_{\mathscr{H}_w}$-lattice $\Lambda_{\mathscr{H}_w}$ of $\mathscr{H}_w$ such that the Kummer homomorphism $\hat{\phi}_v : \widehat{\Lambda}_{\mathscr{H}_w} \to \widehat{\Lambda}_{\mathscr{G}_v}$ maps $\Lambda_{\mathscr{H}_w}$ into $\Lambda_{\mathscr{G}_v}$. Moreover, since $\hat{\phi}$ maps Γ isomorphically onto its image Δ, and j_v maps Δ isomorphically onto its image Δ_v, we get that since $j_v \circ \hat{\phi}$ and $j_w \circ \hat{\phi}_v$ coincide on Γ, it follows that j_w maps Γ isomorphically onto its image Γ_w, and that $\hat{\phi}_v$ maps Γ_w isomorphically onto Δ_v. Therefore, w satisfies mutatis mutandis the conditions (j), (jj), above with respect to Γ. Hence $\Gamma_w \cap \Lambda_{\mathscr{H}_w}$ is a lattice in Γ_w. Hence there exists a unique divisorial $\widehat{U}_{\mathscr{H}}$-lattice $\Lambda_{\mathscr{H}}$ of $\mathscr{H}$ such that $\Gamma\cap\Lambda_{\mathscr{H}}$ is mapped isomorphically onto $\Gamma_w\cap\Lambda_{\mathscr{H}_w}$.

Claim 3. In both cases above, $\hat{\phi}$ maps $\Lambda_{\mathscr{H}}$ into $\Lambda_{\mathscr{G}}$.

First, with the notation from above, it is clear by the discussion above that $\hat{\phi}$ maps $\Gamma\cap\Lambda_{\mathscr{H}}$ isomorphically onto $\Delta\cap\Lambda_{\mathscr{G}}$. Now let Γ' be a finite-corank $\mathbb{Z}_\ell$-module such that $\Gamma'\cap\widehat{U}_{\mathscr{H}}=1$ and $\Gamma\subseteq\Gamma'$. Let $\Lambda'_{\mathscr{H}}$ be the divisorial $\widehat{U}_{\mathscr{H}}$-lattice given by the construction above when starting with Γ' instead of Γ. Then we have that $\Gamma\cap\Lambda'_{\mathscr{H}}$ is a lattice in Γ, which is ℓ-adically equivalent to $\Gamma\cap\Lambda_{\mathscr{H}}$. Hence $\hat{\phi}(\Gamma\cap\Lambda_{\mathscr{H}})$ and $\hat{\phi}(\Gamma\cap\Lambda'_{\mathscr{H}})$ are ℓ-adically equivalent lattices in $\Delta=\hat{\phi}(\Gamma)$, and both of them are contained in $\Lambda_{\mathscr{G}}$. Hence $\hat{\phi}(\Gamma\cap\Lambda_{\mathscr{H}})=\hat{\phi}(\Gamma\cap\Lambda'_{\mathscr{H}})$, thus $\Gamma\cap\Lambda_{\mathscr{H}}=\Gamma\cap\Lambda'_{\mathscr{H}}$. Hence finally $\Lambda_{\mathscr{H}}$ and $\Lambda'_{\mathscr{H}}$ are equal. In other words, for every finite-corank $\mathbb{Z}_\ell$-module Γ' of $\mathscr{H}$ as above we have that if $\Gamma\subseteq\Gamma'$, then $\Gamma'\cap\Lambda_{\mathscr{H}}$ is mapped into $\Lambda_{\mathscr{G}}$. But then $\Lambda_{\mathscr{H}}=\widehat{U}_{\mathscr{H}}+\cup_{\Gamma'}(\Gamma'\cap\Lambda_{\mathscr{H}})$ is mapped into $\Lambda_{\mathscr{G}}$, as claimed.

To (2): Since Φ is an isomorphism, it follows that $\delta_{\mathscr{G}}=\delta=\delta_{\mathscr{H}}$, hence Φ gives rise to a bijection of the multi-indices $\mathfrak{v}$ and $\mathfrak{w}$ of length $\delta-1$ of $\mathscr{G}$, respectively of $\mathscr{H}$; and if $\mathfrak{v}$ and $\mathfrak{w}$ are such indices, then the residual morphism $\Phi_{\mathfrak{v}}:\mathscr{G}_{\mathfrak{v}}\to\mathscr{H}_{\mathfrak{w}}$ is by definition an isomorphism of compete curve-like abstract decomposition graphs. Thus by assertion 1, it is sufficient to prove that all $\Phi_{\mathfrak{v}}:\mathscr{G}_{\mathfrak{v}}\to\mathscr{H}_{\mathfrak{w}}$ as above are divisorial. Let $(\sigma_v)_v$ be a distinguished system of inertia generators for $G_{\mathfrak{v}}$, where the v are the 1-edges of $\mathscr{G}_{\mathfrak{v}}$. If w is the 1-edge of $\mathscr{G}_{\mathfrak{w}}$ corresponding to v, then setting $\tau_w:=\Phi_{\mathfrak{v}}(\sigma_v)$, the system $(\tau_w)_w$ is a distinguished system of inertia generators of $\mathscr{H}_{\mathfrak{w}}$. In particular, condition (a) from Remark/Definition 28 (5), is satisfied. Thus $\Phi_{\mathfrak{v}}$ is divisorial by definition, etc. □

4.2 Rational quotients and geometric like abstract decomposition graphs

We begin by first defining rational quotients of divisorial abstract decomposition graphs. The point is that (divisorial) abstract decomposition graphs which arise from geometry have "sufficiently many" rational quotients; and morphisms of (divisorial) abstract decomposition graphs arising from geometry are compatible with the rational quotients. This suggests that for applications, one should consider/study divisorial abstract decomposition graphs endowed with "sufficiently many" rational quotients, and morphisms of such enriched structures.

To begin with, let $\mathscr{G}_\alpha$ be a level-one complete curve-like abstract decomposition graph. Recall the notation from Construction 5, Case $\delta=1$: For every distinguished system of generators $\mathfrak{T}_\alpha=(\tau_v)_v$ of $\mathscr{G}_\alpha$, we have an exact sequence

$$0\to\widehat{U}_{\mathscr{G}_\alpha}\hookrightarrow\Lambda_{\mathfrak{T}_\alpha}\xrightarrow{j^{\mathscr{G}_\alpha}}\mathrm{Div}_{\mathfrak{T}_\alpha}\xrightarrow{\mathrm{can}}\mathfrak{Cl}_{\mathfrak{T}_\alpha}\cong\mathbb{Z}_{(\ell)}\to 0\,.$$

Definition/Remark 31 With the notation from above we define:

(1) A level-one divisorial abstract decomposition graph $\mathscr{G}_\alpha$ is called rational if $\widehat{U}_{\mathscr{G}_\alpha} = (0)$ for some (thus every) distinguished system of inertia generators $\mathfrak{T}_\alpha$ of $\mathscr{G}_\alpha$, as introduced in Construction 5, Case $\delta = 1$.

We notice the following: Since $\widehat{U}_{\mathscr{G}_\alpha} = (0)$, every $\widehat{U}_{\mathscr{G}_\alpha}$-lattice in $\mathscr{G}_\alpha$ is actually a lattice in $\widehat{\Lambda}_{\mathscr{G}_\alpha}$. Let $\mathfrak{T}_\alpha = (\tau_v)_v$ be a distinguished system of inertia generators, and $\mathfrak{B}_\alpha = (\varphi_v)_v$ the corresponding $\mathbb{Z}_{(\ell)}$-basis of $\mathrm{Div}_{\mathfrak{T}_\alpha}$. An element of the form

$$\boldsymbol{x} = \varphi_{v'} - \varphi_v$$

is called a generating element of $\Lambda_{\mathscr{G}_\alpha}$. We set $(\boldsymbol{x})_0 := v'$ and $(\boldsymbol{x})_\infty := v$, and call these the zero, respectively the pole, of $\boldsymbol{x}$. Further, we define

$$\mathscr{P}_v = \{\, \boldsymbol{x} \in \Lambda_{\mathfrak{T}_\alpha} \mid \boldsymbol{x} \text{ generating, and } (\boldsymbol{x})_\infty = v \,\} = \{\, \varphi_{v'} - \varphi_v \mid \text{ all } v' \neq v \,\},$$

and call it a generating set at v for $\Lambda_{\mathfrak{T}_\alpha}$. Clearly, $\mathscr{P}_v$ defines a $\mathbb{Z}_{(\ell)}$-basis of $\Lambda_{\mathfrak{T}_\alpha}$ for every v. And if $\mathfrak{T}'_\alpha = \mathfrak{T}^\varepsilon_\alpha$ is another distinguished system of inertia generators, and $\mathscr{P}'_v$ is correspondingly defined, then $\varepsilon \in \mathbb{Z}_\ell^\times$ is the unique ℓ-adic unit such that $\varepsilon \cdot \mathscr{P}'_v = \mathscr{P}_v$.

(2) Let $\mathscr{G}$ be a level-δ divisorial abstract decomposition graph, where $\delta > 0$. We will consider quotients $\Phi_\alpha : \mathscr{G} \to \mathscr{G}_\alpha$ of $\mathscr{G}$ together with their Kummer homomorphisms $\hat{\phi}_\alpha : \widehat{\Lambda}_{\mathscr{G}_\alpha} \to \widehat{\Lambda}_{\mathscr{G}}$, and in order to simplify notation we set

$$\widehat{\Lambda}_\alpha := \hat{\phi}_\alpha(\widehat{\Lambda}_{\mathscr{G}_\alpha}) \subseteq \widehat{\Lambda}_{\mathscr{G}}.$$

Further, for every multi-index $\mathfrak{v}$ of $\mathscr{G}$, let $j_{\mathfrak{v}} : \widehat{U}_{\mathfrak{v}} \to \widehat{\Lambda}_{\mathfrak{v}}$ be the canonical reduction homomorphism; see Remark 26 for definitions.

With the above notation, we say that $\Phi_\alpha : \mathscr{G} \to \mathscr{G}_\alpha$ is a rational quotient of $\mathscr{G}$, if $\mathscr{G}_\alpha$ is rational, and Φ_α is divisorial and satisfies the following:

(i) For all multi-indices $\mathfrak{v}$ the following hold: If $j_{\mathfrak{v}}$ is non-trivial on $\widehat{\Lambda}_\alpha \cap \widehat{U}_{\mathfrak{v}}$, then $\widehat{\Lambda}_\alpha \subset \widehat{U}_{\mathfrak{v}}$, and $j_{\mathfrak{v}}$ is injective on $\widehat{\Lambda}_\alpha$, or equivalently, $j_{\mathfrak{v}} \circ \hat{\phi}_\alpha$ is injective on $\widehat{\Lambda}_{\mathscr{G}_\alpha}$.

(ii) For every finite $\mathbb{Z}_\ell$-module $\Delta \subset \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$ with $\widehat{U}_{\mathscr{G}} \subseteq \Delta$, there exist 1-edges v such that $j^v(\widehat{\Lambda}_\alpha) \neq 0$ and $\ker(\Delta \xrightarrow{j_v} \widehat{\Lambda}_{\mathscr{G}_v}) = \Delta \cap \widehat{\Lambda}_\alpha$.

Fact 32 *Let $\Phi_\alpha : \mathscr{G} \to \mathscr{G}_\alpha$ be a rational quotient. Then $\widehat{U}_{\mathscr{G}} \cap \widehat{\Lambda}_\alpha = 0$, and one has:*

(1) *Let $\Lambda_{\mathscr{G}}$ be a divisorial $\widehat{U}_{\mathscr{G}}$-lattice in $\widehat{\Lambda}$. Then there exists a unique divisorial lattice $\Lambda_{\mathscr{G}_\alpha}$ in $\widehat{\Lambda}_{\mathscr{G}_\alpha}$ such that $\hat{\phi}_\alpha(\Lambda_{\mathscr{G}_\alpha})$ is contained in $\Lambda_{\mathscr{G}}$. Moreover, the images $\Lambda_\alpha := \hat{\phi}_\alpha(\Lambda_{\mathscr{G}_\alpha})$ can be recovered from $\widehat{\Lambda}_\alpha = \hat{\phi}_\alpha(\widehat{\Lambda}_{\mathscr{G}_\alpha})$ and $\Lambda_{\mathscr{G}}$ as follows:*

$$(*) \qquad \Lambda_\alpha := \hat{\phi}_\alpha(\Lambda_{\mathscr{G}_\alpha}) = \widehat{\Lambda}_\alpha \cap \Lambda_{\mathscr{G}}.$$

(2) *One can recover Λ_α from $\Lambda_{\mathscr{G}}$ using the maps j^v and j_v as follows:*

$$(**) \quad \Lambda_\alpha = \{x \in \Lambda_{\mathscr{G}} \mid \textit{For all } v \textit{ with } j^v(\widehat{\Lambda}_\alpha) \neq 0 \textit{ and } j^v(x) = 0, \textit{ one has } j_v(x) = 0\}.$$

Proof. First, since $\mathscr{G}_\alpha$ is rational, by definition we have $\widehat{U}_{\mathscr{G}_\alpha} = 0$. But then by Remark 28 (3), one has $0 = \hat{\phi}(\widehat{U}_{\mathscr{G}_\alpha}) = \widehat{U}_{\mathscr{G}} \cap \widehat{\Lambda}_\alpha$, as claimed.

To (1): Since Φ_α defines $\mathscr{G}_\alpha$ as a rational quotient of $\mathscr{G}$, it is divisorial (by definition), and $\widehat{U}_{\mathscr{G}_\alpha} = 0$. Hence we can conclude by applying Remark 28 (5).

To (2): Clearly, if $x \in \Lambda_\alpha$, then it satisfies the hypothesis from $(**)$, i.e., for all v with $j^v(\widehat{\Lambda}_\alpha) \neq 0$ and $j^v(x) = 0$ one has $j_v(x) = 0$. For the converse, let $x \in \Lambda_{\mathscr{G}}$ satisfy hypothesis $(**)$, i.e., be such that for all v with $j^v(\widehat{\Lambda}_\alpha) \neq 0$ and $j^v(x) = 0$ one has $j_v(x) = 0$. Since $\widehat{U}_{\mathscr{G}} \cap \widehat{\Lambda}_\alpha = 0$, by condition (ii) in the definition of Φ_α, it follows that there exist v such that $j^v(\widehat{\Lambda}_\alpha) \neq 0$ and j_v is injective on $\widehat{U}_{\mathscr{G}}$. Therefore, by the hypothesis $(**)$, it follows that $x \notin \widehat{U}_{\mathscr{G}}$. By contradiction, suppose that $x \notin \Lambda_\alpha$. Let $\Delta = \widehat{U}_{\mathscr{G}} + \mathbb{Z}_\ell x$. Since $x \in \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$, we have $\Delta \subset \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$, and since $x \notin \widehat{U}_{\mathscr{G}}$, the inclusion $\widehat{U}_{\mathscr{G}} \subset \Delta$ is strict.

Case (a). $\Delta \cap \widehat{\Lambda}_\alpha = (0)$.

Then by property (ii) of Φ_α it follows that there exists v such that $j^v(\widehat{\Lambda}_\alpha) \neq 0$, and $\Delta \subseteq \widehat{U}_v$ and j_v is injective on Δ. In particular, $x \in \widehat{U}_v$ and $j_v(x)$ is non-trivial. Contradiction!

Case (b). $\Delta \cap \widehat{\Lambda}_\alpha \neq (0)$.

Then there exist $u \in \widehat{U}_{\mathscr{G}}$, $b \in \mathbb{Z}_\ell$ and $z \in \widehat{\Lambda}_\alpha$, $z \neq 0$, such that $u + bx = z$. In particular, since $z \in \Delta \subset \Lambda_{\mathscr{G}} \subset \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$, we have $z \in \widehat{\Lambda}_{\mathscr{G},\mathrm{fin}}$. Further, $b \neq 0$, because $\widehat{U}_{\mathscr{G}} \cap \widehat{\Lambda}_\alpha = 0$. Setting $\Delta_0 := \widehat{U}_{\mathscr{G}}$, we have $\Delta_0 \subset \Delta$, and $\Delta_0 \cap \widehat{\Lambda}_\alpha = 0$. Hence there exists v such that $j^v(\widehat{\Lambda}_\alpha) \neq 0$, and $\Delta \subseteq \widehat{U}_v$, and j_v is injective on $\Delta_0 = \widehat{U}_{\mathscr{G}}$. Thus we have

$$j_v(-u) = j_v(z-u) = b\, j_v(x),$$

hence $j_v(u) \neq 0$ iff $j_v(x) \neq 0$, because $b \neq 0$. First, if $u \neq 0$, then $j_v(u) \neq 0$, hence $j_v(x) \neq 0$. Since $j^v(\widehat{\Lambda}_\alpha) \neq 0$, this contradicts the hypothesis $(**)$. Second, if $u = 0$, then $bx = z \in \widehat{\Lambda}_\alpha$. Since $\widehat{\Lambda}_{\mathscr{G}} / \widehat{\Lambda}_\alpha$ is torsion-free, it follows that $x \in \widehat{\Lambda}_\alpha$, as claimed.

□

Definition 33 Let $\mathscr{G}$ be a divisorial abstract decomposition graph, and let $\Lambda_{\mathscr{G}} \subset \widehat{\Lambda}_{\mathscr{G}}$ be a fixed divisorial $\widehat{U}_{\mathscr{G}}$-lattice. Let $\mathfrak{A}_0 = \{\Phi_\alpha\}_\alpha$ be the set of rational quotients of $\mathscr{G}$. For every subset $\mathfrak{A} \subseteq \mathfrak{A}_0$, we define

$$\Lambda_{\mathfrak{A}} = \textstyle\sum_{\Phi_\alpha \in \mathfrak{A}} \Lambda_\alpha$$

as the $\mathbb{Z}_{(\ell)}$-submodule of $\Lambda_{\mathscr{G}} \subset \widehat{\Lambda}_{\mathscr{G}}$ generated by all the Λ_α with $\Phi_\alpha \in \mathfrak{A}$.

(1) We say that $\mathfrak{A}$ is an ample set of rational quotients of $\mathscr{G}$, if the following hold:

(i) For all α, α' one has that if $\Phi_\alpha \neq \Phi_{\alpha'}$, then $\widehat{\Lambda}_\alpha \cap \widehat{\Lambda}_{\alpha'} = (0)$.
(ii) $\Lambda_{\mathfrak{A}} \cap \widehat{U}_{\mathscr{G}} = (0)$ and $\Lambda_{\mathfrak{A}}$ is ℓ-adically dense in $\widehat{\Lambda}_{\mathscr{G}}$.

(2) Suppose that $\mathfrak{A}$ is an ample set of rational quotients of $\mathscr{G}$. We will say that $\mathscr{G}$ is geometric like with respect to $\mathfrak{A}$ if for every α, α' there exists a multi-index $\mathfrak{v}$ of $\mathscr{G}$ such that:

(j) $\widehat{\Lambda}_\alpha$ and $\widehat{\Lambda}_{\alpha'}$ are contained in $\widehat{U}_{\mathfrak{v}}$.
(jj) $j_{\mathfrak{v}}$ maps $\widehat{\Lambda}_\alpha$ and $\widehat{\Lambda}_{\alpha'}$ injectively into $\Lambda_{\mathscr{G}_{\mathfrak{v}}}$, and $j_{\mathfrak{v}}(\widehat{\Lambda}_\alpha) = j_{\mathfrak{v}}(\widehat{\Lambda}_{\alpha'})$.

(3) In the above context, we will call $\Lambda_{\mathfrak{A}}$ an $\mathfrak{A}$-arithmetical lattice. Its ℓ-adic equivalence class depends in general on $\mathfrak{A}$, and not only on equivalence class of $\Lambda_{\mathscr{G}}$. Further,

$$\widehat{U}_{\mathscr{G}} + \Lambda_{\mathfrak{A}} \subseteq \Lambda_{\mathscr{G}}$$

is a $\widehat{U}_{\mathscr{G}}$-lattice in $\widehat{\Lambda}_{\mathscr{G}}$, and therefore $\Lambda_{\mathscr{G}} \,/\, (\widehat{U}_{\mathscr{G}} + \Lambda_{\mathfrak{A}})$ is a torsion free divisible group, hence a $\mathbb{Q}$-vector space. But in general, $\widehat{U}_{\mathscr{G}} + \Lambda_{\mathfrak{A}}$ is not necessarily a divisorial $\widehat{U}_{\mathscr{G}}$-lattice.

Definition/Remark 34 Let $\mathscr{G}$ and $\mathscr{H}$ be geometric-like abstract decomposition graphs with respect to some sets of rational quotients $\mathfrak{A}_0 = \{\Phi_\alpha\}_\alpha$, respectively $\mathfrak{B}_0 = \{\Phi_\beta\}_\beta$, and let a proper morphism $\Phi : \mathscr{G} \to \mathscr{H}$ of level $\delta := \delta_H$ be given.

(1) We say that Φ is compatible with rational quotients if there exist ample subsets $\mathfrak{A} \subseteq \mathfrak{A}_0$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$ satisfying the following: First, $\mathscr{G}$ and $\mathscr{H}$ are geometric-like with respect to $\mathfrak{A}$, respectively $\mathfrak{B}$. Second, for each $\Psi_\beta \in \mathfrak{B}$ there exist $\Phi_\alpha \in \mathfrak{A}$ and an isomorphism $\Phi_{\alpha\beta} : \mathscr{G}_\alpha \to \mathscr{H}_\beta$ such that the following diagram is commutative:

$$(*) \qquad \begin{array}{ccc} \mathscr{G} & \xrightarrow{\Phi} & \mathscr{H} \\ \downarrow{\scriptstyle \Phi_\alpha} & & \downarrow{\scriptstyle \Psi_\beta} \\ \mathscr{G}_\alpha & \xrightarrow{\Phi_{\alpha\beta}} & \mathscr{H}_\beta \end{array}$$

(2) We observe that in the above context, for every $\Psi_\beta \in \mathfrak{B}$ there exists a unique Φ_α satisfying hypothesis $(*)$. Indeed, if $\Phi_{\alpha'}$ together with $\Phi_{\alpha'\beta}$ also satisfy hypothesis $(*)$, then $\widehat{\Lambda}_{\mathscr{G}_\alpha} = \hat{\phi}_{\alpha\beta}(\widehat{\Lambda}_{\mathscr{H}_\beta})$, and therefore we get

$$\widehat{\Lambda}_\alpha := \hat{\phi}_\alpha(\widehat{\Lambda}_{\mathscr{G}_\alpha}) = \hat{\phi}_\alpha\big(\hat{\phi}_{\alpha\beta}(\widehat{\Lambda}_{\mathscr{H}_\beta})\big) = \hat{\phi}\big(\hat{\psi}_\beta(\widehat{\Lambda}_{\mathscr{H}_\beta})\big) = \hat{\phi}(\widehat{\Lambda}_\beta).$$

Since the same is true correspondingly for α', we finally get $\widehat{\Lambda}_\alpha = \hat{\phi}(\widehat{\Lambda}_\beta) = \widehat{\Lambda}_{\alpha'}$. But then by Definition 33 (1) (i) it follows that $\Phi_\alpha = \Phi_{\alpha'}$, as claimed.

In the above context, we say that α corresponds to β if the hypothesis $(*)$ is satisfied for Ψ_β and Φ_α. Thus α corresponds to β if and only if $\hat{\phi}(\widehat{\Lambda}_\beta) = \widehat{\Lambda}_\alpha$.

Proposition 35 *In the above context, let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-δ proper morphism of geometric-like abstract decomposition graphs which is compatible with the rational quotients $\mathfrak{A}$ and $\mathfrak{B}$, where $\delta := \delta_H$. Then Φ is divisorial.*

(1) *More precisely, let $\hat{\phi} : \widehat{\Lambda}_{\mathscr{H}} \to \widehat{\Lambda}_{\mathscr{H}}$ be the Kummer homomorphism of Φ. Let $\Lambda_{\mathfrak{B}}$ be an arithmetical lattice for $\mathscr{H}$ defined by $\mathfrak{B}$. Then there exists a unique arithmetical lattice $\Lambda_{\mathfrak{A}}$ for $\mathscr{G}$ defined by $\mathfrak{A}$ such that $\hat{\phi}(\Lambda_{\mathfrak{B}}) \subseteq \Lambda_{\mathfrak{A}}$, and one has*

$$\hat{\phi}(\Lambda_{\mathfrak{B}}) = \hat{\phi}(\widehat{\Lambda}_{\mathscr{H}}) \cap \Lambda_{\mathfrak{A}}.$$

(2) *Suppose that $\hat{\phi}(\Lambda_{\mathfrak{B}}) \subseteq \Lambda_{\mathfrak{A}}$, and for each α and β consider the unique divisorial lattices $\Lambda_{\mathscr{H}_\beta} \subset \widehat{\Lambda}_{\mathscr{H}_\beta}$ and $\Lambda_{\mathscr{G}_\alpha} \subset \widehat{\Lambda}_{\mathscr{G}_\alpha}$ such that $\Lambda_\beta := \hat{\phi}_\beta(\Lambda_{\mathscr{H}_\beta}) = \hat{\phi}_\beta(\widehat{\Lambda}_{\mathscr{H}_\beta}) \cap \Lambda_{\mathfrak{B}}$ and $\Lambda_\alpha := \hat{\phi}_\alpha(\Lambda_{\mathscr{G}_\alpha}) = \hat{\phi}_\alpha(\widehat{\Lambda}_{\mathscr{G}_\alpha}) \cap \Lambda_{\mathfrak{A}}$. Then for all α, β it follows that $\Phi_\alpha \in \mathfrak{A}$ corresponds to $\Psi_\beta \in \mathfrak{B}$ if and only if $\hat{\phi}_{\alpha\beta}(\Lambda_{\mathscr{H}_\beta}) = \Lambda_{\mathscr{G}_\alpha}$ and $\hat{\phi}(\Lambda_\beta) = \Lambda_\alpha$.*

Proof. It is clear that assertion (2) follows from assertion (1) and previous discussion. Therefore we will concentrate on the proof of assertion (1).

First recall that by Definition/Remark 34 (2), we have that α corresponds to β if and only if $\hat{\phi}(\widehat{\Lambda}_\beta) = \widehat{\Lambda}_\alpha$. Using this we deduce the following:

- $\hat{\phi}$ maps $\widehat{\Lambda}_{\mathfrak{B}} := \sum_{\Psi_\beta \in \mathfrak{B}} \widehat{\Lambda}_\beta$ into $\widehat{\Lambda}_{\mathfrak{A}} := \sum_{\Phi_\alpha \in \mathfrak{A}} \widehat{\Lambda}_\alpha$.
- Let $\Lambda_{\mathfrak{B}}$ and $\Lambda_{\mathfrak{A}}$ be fixed arithmetical lattices of $\mathscr{H}$, respectively $\mathscr{G}$. For a given β, choose α corresponding to it. By Definition/Remark 31 (3) above, and with the notation from there we have that there exists a unique divisorial lattice $\Lambda_{\mathscr{G}_\alpha}$ in $\widehat{\Lambda}_{\mathscr{G}_\alpha}$ such that $\hat{\phi}_\alpha$ maps $\Lambda_{\mathscr{G}_\alpha}$ into $\Lambda_{\mathfrak{A}}$, and actually $\hat{\phi}_\alpha(\Lambda_{\mathscr{G}_\alpha}) = \widehat{\Lambda}_\alpha \cap \Lambda_{\mathfrak{A}}$. And correspondingly, the same is true for β, i.e., there exists a unique $\Lambda_{\mathscr{H}_\beta}$ in $\widehat{\Lambda}_{\mathscr{H}_\beta}$ such that $\hat{\psi}_\beta(\Lambda_{\mathscr{H}_\beta}) = \widehat{\Lambda}_\beta \cap \Lambda_{\mathfrak{B}}$.

Since α corresponds to β, with the notation from Definition 34, let $\hat{\phi}_{\alpha\beta}$ be the Kummer isomorphism defined by $\Phi_{\alpha\beta}$. Then $\hat{\phi}_{\alpha\beta}(\Lambda_{\mathscr{H}_\beta})$ is a divisorial lattice in $\widehat{\Lambda}_{\mathscr{G}_\alpha}$. Thus there exists an ℓ-adic unit $\varepsilon_{\alpha\beta}$ such that

$$\hat{\phi}_{\alpha\beta}(\Lambda_{\mathscr{H}_\beta}) = \varepsilon_{\alpha\beta} \cdot \Lambda_{\mathscr{G}_\alpha}.$$

On the other hand, the commutativity of the diagram $(*)$ from Definition/Remark 34 translated in terms of Kummer homomorphisms means that the above equality is equivalent to the following: For all β and its corresponding α one has

$$(\alpha\beta) \qquad \hat{\phi}(\Lambda_\beta) = \varepsilon_{\alpha\beta} \cdot \Lambda_\alpha.$$

Let β and β', and the corresponding α and α' be given. Hence $\hat{\phi}$ maps $\widehat{\Lambda}_\beta$ and $\widehat{\Lambda}_{\beta'}$ isomorphically onto $\widehat{\Lambda}_\alpha$, respectively $\widehat{\Lambda}_{\beta'}$. Since $\mathscr{G}$ is geometric-like with respect to the family of rational projections $\mathfrak{A}$, it follows that there exists some multi-index $\mathfrak{v}$ of $\mathscr{G}$ which has the properties (j), (jj), of Definition 33 (2).

Before moving on, we recall that by Fact 8 (2), the fixed divisorial $\widehat{U}_{\mathscr{G}}$-lattice $\Lambda_{\mathscr{G}}$ of $\mathscr{G}$ defines uniquely a $\mathfrak{v}$-residual $\widehat{U}_{\mathscr{G}_\mathfrak{v}}$-lattice $\Lambda_{\mathscr{G}_\mathfrak{v}}$ by setting

$$\Lambda_{\mathscr{G}_\mathfrak{v}} := \widehat{U}_{\mathscr{G}_\mathfrak{v}} + j_\mathfrak{v}(\Lambda_{\mathscr{G}} \cap \widehat{U}_\mathfrak{v}).$$

We further remark that condition (j) from Definition 33 (2) implies that $\Phi_\alpha(T_\mathfrak{v}) = 1$. Hence Φ_α gives rise to a residual morphism $\Phi_{\mathfrak{v}\alpha} : \mathscr{G}_\mathfrak{v} \to \mathscr{G}_\alpha$. And if $\hat{\phi}_{\mathfrak{v}\alpha} : \widehat{\Lambda}_{\mathscr{G}_\alpha} \to \Lambda_{\mathscr{G}_\mathfrak{v}}$ is the Kummer homomorphism of $\Phi_{\mathfrak{v}\alpha}$, then $j_\mathfrak{v} \circ \hat{\phi}_\alpha = \hat{\phi}_{\mathfrak{v}\alpha}$. Therefore we have

$$j_\mathfrak{v}(\widehat{\Lambda}_\alpha) = \hat{\phi}_{\mathfrak{v}\alpha}(\Lambda_{\mathscr{G}_\alpha}), \quad j_\mathfrak{v}(\Lambda_\alpha) = \hat{\phi}_{\mathfrak{v}\alpha}(\Lambda_{\mathscr{G}_\alpha}).$$

Now since Φ_α is divisorial, $\Phi_{\mathfrak{v}\alpha}$ is so by definition. Hence by Remark 28, 5), we have:

$$\hat{\phi}_{\mathfrak{v}\alpha}(\Lambda_{\mathscr{G}_\alpha}) = \hat{\phi}_{\mathfrak{v}\alpha}(\widehat{\Lambda}_{\mathscr{G}_\alpha}) \cap \Lambda_{\mathscr{G}_\mathfrak{v}}.$$

Thus combining the assertions above, we finally get

$$j_{\mathfrak{v}}(\Lambda_\alpha) = j_{\mathfrak{v}}(\widehat{\Lambda}_\alpha) \cap \Lambda_{\mathscr{G}_{\mathfrak{v}}}\,.$$

On the other hand, both α and α' satisfy condition j) from Definition 33 (2). Hence by symmetry, the equalities above hold correspondingly for α' too. And since by condition jj) of Definition 33 (2), one has $\hat{\phi}_{\mathfrak{v}\alpha}(\widehat{\Lambda}_{\mathscr{G}_\alpha}) =: \widehat{\Lambda}_{\mathfrak{v},\alpha\alpha'} := \hat{\phi}_{\mathfrak{v}\alpha'}(\widehat{\Lambda}_{\alpha'})$, we get

$$(\alpha) \qquad\qquad j_{\mathfrak{v}}(\Lambda_\alpha) = \widehat{\Lambda}_{\mathfrak{v},\alpha\alpha'} \cap \Lambda_{\mathscr{G}_{\mathfrak{v}}} = j_{\mathfrak{v}}(\Lambda_{\alpha'})\,.$$

On the other hand, since Φ is proper, there exists some $\mathfrak{w}$ corresponding to $\mathfrak{v}$. Recall the second diagram in Remark 26 (3), from which we bring forward

$$\begin{array}{ccc} \widehat{U}_{\mathfrak{w}} & \xrightarrow{j_{\mathfrak{w}}} & \widehat{\Lambda}_{\mathscr{H}_{\mathfrak{w}}} \\ \downarrow{\scriptstyle\hat{\phi}} & & \downarrow{\scriptstyle\hat{\phi}_{\mathfrak{v}}} \\ \widehat{U}_{\mathfrak{v}} & \xrightarrow{j_{\mathfrak{v}}} & \widehat{\Lambda}_{\mathscr{G}_{\mathfrak{v}}} \end{array}$$

and recall that $\hat{\phi}$, $\hat{\phi}_{\mathfrak{v}}$ are injective. Since $\widehat{\Lambda}_\beta = \hat{\phi}(\widehat{\Lambda}_{\mathscr{G}_\alpha})$, $\widehat{\Lambda}_{\beta'} = \hat{\phi}(\widehat{\Lambda}_{\mathscr{G}_{\alpha'}})$, we get:

(c) $\widehat{\Lambda}_\beta, \widehat{\Lambda}_{\beta'} \subset \widehat{U}_{\mathfrak{w}}$.

(d) $\widehat{\Lambda}_\beta$ and $\widehat{\Lambda}_{\beta'}$ are mapped by $j_{\mathfrak{w}} : \widehat{U}_{\mathfrak{w}} \to \widehat{\Lambda}_{\mathscr{H}_{\mathfrak{w}}}$ injectively into $\widehat{\Lambda}_{\mathscr{H}_{\mathfrak{w}}}$, and have equal images $j_{\mathfrak{w}}(\widehat{\Lambda}_\beta) =: \widehat{\Lambda}_{\mathfrak{w},\beta\beta'} := j_{\mathfrak{w}}(\widehat{\Lambda}_{\beta'})$.

And note that $\hat{\phi}_{\mathfrak{v}}$ maps $\Lambda_{\mathfrak{w},\beta\beta'}$ isomorphically onto $\Lambda_{\mathfrak{v},\alpha\alpha'}$. Then going through the same steps as above and using notation correspondingly, we get as above

$$(\beta) \qquad\qquad j_{\mathfrak{w}}(\Lambda_\beta) = j_{\mathfrak{w}}(\Lambda_{\beta'})\,.$$

We conclude the proof of the proposition as follows: For β, β' and α, α' corresponding to them, with the notation from above, we have by relation (α) above,

$$\varepsilon_{\alpha\beta} \cdot \Lambda_\alpha = \hat{\phi}(\Lambda_\beta) \quad \text{and} \quad \varepsilon_{\alpha'\beta'} \cdot \Lambda_{\alpha'} = \hat{\phi}(\Lambda_{\beta'})$$

for some ℓ-adic units $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\alpha'\beta'}$. Applying $j_{\mathfrak{v}}$ to the above equalities, and taking into account that by the commutativity of the diagram above one has $j_{\mathfrak{v}} \circ \hat{\phi} = \hat{\phi}_{\mathfrak{v}} \circ j_{\mathfrak{w}}$ on $\widehat{U}_{\mathfrak{w}}$, thus on $\Lambda_\beta, \Lambda_{\beta'} \subset \widehat{U}_{\mathfrak{w}}$, we finally get

$$j_{\mathfrak{v}}(\varepsilon_{\alpha\beta} \cdot \Lambda_\alpha) = j_{\mathfrak{v}}\big(\hat{\phi}(\Lambda_\beta)\big) = (j_{\mathfrak{v}} \circ \hat{\phi})(\Lambda_\beta)\big) = (\hat{\phi}_{\mathfrak{v}} \circ j_{\mathfrak{w}})(\Lambda_\beta)\big) = \hat{\phi}_{\mathfrak{v}}\big(j_{\mathfrak{w}}(\Lambda_\beta)\big)$$

and correspondingly

$$j_{\mathfrak{v}}(\varepsilon_{\alpha'\beta'} \cdot \Lambda_{\alpha'}) = j_{\mathfrak{v}}\big(\hat{\phi}(\Lambda_{\beta'})\big) = (j_{\mathfrak{v}} \circ \hat{\phi})(\Lambda_{\beta'})\big) = (\hat{\phi}_{\mathfrak{v}} \circ j_{\mathfrak{w}})(\Lambda_{\beta'})\big) = \hat{\phi}_{\mathfrak{v}}\big(j_{\mathfrak{w}}(\Lambda_{\beta'})\big).$$

On the other hand, $j_{\mathfrak{w}}(\Lambda_\beta) = j_{\mathfrak{w}}(\Lambda_{\beta'})$ by remark (β) above; hence the last two terms of the equalities above are equal. Thus we get

$$j_{\mathfrak{v}}(\varepsilon_{\alpha\beta} \cdot \Lambda_\alpha) = j_{\mathfrak{v}}(\varepsilon_{\alpha'\beta'} \cdot \Lambda_{\alpha'}), \quad \text{hence} \quad \varepsilon_{\alpha\beta} \cdot j_{\mathfrak{v}}(\Lambda_\alpha) = \varepsilon_{\alpha'\beta'} \cdot j_{\mathfrak{v}}(\Lambda_{\alpha'}).$$

On the other hand, $j_{\mathfrak{v}}(\Lambda_\alpha) = j_{\mathfrak{v}}(\Lambda_{\alpha'})$, by equalities (α) above. Thus finally

$$\varepsilon_{\alpha\beta} \cdot j_{\mathfrak{v}}(\Lambda_\alpha) = \varepsilon_{\alpha'\beta'} \cdot j_{\mathfrak{v}}(\Lambda_\alpha)\,.$$

Next recall that if $\hat{\phi}_{\mathfrak{v}\alpha} : \widehat{\Lambda}_{\mathscr{G}_\alpha} \to \widehat{\Lambda}_{\mathscr{G}_{\mathfrak{v}}}$ is the Kummer homomorphism of the residual morphism $\Phi_{\mathfrak{v}\alpha} : \mathscr{G}_{\mathfrak{v}} \to \mathscr{G}_\alpha$, then we have $j_{\mathfrak{v}}(\Lambda_\alpha) = \hat{\phi}_{\mathfrak{v}\alpha}(\Lambda_{\mathscr{G}_\alpha})$, and the latter is a $\widehat{U}_{\mathscr{G}_{\mathfrak{v}}}$-sublattice of $\Lambda_{\mathscr{G}_{\mathfrak{v}}}$. Hence finally $\varepsilon_{\alpha\beta}/\varepsilon_{\alpha'\beta'}$ must be a rational ℓ-adic unit. Since β, β' were arbitrary, we conclude that for every fixed β_0 and the corresponding α_0, after setting $\varepsilon := \varepsilon_{\alpha_0\beta_0}$, one has $\hat{\phi}(\Lambda_\beta) = \varepsilon \cdot \Lambda_\alpha$. Equivalently, $\hat{\phi}$ maps $\Lambda_{\mathfrak{B}} = \sum_\beta \Lambda_\beta$ into $\varepsilon \cdot \Lambda_{\mathfrak{A}} = \varepsilon \cdot \sum_\beta \Lambda_\alpha$. $\square$

5 Morphisms arising from algebraic geometry

5.1 Morphisms

Let k and l be algebraically closed fields of characteristic $\neq \ell$. Let $K|k$ and $L|l$ be function fields, and let

$$\imath : L|l \hookrightarrow K|k$$

be an embedding of function fields such that l is mapped isomorphically onto k, and $K|\imath(L)$ is a separable field extension; see e.g., Lang [17] for a thorough discussion of this situation.

As defined in the introduction, let $\mathscr{D}_K^{\rm tot}$ and $\mathscr{D}_L^{\rm tot}$ be the total graphs of prime divisors on K, respectively on L. Then $\imath$ gives rise in a canonical way to a morphism of the total prime divisor graphs

$$\varphi_\imath : \mathscr{D}_K^{\rm tot} \to \mathscr{D}_L^{\rm tot}\,.$$

The precise definition of $\varphi_\imath$ is as follows: First let v be a prime divisor of $K|k$. Then either the restriction $v_L := v|_L$ of v to $L|l$ is the trivial valuation w_0 of $L|l$, or v_L is a prime divisor of $L|l$ otherwise. In both cases, $\imath$ gives rise to an embedding of the residue function fields

$$\imath_v : Lv_L|\,l \hookrightarrow Kv\,|\,k\,.$$

Inductively, we deduce from this that if $\mathfrak{v} = v_r \circ \cdots \circ v_1$ is a prime r-divisor of $K|k$ as defined in the introduction, then $\mathfrak{w} := \mathfrak{v}|_L$ is a prime s-divisor of $L|l$ for some non-negative integer $s \leq r$. Moreover, by general valuation theory, it follows that every generalized prime divisor of $L|l$ is the restriction of some generalized prime divisor of $K|k$; hence $\varphi_\imath$ is surjective, etc.

The situation will become clearer after we analyze in more detail how *geometric prime divisor graphs* $\mathscr{D}_K$ of $K|k$ behave under $\varphi_\imath$.

First, observe that if $K|\imath(L)$ is finite, then for every generalized prime divisor $\mathfrak{w}$ of $L|l$, its fiber is finite of cardinality bounded by $[K : \imath(L)]$. From this one immediately deduces that the image of every geometric decomposition graph for $K|k$ under $\varphi_\imath$ is a geometric decomposition graph for $L|l$, etc.

Therefore, let us assume from now on that $K|\imath(L)$ is not algebraic. Then denoting by $K_1|k$ the relative algebraic closure of $\imath(L)$ in K, we have that $K_1|\imath(L)$ is finite separable, and $K|K_1$ is a regular function field extension. The situation of $L|l \hookrightarrow K_1|k$ was explained above. Thus mutatis mutandis, let $K|\imath(L)$ be a *regular field extension.*

Lemma 36 *Let X be a projective normal model for $K|k$, and $D \supseteq D_X$ a set of prime divisors with $D \backslash D_X$ finite. Then there exist a projective normal model $\tilde{X}$ for $K|k$ and a dominant k morphism $\phi : \tilde{X} \to X$ such that $D \subset D_{\tilde{X}}$; hence D is geometric.*

Proof. Clear. □

Using the lemma above, we have that there exist projective normal models $X \to k$ for $K|k$ such that D_X contains the 1-edges of $\mathscr{D}_K$ and X is complete regular-like. And correspondingly, the same holds for $L|l$ and $\mathscr{D}_L$. On the other hand, the regular embedding of function fields $\imath : L|l \to K|k$ is the generic fiber of a dominant rational map $f : X \dashrightarrow Y$ which factors through $\imath : l \to k$. And note that since X and Y are normal, f is defined at all points x_1 of codimension one of X. Moreover, replacing $X \to k$ by a properly chosen blowup, and normalizing the resulting k-variety, we can suppose that $f : X \to Y$ is a k-morphism of projective normal varieties. And since $K|\imath(L)$ is a regular field extension, it follows that $f : X \to Y$ has geometric generic integral fibers. Hence by the characterization of (the dimension of) the fibers the following hold:

• At almost all points x_1 of codimension one in X, $f(x_1)$ is either the generic point of Y, or $y_1 = f(x_1)$ is a point of codimension one of Y otherwise.

• On a Zariski open subset $V \subset Y$, the fiber X_y at $y \in V$ is irreducible, and if $\overline{X}_y \subseteq X$ is the Zariski closure, one has the following:

$$\operatorname{codim}(y) + \dim(\overline{X}_y) = \dim(X)\,.$$

Hence for almost all points y_1 of codimension one in Y, the closure of the fiber $\overline{X}_{y_1}$ is irreducible and has $\dim(\overline{X}_{y_1}) = \dim(X) - 1$. Equivalently, $\overline{X}_{y_1}$ is a Weil prime divisor of X, and its generic point x_1 has codimension one in X and is mapped to y_1.

In birational terms this means the following: For every prime divisor $v = v_{x_1} \in D_X$ let $w := v|_L = \varphi_\imath(v)$ be its restriction to L. Then the center of w on Y is $y_1 = f(x_1)$, and one of the following holds:

(a) w is the trivial valuation of $L|l$. This is so iff y_1 is the generic point of Y.

(b) w is a prime divisor of $L|l$. Then either y_1 has codimension one in Y, and if so, then w is the Weil prime divisor defined by y_1, or y_1 has codimension > 1.

In particular, we see that the following hold: First, all $w \in D_Y$ have preimages v in D_X, and for almost all w the preimage v is unique. Second, there are at most finitely many "exceptional" $v \in D_X$ for which $\varphi_\imath(v)$ does lie in D_Y. Let Σ_f be that set.

We now claim that for the given projective models $X \to k$ and $Y \to l$ as above, there exist quasi-projective normal models $\tilde{X} \to k$ and $\tilde{Y} \to l$ dominating $X \to k$ and $Y \to l$, and a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ above $f : X \to Y$, and having the following property:

$$(*) \qquad \varphi_\imath(D_{\tilde X}\cup\{v_0\}) = D_{\tilde Y}\cup\{w_0\},$$

where v_0 and w_0 are the trivial valuations. In particular, $D_X \subseteq D_{\tilde X}$ and $D_Y \subseteq D_{\tilde Y}$. Indeed, if $\varphi_\imath(D_X\cup\{v_0\}) = D_Y\cup\{w_0\}$, i.e., if the exceptional set Σ_f is empty, then there is nothing to prove. Hence consider some $v := v_{x_1} \in \Sigma_f$ such that the center y_v of $w = \varphi_\imath(v)$ has codimension > 1. Let $Y_v \subset Y$ be the closure of y_v in Y. Setting $Y_1 := Y$, and $Z_1 := Y_v$, we consider a sequence of blowups $\dots \to Y_{n+1} \to Y_n \to \dots$ as follows: $Z_n \subset Y_n$ is the closure of the center of w on Y_n. We stop if Z_n has codimension 1, and blow up Z_n otherwise. Then the above sequence is finite. Moreover, if $\mathrm{codim}(Z_n) > 1$, then $Y_{n+1} \to Y_n$ is an isomorphism outside Z_n. But then if the process above stops say at Y_n, it follows that Z_n is the center of w on Y_n, and $\mathrm{codim}(Z_n) = 1$. An easy Noether induction shows that one gets models $\tilde Y'$ dominating Y such that $\varphi_\imath(D_X) \subseteq D_{\tilde Y'}\cup\{w_0\}$. On the other hand, $f : X \to Y$ can be interpreted as a dominant rational map $\tilde f : X \dashrightarrow \tilde Y'$. Since X is normal, and $\tilde Y'$ is complete, $\tilde f$ is defined at all points $v \in D_X$ and maps these points into $D_{\tilde Y}$ by the discussion above. To conclude, let $S_Y \subset \tilde Y'$ be the Zariski closure of the (finite) complement of $D_{\tilde Y'}\backslash\varphi_\imath(D_X)$, and S_X the preimage of S_Y under f. Finally, set $\tilde Y := \tilde Y'\backslash S_Y$, and $\tilde X = U(f)\backslash S_X$, where $U(f)$ is the the domain of f. Then by the choices made, it follows that f defines a dominant morphism $\tilde f : \tilde X \to \tilde Y$ which has the required property $(*)$.

Now using the fact $(*)$ above and proceeding by induction on the transcendence degree of the residual function fields $L\mathfrak{w}|l \hookrightarrow K\mathfrak{v}|k$, a straightforward Noether induction argument shows finally the following.

Proposition 37 *In the above context, let $\mathscr{D}_K \subset \mathscr{D}_K^{\mathrm{tot}}$ and $\mathscr{D}_L \subset \mathscr{D}_L^{\mathrm{tot}}$ be geometric graphs of prime divisors for $K|k$, respectively $L|l$. Then there exists a unique maximal geometric subgraph $\mathscr{D}'_K \subset \mathscr{D}_K$ such that $\varphi_\imath$ defines by restriction a morphism of graphs of prime divisors*

$$\varphi_\imath : \mathscr{D}'_K \to \mathscr{D}_L.$$

Moreover, for given geometric graphs $\mathscr{D}_K \subset \mathscr{D}_K^{\mathrm{tot}}$ and $\mathscr{D}_L \subset \mathscr{D}_L^{\mathrm{tot}}$ as above, there exist geometric graphs of prime divisors $\mathscr{D}_K^0 \supseteq \mathscr{D}_K$ and $\mathscr{D}_L^0 \supseteq \mathscr{D}_L$ for $K|k$, respectively $L|l$, such that $\varphi_\imath$ defines by restriction a surjective morphism of graphs of prime divisors

$$\varphi_\imath : \mathscr{D}_K^0 \to \mathscr{D}_L^0.$$

Using Galois theory and decomposition theory of valuations, the above facts have the following translation in terms of abstract decomposition graphs: Let $\imath' : L' \to K'$ be a prolongation of $\imath : L|l \to K|k$ to L', and let

$$\Phi_\imath : \Pi_K \to \Pi_L$$

be the corresponding canonical projection of Galois groups. Then since $\imath : L|l \to K|k$ is a morphism of function fields, it follows that the relative algebraic closure L_1 of $L|l$ in $K|k$ is a finite extension of L, thus a function field over l. But then it follows that $\Phi_\imath$ is an open homomorphism.

Moreover, if $\varphi_\imath(v) = w$, and v' is a prolongation of v to K', then the restriction w' of v' to L' satisfies, first, that w' is a prolongation of w to L'. Second, let $T_v \subset Z_v$ and $T_w \subset Z_w$ be the corresponding decomposition groups. Then $\Phi_\imath(Z_v) \subset Z_w$ and $\Phi_\imath(T_v) \subset T_w$ are open subgroups. (This discussion includes the case that w is the trivial valuation of L.) Moreover, if w is non-trivial, then $wL \subset vK$ has finite index $e(v|w)$. Hence we have commutative diagrams of the form

$$\begin{array}{ccl} L & \xrightarrow{w} & wL \subset \widehat{wL} = \mathrm{Hom}\big(T_w, \mathbb{Z}_\ell\big) \\ \downarrow \imath & & \qquad\quad \downarrow e(v|w) \\ K & \xrightarrow{v} & vK \subset \widehat{vK} = \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big) \end{array}$$

Therefore, if γ_w and γ_v are the unique positive generators of vK, respectively wL, then γ_w is mapped to $e(v|w)\cdot\gamma_v$. Thus if $\tau_v \in T_v$ and $\tau_w \in T_w$ are the arithmetical inertia generators as defined/introduced at Remark 19 (2), then from the commutativity of the above diagrams and definitions it follows that $\Phi_\imath(\tau_v) = \tau_w^{e(v|w)}$.

Now combining these observations with Proposition 37 above and Remark 28, especially (5), we obtain the following by merely applying the definitions:

Proposition 38 *In the notation from Proposition 37, the embedding of function fields $\imath : L|l \hookrightarrow K|k$ and the resulting canonical homomorphism $\Phi_\imath : \Pi_K \to \Pi_L$ give rise in a natural way to a level-*td$(L|l)$ *morphism $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_L}$ of the corresponding abstract decomposition graphs.*

(1) *Moreover, if $\varphi_\imath : \mathcal{D}_K \to \mathcal{D}_L$ is a proper morphism of graphs of prime divisors, then the corresponding $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_L}$ is a proper morphism of abstract decomposition graphs.*

(2 *Further, if $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\mathcal{D}_L}$ is proper, and both $\mathcal{G}_{\mathcal{D}_K}$ and $\mathcal{G}_{\mathcal{D}_L}$ are complete regular-like, hence divisorial by Proposition 23, then $\Phi_\imath$ is divisorial.*

(3) *The Kummer homomorphism $\hat{\phi} : \widehat{L} \to \widehat{K}$ of Φ is actually the ℓ-adic completion of the embedding of function fields $\imath : L|l \hookrightarrow K|k$. In particular, $\imath$ defines $\Phi_\imath$ uniquely.*

(4) *Moreover, $\Phi_\imath$ defines $\imath$ uniquely up to Frobenius twists.*

Proof. Assertions (1), (2), and (3) follow from the discussion above.

To (4): Recall that in the introduction we considered an identification $\imath_K : \mathbb{T}_{\ell,K} \to \mathbb{Z}_\ell$ of the ℓ-adic Tate module of K with $\mathbb{Z}_\ell$, and via that identification one gets the identification $\widehat{K} = \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell)$. Explicitly, this identification works as follows: For each $x \in K^\times$, let $\delta(x) : \Pi_K \to \mathbb{T}_{\ell,K}$ be the corresponding character defined in Kummer theory. Then $\delta_x := \imath_K \circ \delta(x)$ is the homomorphism $\delta_x : \Pi_K \to \mathbb{Z}_\ell$ defined by x. Given the embedding $\imath : L|l \hookrightarrow K|k$, by the functoriality of Kummer Theory one has $\delta\big(\imath(y)\big) = \imath \circ \delta(x) \circ \Phi$. Therefore, if we choose the identifications $\imath_K : \mathbb{T}_{\ell,K} \to \mathbb{Z}_\ell$, $\imath_L : \mathbb{T}_{\ell,L} \to \mathbb{Z}_\ell$ compatible with $\imath$, i.e., such that $\imath_L = \imath_K \circ \imath$, it follows that one has $\delta_{\imath(y)} = \delta_u \circ \Phi$; hence the Kummer homomorphism defined by $\Phi_\imath$ is

$$\hat{\phi} : \widehat{L} = \mathrm{Hom}(\Pi_L, \mathbb{Z}_\ell) \to \mathrm{Hom}(\Pi_K, \mathbb{Z}_\ell) = \widehat{K}, \quad \delta_y \mapsto \delta_{\imath(y)}$$

and therefore, $\hat{\phi}$ is exactly the ℓ-adic completion of the embedding $\imath : L^\times \to K^\times$.

Now let $\imath' : L|l \hookrightarrow K|k$ be a further embedding of function fields such that $\Phi_{\imath'} = \Phi_\imath$. Then choosing $\imath'_K : \mathbb{T}_{\ell,L} \to \mathbb{Z}_\ell$ such that $\imath'_L = \imath_K \circ \imath'$, it follows that the Kummer homomorphism $\hat{\phi}'$ of $\Phi_{\imath'} = \Phi_\imath$ in this new setting is the ℓ-adic completion of $\imath'$. On the other hand, there exists an ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$ such that $\imath'_L = \varepsilon \cdot \imath_L$. If so, then we have $\hat{\phi}' = \varepsilon \cdot \hat{\phi}$ on $\widehat{L}$. Since $\hat{\phi}$ is the ℓ-adic completion of $\imath$, and $\hat{\phi}'$ is the ℓ-adic completion of $\imath'$, if we denote by $j_K : K^\times \to \widehat{K}$ the ℓ-adic completion homomorphisms, we have

$$j_K\big(\imath'(y)\big) = \varepsilon \cdot j_K\big(\imath(y)\big), \quad y \in L^\times .$$

Therefore, ε must be a rational ℓ-adic unit, say $\varepsilon = m/n$ with n, m natural numbers relatively prime to ℓ. Equivalently, there exists $a_y \in k$ such that $\imath'(y) = a_y \imath(y)^{m/n}$ in K, hence $\imath'(y)$ is of the form $\imath'(y) = u^{m/n}$ in K, as k is algebraically closed. But then $\imath'(y)$ is an n^{th} power in K. Since this is the case for all $\imath'(y) \in \imath'(L)$, it finally follows that $n = p^k$ is a power of the characteristic exponent p of k and l. By symmetry, the same is true for m. Hence finally ε is a power of the characteristic exponent of k and l. Equivalently, $\imath'$ is a Frobenius twist of $\imath$. □

Before studying the rational quotients in more detail in the next subsection, we mention the following weak version of the main result mentioned in the introduction. Recall that by Proposition 23 and in the notations from there one has: Let $\mathscr{G}_{\mathscr{D}_K}$ be a complete regular-like decomposition graph for $K|k$ having D_K as the set of 1-indices, the canonical sequence $1 \to \widehat{U}_K \to \Lambda_K \to \mathrm{Div}(D_K)_{(\ell)} \to \mathfrak{Cl}_{D_K} \to 0$ can be recovered from $\mathscr{G}_{\mathscr{D}_K}$ up to multiplication by ℓ-adic units $\varepsilon \in \mathbb{Z}_\ell^\times$.

Proposition 39 *Let $K|k$ and $L|l$ be function fields over algebraically closed fields of characteristic $\neq \ell$, and $\Phi : \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{G}_{\mathscr{D}_L^{\mathrm{tot}}}$ be an isomorphism. The following hold:*

(1) *For every geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$ for $K|k$ there exists a geometric decomposition graph $\mathscr{G}_{\mathscr{D}_L}$ for $L|l$ such that Φ defines an isomorphism $\mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_L}$, and $\mathscr{G}_{\mathscr{D}_K}$ is complete regular-like (hence abstract divisorial) iff $\mathscr{G}_{\mathscr{D}_L}$ is so.*

(2) *Let $\Phi : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_L}$ be an isomorphism as above, $\mathscr{G}_{\mathscr{D}_K}$ and $\mathscr{G}_{\mathscr{D}_L}$ be complete regular-like decomposition graphs with sets of 1-edges D_K, respectively D_L. Then Φ is divisorial, and there exists $\varepsilon \in \mathbb{Z}_\ell^\times$ such that the Kummer isomorphism $\hat{\phi}$ of Φ makes the diagram below commutative*

$$\begin{array}{ccccccccc}
0 \to & \widehat{U}_L & \longrightarrow & \Lambda_L & \xrightarrow{\mathrm{div}_{D_L}} & \mathrm{Div}(D_L)_{(\ell)} & \longrightarrow & \mathfrak{Cl}_L & \to 0 \\
& \downarrow{\scriptstyle \varepsilon\cdot\hat{\phi}} & & \downarrow{\scriptstyle \varepsilon\cdot\hat{\phi}} & & \downarrow{\scriptstyle \varepsilon\cdot\mathrm{div}_\Phi} & & \downarrow{\scriptstyle \varepsilon\cdot\mathrm{can}} & \\
0 \to & \widehat{U}_K & \longrightarrow & \Lambda_K & \xrightarrow{\mathrm{div}_{D_K}} & \mathrm{Div}(D_K)_{(\ell)} & \longrightarrow & \mathfrak{Cl}_K & \to 0
\end{array}$$

Proof. Assertion (1) follows immediately by sorting through the proof of Propositions 22, as the group-theoretical recipe given there is invariant under group isomorphisms. Assertion (2) follows immediately from assertion (1) above, combined with Propositions 23 and 30. □

5.2 Rational quotients

Next we turn our attention to rational projections of abstract decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ as above. Let $t \in K$ be an arbitrary non-constant function, and let κ_t be the relative algebraic closure of $k(t)$ in K. Then $\kappa_t|k$ is a function field in one variable. We endow $\kappa_t|k$ with its unique complete normal model $X_t \to k$, which is also projective, and consider the corresponding graph of prime divisors $\mathscr{D}_{\kappa_t}$ for κ_t and the resulting complete regular-like decomposition graph $\mathscr{G}_{\kappa_t}$ for Π_{κ_t}. Then $\mathscr{G}_{\kappa_t}$ has level $\delta = 1$ and is divisorial, by Proposition 23. Moreover, if g_t is the geometric genus of X_t, then we have:

(a) $\widehat{\mathfrak{Cl}}_{\mathscr{G}_{\kappa_t}} \cong \mathbb{Z}_\ell$.

(b) $\widehat{U}_{\mathscr{G}_{\kappa_t}} \cong \mathbb{Z}_\ell^{2g_t}$ as the ℓ-adic dual of $\Pi_1(X_t) \cong \mathbb{Z}_\ell^{2g}$, thus g_t is encoded in $\mathscr{G}_{\kappa_t}$.

(c) The canonical (surjective) projection $\Phi_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ defines a level-one morphism of abstract decomposition graphs $\Phi_{\kappa_t} : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\kappa_t}$, by Proposition 38.

Before going into the details of characterizing the rational projections, let us mention the following fact for later use:

Proposition 40 *With the above notation, suppose that $\kappa_t \hookrightarrow K$ is a regular field extension, i.e., K is separably generated over κ_t, and $K \cap \overline{\kappa_t} = \kappa_t$. For normal models X and X_t of $K|k$, respectively of $\kappa_t|k$, let $f : X \dashrightarrow X_t$ be a rational map defining $\kappa_t \hookrightarrow K$. Then:*

(1) *There exists on open subset $U \subset X_t$ such that the fibers $f_x : X_x \dashrightarrow \kappa(s) = k$ of f at $s \in U(k)$ are integral, and the generic point x_s of X_s is the center of the unique prime divisor $v_s \in D_X$ which restricts to $s \in X_t$. In particular, $v_s(K) = v_s(\kappa_t)$.*

(2) *Let $L|K$ be a finite separable extension of K which is linearly disjoint from $\overline{\kappa_t}$ over K, i.e., $L \cap \overline{\kappa_t} = \kappa_t$. Then for almost all $s \in U(k)$ the prime divisor v_s has a unique prolongation w_s to L, and moreover, $w_s|v_s$ is totally inert, i.e., $Lw_s\,|\,Kv_s$ is separable and $[Lw_s : Kv_s] = [L : K]$.*

(3) *Let $\Delta \subset \widehat{K}$ be a $\mathbb{Z}_\ell$-submodule of finite corank such that $\Delta \cap \widehat{\kappa}_t = 1$. Then for almost all $s \in U(k)$ one has that $\Delta \subset \widehat{U}_{v_s}$ and j_{v_s} maps Δ injectively into $\widehat{Kv}$.*

Proof. To (1): Since $K|\kappa_t$ is a regular, it follows that the generic fiber $f_{\kappa_t} : X_{\kappa_t} \dashrightarrow \kappa_t$ of f is geometrically integral. Hence the fiber $f_s : X_s \to \kappa(s)$ of f is geometrically integral for s in a Zariski open subset $s \in U \subset X_t$. The remaining facts are just the (valuation-theoretical) birational interpretation of this fact.

To (2): This is just a souped-up version of assertion (1), using the fact that since $L|K$ is separable, the fundamental equality $[L : K] = \sum_i e(w_i|v) f(w_i|v)$ is satisfied for every discrete valuation v of K and the set of its prolongations $w_i|v$ to L.

To (3): Choose some projective normal model $X \to k$ of $K|k$ such that the rational map $X \dashrightarrow X_t$ is defined on the whole X, and $\Pi_{1,K} = \Pi_{1,D_X}$. Then $K^\times/\kappa_t^\times$ embeds in the divisor group $\mathrm{Div}(X_{\kappa_t})$ of the generic fiber $X_{\kappa_t} \to \kappa_t$ of $X \to X_t$. Hence $K^\times/\kappa_t^\times$ is a free abelian group, and we have an exact sequence of free abelian groups

$$1 \to \kappa_t^\times/k^\times \to K^\times/k^\times \to K^\times/\kappa_t^\times \to 1,$$

and its ℓ-adic completion $1 \to \widehat{\kappa_t} \to \widehat{K} \to \widehat{\mathscr{K}} \to 1$, where $\mathscr{K} := K^\times/\kappa_t^\times$. Note that since $\Delta \cap \widehat{\kappa_t} = 1$ by hypothesis, the map $\widehat{K} \to \widehat{\mathscr{K}}$ is injective on Δ. For $n = \ell^e$, consider the exact sequence $1 \to \kappa_t^\times/n \to K^\times/n \to \mathscr{K}/n \to 1$, and let $\Delta_n \subset K^\times/n$ be the image of Δ in $K^\times/n$. Then Δ is the projective limit of $(\Delta_n)_n$. Further, setting $\Xi_n := \Delta_n \cap (\kappa_t^\times/n)$, the projective limit of $(\Xi_n)_n$ equals $\Delta \cap \widehat{\kappa_t} = 1$. Hence for every n_0 there exists $n > n_0$ such that the image of $\Xi_n \to \Xi_{n_0}$ is trivial.

The Kummer theory interpretation of the facts above is: Let $K_n := K[\sqrt[n]{\Delta_n}]$ be the corresponding $\mathbb{Z}/n$ elementary abelian extension of K. Then $\mathrm{Gal}(K_n|K)$ is isomorphic to $\mathrm{Hom}(\Delta_n, \mu_n)$, and setting $\kappa_n := K_n \cap \overline{\kappa_t}$ one actually has $\kappa_n = \kappa_t[\sqrt[n]{\Xi_n}]$. And further, $\mathrm{Gal}(\kappa_n|\kappa_t)$ is canonically isomorphic to $\mathrm{Hom}(\Xi_n, \mu_n)$, and the canonical projection $\mathrm{Gal}(K_n|K) \to \mathrm{Gal}(\kappa_n|\kappa_t)$ is given by $\mathrm{Hom}(\Delta_n, \mu_n) \to \mathrm{Hom}(\Xi_n, \mu_n)$, which is defined by the inclusion $\Xi_n \hookrightarrow \Delta_n$. In particular, setting $M_n := K\kappa_n$, it follows that $K_n|M_n$ is a $\mathbb{Z}/n$ elementary abelian extension with $\mathrm{Gal}(K_n|M_n)$ canonically isomorphic to $\mathrm{Hom}(\Delta_n/\Xi_n, \mu_n)$.

Now recall that Δ is the projective limit of $(\Delta_n)_n$; hence for n_0 sufficiently large, the map $\Delta \to \Delta/\ell$ factors through $\Delta \to \Delta_{n_0}$. Second, for any fixed n_0, if $n > n_0$ is sufficiently large, the image of $\Xi_n \to \Xi_{n_0}$ is trivial. Therefore, the canonical map $\Delta_n \to \Delta_{n_0}$ factors through Δ_n/Ξ_n; and therefore, the canonical map $\Delta \to \Delta/\ell$ factors through Δ_n/Ξ_n. We conclude that if $\delta > 0$ is the rank of the finite free $\mathbb{Z}_\ell$-module Δ, i.e., $\Delta \cong \mathbb{Z}_\ell^\delta$, then $\mathrm{Gal}(K_n|M_n)$ has $(\mathbb{Z}/\ell)^\delta$ as a quotient.

In order to simplify and fix notation, for $n > n_0$ as above, set $M := M_n$, $L := K_n$, and $\kappa := \kappa_n$; hence $M = K\kappa$ and $L|\kappa$ is a regular field extension, because $\kappa_n = K_n \cap \overline{\kappa_t}$. And denoting by $\Delta_M \subset M^\times/n$ the image of Δ in $M^\times/n$, one has $L = M[\sqrt[n]{\Delta_M}]$, and in particular $\Delta_M \cong \Delta_n/\Xi_n$ by the fact that $\mathrm{Gal}(L|M)$ is canonically isomorphic to both $\mathrm{Hom}(\Delta_n/\Xi_n, \mu_n)$ and $\mathrm{Hom}(\Delta_M, \mu_n)$. In particular, Δ_M has $(\mathbb{Z}/\ell)^\delta$ as a quotient.

Changing gears, let $Z_t \to X_t$ be the normalization of X_t in the function field extension $\kappa_t \hookrightarrow \kappa$, and $Z \to X$ the normalization of X in the field extension $K \hookrightarrow M$. Then the morphism $X \to X_t$ is dominated by $Z \to Z_t$, and the following holds: Since $M|\kappa$ is a regular field extension, the generic fiber of $Z \to Z_t$ is geometrically integral. Therefore, there exists an open subvariety $V \subset Z_t$ such that for all $s \in V(k)$, the fiber $Z_s \to \kappa(s) = k$ is integral. Consequently, the prime divisor v_s of M defined by the Weil prime divisor $Z_s \subset Z$ is the unique prime divisor in D_Z which restricts to the point $s \in V(k)$.

Further, let $Y \to Z$ be the normalization of Z in $M \hookrightarrow L$. Then arguing as above, it follows that for almost all $s \in V(k)$, the fiber $Y_s \to \kappa(s) = k$ of $Y \to Z_t$ at $s \in V(k)$ is integral, and the prime divisor w_s of $L|k$ defined by the Weil prime divisor $Y_s \subset Y$ is the only prime divisor in D_Y which restricts to the point $s \in V(k)$.

But then w_s must restrict to v_s too, and moreover, w_s is the only prolongation of v_s from M to L and $w_s|v_s$ is inert. By the fundamental equality we conclude that

$$\mathrm{Gal}(L|M) = Z_{w_s|v_s} \to \mathrm{Gal}\big(Lw_s|Mv_s\big)$$

is an isomorphism. By general valuation theory one has $Lw_s = Mv_s[\sqrt[n]{\Delta_M v_s}]$, where $\Delta_M v_s$ is the image of $\Delta_M \subset M^\times/n$ under the residue map $U_{v_s}/n \to Mv_s/n$ induced by $j_{v_s} : U_{v_s} \to Mv_s$. By Kummer theory applied to both $L|M$ and $Lw_s|Mv_s$, we conclude that $\mathrm{Gal}(L|M) \to \mathrm{Gal}(Lw_s|Mv_s)$ is an isomorphism iff $\Delta_M \to \Delta_M v_s$ is an isomorphism. From this we finally conclude that $(\mathbb{Z}/\ell)^\delta$ is a quotient of $\Delta_M v_s$. Therefore from the commutativity of the diagram of surjective morphisms

$$\begin{array}{ccccc} \Delta & \to & \Delta_n & \to & \Delta_M \\ \downarrow j_v & & \downarrow j_v & & \downarrow j_{v_s} \\ j_v(\Delta) & \to & \Delta_n v & \to & \Delta_M v_s \end{array}$$

it follows that $j_v(\Delta)$ has $(\mathbb{Z}/\ell)^\delta$ as a quotient, because $\Delta_M v_s$ does so. But then since $\Delta \cong \mathbb{Z}_\ell^\delta$, and $j_v(\Delta)$ has no torsion, being a submodule of the torsion free $\mathbb{Z}_\ell$-module $\widehat{Kv}$, it follows that j_v maps Δ isomorphically onto $j_v(\Delta)$. $\square$

Proposition 41 *Let $\mathcal{G}_{\mathcal{D}_K}$ be a complete regular-like decomposition graph, which we view as a divisorial abstract decomposition graph, as indicated in Proposition 23. Then with the above notation, and that of Definition/Remark 31, for $t \in K$ the following are equivalent:*

(i) $\Phi_{\kappa_t} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\kappa_t}$ *is a rational quotient of* $\mathcal{G}_{\mathcal{D}_K}$.

(ii) κ_t *is a rational function field.*

Proof. To (i) $\Rightarrow$ (ii): First recall that $\widehat{U}_{\mathcal{G}_{\kappa_t}} \cong \mathbb{Z}_\ell^{2g_t}$, where g_t is the genus of X_t. Hence $\mathcal{G}_{\kappa_t}$ is rational if and only if $g = 0$, or equivalently, κ_t is a rational function field.

For (ii) $\Rightarrow$ (i), we first claim that $\mathcal{G}_{\kappa_t}$ is rational. Indeed, by the discussion above, $\widehat{U}_{\mathcal{G}_{\kappa_t}} = 0$ and $\widehat{\mathfrak{Cl}}_{\mathcal{G}_{\kappa_t}} \cong \mathbb{Z}_\ell$, thus the claim. Next we claim that $\Phi_{\kappa_t} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\kappa_t}$ defines $\mathcal{G}_{\kappa_t}$ as a quotient of $\mathcal{G}_{\mathcal{K}}$. Indeed, first $\Phi_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ is surjective by the definition of κ_t. Thus it is left to show that Φ_{κ_t} is proper, i.e., to show that if $\Phi_{\mathfrak{v}} : \mathcal{G}_{\mathfrak{v}} \to \mathcal{G}_{\kappa_t}$ is a level-one residual morphism for Φ_{κ_t}, then the following hold:

(a) Each 1-index of $\mathcal{G}_{\mathfrak{v}}$ is mapped under $\Phi_{\mathfrak{v}}$ to some multi-index of $\mathcal{G}_{\kappa_t}$.

(b) Each multi-index of $\mathcal{G}_{\kappa_t}$ corresponds to some multi-index of $\mathcal{G}_{\mathfrak{v}}$.

To prove (a), let $X_{\mathfrak{v}} \to k$ be a normal model of $K_{\mathfrak{v}}|k$ such that $D_{X_{\mathfrak{v}}}$ is the set of all the 1-vertices of $\mathcal{G}_{\mathfrak{v}}$. Then the embedding of k function fields $\iota : \kappa_t \hookrightarrow K_{\mathfrak{v}}$ is defined by some dominant rational map $f : X_{\mathfrak{v}} \dashrightarrow X_t$. Since $X_{\mathfrak{v}}$ is normal and X_t is complete, it follows that f is defined at all points of codimension 1. This means that for every $v \in D_{X_{\mathfrak{v}}}$, if v is trivial on κ_t, then $f(v)$ is the generic point of X_t, and hence v is mapped to the trivial valuation of $\mathcal{G}_{\kappa_t}$; and if v is non-trivial on κ_t, then $f(v)$ is a closed point in X_t.

To prove (b), we proceed by induction on $d_{\mathfrak{v}} = \mathrm{td}(K_{\mathfrak{v}}|k)$. If $d_{\mathfrak{v}} = 1$, then $X_{\mathfrak{v}}$ is a normal curve. Since $\mathcal{G}_{\mathcal{D}_K}$ was assumed to be divisorial, $\mathcal{G}_{\mathfrak{v}}$ is divisorial too by definition. Hence by Proposition 23 (1), $X_{\mathfrak{v}} \to k$ is a complete normal curve. But then the dominant rational map $f : X_{\mathfrak{v}} \dashrightarrow X_t$ is a surjective morphism. Finally, if $d_{\mathfrak{v}} > 1$, then there exist “many” $v \in D_{X_{\mathfrak{v}}}$ which are trivial on κ_t. But then $\Phi_{\mathfrak{v}}$ gives rise to a level-one residual morphism $\Phi_v : \mathcal{G}_v \to \mathcal{G}_{\kappa_t}$ of $\mathcal{G}_{\mathfrak{v}}$. Since $\mathrm{td}(Kv|k) < d_{\mathfrak{v}}$, by induction Φ_v is proper. On the other hand, the set of multi-indices $\mathcal{V}_{\mathcal{G}_v}$ of $\mathcal{G}_v$ is contained in the set of multi-indices $\mathcal{V}_{\mathcal{G}_{\mathfrak{v}}}$. Hence finally, every vertex of $\mathcal{G}_{\kappa_t}$ corresponds to some vertex of $\mathcal{G}_{\mathfrak{v}}$.

Finally, it is left to check properties (i), (ii), Remark/Definition 31 (2).

Checking property (i) form Remark/Definition 31 (2): If $j_{\mathfrak{v}}(\widehat{U}_v \cap \widehat{\kappa}_t)$ is non-trivial, then $j_{\mathfrak{v}}$ maps $\widehat{\kappa}_t$ injectively into $\widehat{K_{\mathfrak{v}}}$. Indeed, let $\mathfrak{v} = v_r \circ \cdots \circ v_1$ with v_i prime divisors. Then if $\mathfrak{v}$ is not trivial on $\kappa_t^\times$, then $\kappa_t\mathfrak{v} = k$, and hence $j_{\mathfrak{v}}$ is trivial on $\widehat{\kappa}_t \cap \widehat{U}_{\mathfrak{v}}$. Second, if $\mathfrak{v}$ is trivial on $\kappa_t^\times$, then $\kappa_t\mathfrak{v} = \kappa_t$; hence $\widehat{\kappa}_t \subseteq \widehat{U}_{\mathfrak{v}}$, and $j_{\mathfrak{v}}$ is injective on $\widehat{\kappa}_t$.

Checking property (ii) form Remark/Definition 31 (2): Let us view K as a function field over the rational function field κ_t; hence $\mathrm{td}(K|\kappa_t) = \mathrm{td}(K|k) - 1 > 0$, and κ_t is relatively algebraically closed in K. Moreover, after replacing κ_t by a finite purely inseparable extension (which is of the form κ_y with $y^{p^e} = t$ for some power p^e of $p = \mathrm{char}(k)$, which does not change the Galois theory of the situation), we can suppose that $K|\kappa_t$ is actually a regular field extension. Let $X \to k$ be any normal model of $K|k$, and let $\mathbb{P}^1_k$ be the projective t-line over k. Since $K|\kappa_t$ is regular, by Fact 40 there exists a a cofinite subset $S \subset k$ such that the prime divisor v_a of $K|k$ defined by the fiber $X_s \subset X$ at the point $s \in \mathbb{P}^1_k$ defined by $a \in S$ restricts to the $(t-a)$-adic valuation of κ_t, and $t-a$ is a uniformizing parameter of v_a. Now recall that $K|\kappa_t$ is a (regular) function field over κ_t with $\mathrm{td}(K|\kappa_t) = \mathrm{td}(K|k) - 1$ positive, and one has an exact sequence of the form

$$1 \to \kappa_t^\times \to K^\times \to K^\times/\kappa_t^\times \to 1.$$

And note that the last group is a free abelian group, since it is contained in the group of Weil prime divisors of any projective normal model $Y \to \kappa_t$ of $K|\kappa_t$. Therefore, $\kappa_t^\times$ has complements, say $\mathscr{K} \subset K^\times$ in $K^\times$, and hence we have $K^\times = \kappa_t^\times \cdot \mathscr{K}$ with $\mathscr{K}$ a free abelian subgroup with $\mathscr{K} \cap \kappa_t^\times = \{1\}$. Moreover, we can "adjust" $\mathscr{K}$ in such a way as to have $v_a(\mathscr{K}) = 0$ for all $a \in S$ as above. Indeed, if $(\alpha_i)_i$ is a $\mathbb{Z}$-basis of $\mathscr{K}$, then replacing each α_i by $\beta_i := \alpha_i \prod_{a\in S}(t-a)^{-v_a(\alpha_i)}$, the resulting system $(\beta_i)_i$ generates freely a $\mathbb{Z}$-submodule $\mathscr{K}_1$ of $\mathscr{K}$ such that $v_a(\mathscr{K}_1) = 0$ for all $a \in S$, and $\mathscr{K}_1$ is a complement of $\kappa_t^\times$ in $K^\times$.

Therefore, we may and will suppose that $\mathscr{K} \subset K^\times$ is a complement of $\kappa_t^\times$ in $K^\times$ such that $v_a(\mathscr{K}) = 0$, or equivalently $\mathscr{K} \subset U_{v_a}$, for all $a \in S$; and taking ℓ-adic completions, $\widehat{\mathscr{K}}$ is a complement of $\widehat{\kappa}_t$ in $\widehat{K}$ with the same property, i.e., $\widehat{\mathscr{K}} \subset \widehat{U}_{v_a}$ for all $a \in S$.

Now let $\Delta \subset \widehat{K}_{\mathrm{fin}}$ be a $\mathbb{Z}_\ell$-submodule of finite corank, which means that $v(\Delta) = 0$ for almost all $v \in D_X$. In particular, this implies that $\Delta \subset \widehat{U}_{v_a}$ for almost all $a \in S$. Let $\Delta_1 \subset \widehat{\mathscr{K}}$ and $\Delta_2 \subset \widehat{\kappa}_t$ be the projections of Δ on $\widehat{\mathscr{K}}$, respectively $\widehat{\kappa}_t$. We claim that both Δ_1 and Δ_2 have finite corank, i.e., $v(\Delta_1) = 0$ and $v(\Delta_2) = 0$ for almost all v. Indeed, let $\Sigma \subset D_X$ be the finitely many $v \in D_X$ such that $v|_{\kappa_t}$ is non-trivial, and $v \neq v_a$ for all $a \in S$. Every $x \in \Delta$ has a unique presentation of the form $x = x_1x_2$ with $x_i \in \Delta_i$, and clearly, $v(x) = v(x_1) + v(x_2)$ for all v. Then for $v \in D_X\backslash\Sigma$ satisfying $v(\Delta) = 0$, since $v(x_1) + v(x_2) = v(x) = 0$, we must have $v(x_1) \neq 0$ iff $v(x_2) \neq 0$ for $v \in D_X$. On the other hand, if $v(x_2) \neq 0$, then $v|_{\kappa_t}$ is non-trivial; hence $v = v_a$ for some $a \in S$, by the fact that $v \in D_X\backslash\Sigma$. And if $v = v_a$ for some $a \in S$, then $v(\widehat{\mathscr{K}}) = 0$;

hence $v(x_1)=0$. We conclude that for $v\in D_X\backslash\Sigma$ we have $v(\Delta)=0$ iff $v(\Delta_i)=0$ for $i=1,2$. Thus both Δ_1 and Δ_2 have finite corank.

Now since $\Delta_1\cap\widehat{\kappa}_t=1$, it follows by Proposition 40 (3) that j_{v_a} maps Δ_1 injectively into $\widehat{Kv_a}$ for almost all $a\in S$. Further, we notice that j_{v_a} is trivial on Δ_2 for almost all $a\in S$. (Indeed, $f\in\kappa_t$ is a v_a-unit iff a is neither a zero nor a pole of f; and if so, then $j_{v_a}(f)=j_{v_a}\big(f(a)\big)=1$, because $j_{v_a}\big(f(a)\big)\in k^\times$, and j_{v_a} is trivial on $k^\times$.) Therefore, for $x=x_1x_2\in\Delta$ with $x_i\in\Delta_i$ as above, one has $j_{v_a}(x)=j_{v_a}(x_1)$. Hence for $x\in\Delta$ as above we have $j_{v_a}(x)=1$ iff $j_{v_a}(x_1)=1$ iff $x\in\Delta_2$, as claimed. □

Notations 42 We introduce notation as follows:

(1) Let $\{\kappa_x=k(x)\mid x \text{ general element of } K\}$ be the set of all the subfields of K generated by general elements $x\in K$. Note that $\kappa_x=\kappa_{x'}$ if and only if x' is a linear transformation $x'=(ax+b)/(cx+d)$ of x.

Further, let $\mathfrak{A}_K=\{\Phi_{\kappa_x}\}_{\kappa_x}$ be the set of the corresponding rational quotients of $\mathscr{G}_{\mathscr{D}_K}$, and note that $\Phi_{\kappa_x}=\Phi_{\kappa_{x'}}$ if and only if $x'=(ax+b)/(cx+d)$ is a linear transformation of x.

(2) In order to simplify notation, we identify κ_x with the corresponding subfield of K. This identification defines a canonical embedding $\widehat{\kappa}_x\hookrightarrow\widehat{K}$ which turns out to be the inflation map defined by the canonical projection $\Phi_{\kappa_x}:\Pi_K\to\Pi_{\kappa_x}$. Therefore, the ℓ-adic completion homomorphism $j_K:K^\times\to\widehat{K}$ then identifies $j_{\kappa_x}(\kappa_x^\times)$ with $j_K(\kappa_x^\times)$ inside $\widehat{K}$.

(3) Let $\imath:L|l\to K|k$ be an embedding of function fields such that $\imath(l)=k$ and $K|\imath(L)$ a separable field extension. Let $\mathfrak{A}_K=\{\Phi_{\kappa_x}\}_{\kappa_x}$ and $\mathfrak{B}_L=\{\Psi_{\kappa_y}\}_{\kappa_y}$ the sets of all rational quotients of $K|k$, respectively $L|l$. Finally, let $\mathfrak{B}_\imath\subseteq\mathfrak{B}_L$ be the set of all Ψ_{κ_y} such that $\imath(\kappa_y)$ is relatively algebraically closed in K. Thus in the context of Proposition 38, by taking into account Fact 41, for $\Psi_{\kappa_y}\in\mathfrak{B}_\imath$ and the corresponding $\Phi_{\kappa_x}\in\mathfrak{A}_K$, one has commutative diagrams in which $\Phi_{\kappa_x\kappa_y}$ is an isomorphism:

$$\begin{array}{ccc}\Pi_K & \xrightarrow{\Phi_\imath} & \Pi_L\\ \downarrow{\scriptstyle\Phi_{\kappa_x}} & & \downarrow{\scriptstyle\Psi_{\kappa_y}}\\ \Pi_{\kappa_x} & \xrightarrow{\Phi_{\kappa_x\kappa_y}} & \Pi_{\kappa_y}\end{array}\qquad\text{and}\qquad\begin{array}{ccc}\mathscr{G}_{\mathscr{D}_K} & \xrightarrow{\Phi_\imath} & \mathscr{G}_{\mathscr{D}_L}\\ \downarrow{\scriptstyle\Phi_{\kappa_x}} & & \downarrow{\scriptstyle\Psi_{\kappa_y}}\\ \mathscr{G}_{\kappa_x} & \xrightarrow{\Phi_{\kappa_x\kappa_y}} & \mathscr{G}_{\kappa_y}\end{array}$$

Fact/Definition 43 In the context from Notation 42 above, the following hold:

Birational Bertini: Let $x,t\in K$ be algebraically independent over k, and let x be separable in K, i.e., x is not a p-power in K, where $p=\text{char}(k)$. Then for all but finitely many $a\in k$, the element $ax+t$ is a general element of K, i.e., $k(ax+t)$ is relatively algebraically closed in K; see e.g. Lang [18], Ch. VIII, Lemma in the proof of Theorem 7, or Roquette [32], §4.

(1) We will use the above "birational Bertini" repeatedly in the following form: Let $x,t\in K$ be fixed algebraically independent functions over k, with x separable, e.g., general. Then the following hold:

(a) $t_a := ax + t$ is a general element for almost all $a \in k$.

(b) $t_{a',a} := t/(a'x + a)$ is a general element for all $a' \in k^\times$ and almost all $a \in k$.

(c) $t_{a'',a',a} := (a''t + a'x + a + 1)/(t + a'x + a)$ is a general element for all $a'' \in k$ and almost all $a', a \in k$.

(2) For $x, t \in K$ as above, the general elements of the form t_a, $t_{a',a}$, $t_{a'',a',a}$, will be called general elements of Bertini type defined by x, t. Further, a set $\Sigma \subset K^\times$ will be called a Bertini set, if for all $x, t \in K$ which are algebraically independent over k, and x is separable, one has $t_a, t_{a',a}, t_{a'',a',a} \in \Sigma$ for all $a'' \in k$, and almost all $a', a \in k$. Clearly, Σ generates the multiplicative group $K^\times$ by assertion (1) (b) above.

We say that a set of rational quotients $\mathfrak{A} \subseteq \mathfrak{A}_K$ is of Bertini type, if $\mathfrak{A}$ has a subset of the form $\mathfrak{A}_\Sigma := \{\Phi_{\kappa_x} \mid x \in \Sigma\}$ for some Bertini set $\Sigma \subset K^\times$.

(3) Next let $\imath : L|l \to K|k$ be an embedding of function fields such that $\imath(l) = k$ and $K|\imath(L)$ separable. Then for every separable element $y \in L$, one has that $x := \imath(y)$ is a separable element of $K|k$. Further, directly from the definition of a general element of Bertini-type one gets the following: Let $u_b, u_{b',b}, u_{b'',b',b} \in L$ be general elements of Bertini type defined by some $y, u \in L$. Then for all $b'' \in l$ and almost all $b', b \in l$, the images $t_b := \imath(y_b)$, $t_{b',b} := \imath(u_{b',b})$, $t_{b'',b',b} = \imath(u_{b'',b',b})$ are general elements of Bertini type in $K|k$ defined by $x := \imath(y)$, $t := \imath(u)$.

(4) From this we deduce that there exist Bertini sets $\Delta \subset L^\times$ and $\Sigma \subset K^\times$ such that $\imath(\Delta) \subseteq \Sigma$. Therefore, for the corresponding Bertini-type sets of rational quotients $\mathfrak{B}_\Delta$ and $\mathfrak{A}_\Sigma$, we have that if $\kappa_y \in \mathfrak{B}_\Delta$, then $\kappa_x := \imath(\kappa_y)$ lies in $\mathfrak{A}_\Sigma$, etc.

Proof. The only assertions which are perhaps not obvious are (1) (b) and c).

To (1) (b): $t_{a',a}$ is general if and only if $1/t_{a',a} = a'(x/t) + a(1/t)$ is general. Now note that if $x/t, 1/t \in K$ are algebraically independent over k, and because x is separable, it follows that at least one of the two elements is separable. Finally apply the "birational Bertini".

To (1) (c): Let $\alpha := 1 - a''$. Then $t_{a'',a',a} = a'' + (\alpha a'x + \alpha a + 1]/(t + a'x + a)$ is a general element if and only if $t' := (\alpha a'x + \alpha a + 1)/(t + a'x + a)$ is so. Note that $t + a'x + a$ is a general element for all $a \in k^\times$ and almost all $a' \in k$ by the "birational Bertini." Hence if $\alpha = 0$, then $t' := 1/(t + a'x + a)$ is a general element. Finally, if $\alpha \neq 0$, then x' is a general element if and only if $1/t' = (t - \frac{1}{\alpha})/(\alpha a'x + \alpha a + 1) + \frac{1}{\alpha}$ is a general element, thus if and only if $(t - \frac{1}{\alpha})/(\alpha a'x + \alpha a + 1)$ is a general element. And the latter is a general element for all $\alpha a + 1 \in k^\times$ and almost all $a' \in k$, by Case (1) (b). $\square$

Proposition 44 *With the above notation, the following hold:*

(1) *Suppose that* $\mathrm{td}(K|k) > 1$, *and let* $\mathcal{G}_{\mathcal{D}_K}$ *be a complete regular-like geometric decomposition graph, which we view as a divisorial abstract decomposition graph. Then endowing* $\mathcal{G}_{\mathcal{D}_K}$ *with a Bertini-type set* $\mathfrak{A} \subseteq \mathfrak{A}_K$ *of rational projections,* $\mathcal{G}_{\mathcal{D}_K}$ *becomes a geometric like abstract decomposition graph satisfying the following:* $K^\times_{(\ell)} := j_K(K^\times) \otimes \mathbb{Z}_{(\ell)}$ *is an* arithmetical lattice *defined by* $\mathfrak{A}$ *inside* $\widehat{K}$, *which we call the canonical arithmetical lattice.*

(2) *Let $\imath : L|l \to K|l$ be an embedding of function fields such that $\imath(l) = k$, and $K|\imath(L)$ is separable. Let $\mathscr{H}_{\mathscr{D}_L}$ be a complete regular-like abstract decomposition graph for $L|l$ such that*

$$\Phi_\imath : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$$

gives rise to a proper morphism of abstract decomposition graphs. Then there exist Bertini type sets $\mathfrak{B}$ of rational quotients for $\mathscr{H}_{\mathscr{D}_L}$ such that $\Phi_\imath$ is compatible with the rational projections $\mathfrak{B}$ and $\mathfrak{A}$.

Proof. To (1): Let us check that $\mathfrak{A}$ satisfies the conditions from Definition 33. Let $X \to k$ be a quasi-projective normal model of $K|k$ such that D_X is the set of all 1-vertices of $\mathscr{G}_{\mathscr{K}}$.

Step 1. $\mathfrak{A}$ is an ample family of rational quotients for $\mathscr{G}_{\mathscr{D}_K}$.

Indeed, first recall that by Fact 43 (2), the set $\Sigma_{\mathfrak{A}}$ generates $K^\times$. Therefore, with the notation from Definition 33 we have $\Lambda_{\mathfrak{A}_\Sigma} = K^\times_{(\ell)}$, hence $\Lambda_{\mathfrak{A}} = K^\times_{(\ell)}$ too. From this we deduce, first, that $\Lambda_{\mathfrak{A}}$ is ℓ-adically dense in $\widehat{K}$, as $j_K(K^\times)$ itself is so. Second, since $\mathscr{D}_K$ was supposed to be complete regular-like, for every non-constant $x \in K$ there exists $v \in D_X$ such that $v(x) \neq 0$. Equivalently, for every non-trivial $x \in K^\times_{(\ell)}$, there exists $v \in D^1_{\mathscr{D}_K}$ such that $v(x) \neq 0$. But this means exactly that $K^\times_{(\ell)} \cap \widehat{U}_{\mathscr{G}_{\mathscr{K}}}$ is trivial. From this discussion, condition (ii) of Definition 33 follows. For condition (i), observe that $\Phi_{\kappa_x} \neq \Phi_{\kappa_{x'}}$ implies that $\kappa_x \neq \kappa_{x'}$. But then $\kappa_x \cap \kappa_{x'} = k$; hence $\widehat{\kappa}_x$ and $\widehat{\kappa}_{x'}$ have trivial intersection inside $\widehat{K}$.

Step 2. $\mathscr{G}_{\mathscr{D}_K}$ endowed with $\mathfrak{A}$ is geometric like.

Indeed, let κ_x and $\kappa_{x'}$ be given. If $\kappa_x = \kappa_{x'}$, then there is nothing to prove. Hence let $\kappa_x \neq \kappa_{x'}$. Since κ_x and $\kappa_{x'}$ are relatively algebraically closed in K, it follows that x, x' are actually algebraically independent over k. Therefore, by the "birational Bertini," it follows that for almost all $a, a' \in k$ we have that $t := ax - a'x'$ gives rise to a dominant rational map $f : X \dashrightarrow \mathbb{P}^1_t$ such that for general points $t = b$, the fiber X_b is a Weil prime divisor of X, and x, x' are non-constant on X_b. The birational translation of this is the following: If $v := v_{X_b} \in D_X$ is the corresponding prime divisor of K, then x, x' are v-units such that $j_v(x)$, $j_v(x')$ are not constant in the residue field Kv of v. But then $\kappa_x^\times$ and $\kappa_{x'}^\times$ consist of non-principal v-units, and are mapped isomorphically into the residue field Kv. Moreover, since $t = ax - a'x'$ has $v(t) > 0$, it follows that $ax \equiv a'x' \,(\bmod\, \mathfrak{m}_v)$, hence $j_v(\kappa_x) = j_v(\kappa_{x'})$. Taking ℓ-adic completions, we deduce from this that conditions j), jj) of Definition 33 are satisfied at v.

To (2): Apply Fact 43 (3), (4), and the commutative diagrams from Notations 42 (3). $\square$

6 Proof of Main Theorem

In this Section we will give a proof of the main theorem from the introduction. We will actually prove a slightly more general result than the main theorem announced in the introduction, in the sense that the first part of the theorem proved below compares complete regular-like geometric decomposition graphs with geometric-like abstract decomposition graphs.

Theorem 45. *Let $K|k$ be a function field with* $\mathrm{td}(K|k)>1$*, and let $\mathscr{G}_{\mathscr{D}_K}$ be a complete regular-like geometric decomposition graph for $K|k$. We endow $\mathscr{G}_{\mathscr{D}_K}$ with a Bertini type set $\mathfrak{A}$ of rational quotients, and view it as a* geometric like abstract decomposition graph.

(1) *Let $\mathscr{H}$ endowed with a family of rational quotients $\mathfrak{B}$ be a geometric like abstract decomposition graph. Then up to multiplication by ℓ-adic units and composition with automorphisms $\Phi_\imath : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_K}$ defined by embedding of function fields $\imath : K|l \to K|k$ such that $K|\imath(K)$ is purely inseparable, there exists at most one isomorphism $\Phi : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}$ of abstract decomposition graphs which is compatible with the rational quotients $\mathfrak{A}$ and $\mathfrak{B}$.*

(2) *Let $L|l$ be a further function field with* $\mathrm{td}(L|l)>1$*, and let $\mathscr{H}_{\mathscr{D}_L}$ be a complete regular-like abstract decomposition graph for $L|l$. We endow $\mathscr{H}_{\mathscr{D}_L}$ with a Bertini-type set $\mathfrak{B}$ of rational quotients, and view it as a* geometric like abstract decomposition graph. *Let*

$$\Phi : \Pi_K \to \Pi_L$$

be an open group homomorphism which defines a proper morphism $\Phi : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$ of abstract decomposition graphs compatible with the rational quotients $\mathfrak{B}$ and $\mathfrak{A}$. Then there exist an ℓ-adic unit ε and an embedding of function fields

$$\imath : L|l \to K|k$$

such that $\Phi = \varepsilon \cdot \Phi_\imath$, where $\Phi_\imath : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$ is the functorial morphism of decomposition graphs defined by $\imath$ as indicated above. Further, $\imath(l) = k$, $\imath$ is unique up to Frobenius twists, and ε is unique up to multiplication by p^n-powers, where $p = \mathrm{char}(k)$ and $n \in \mathbb{Z}$.

Proof. Since (1) follows from (2), it suffices to prove assertion (2).

Recall that by Proposition 44, $K^\times_{(\ell)} := j_K(K^\times) \otimes \mathbb{Z}_{(\ell)}$ and $L^\times_{(\ell)} := j_L(L^\times) \otimes \mathbb{Z}_{(\ell)}$ are arithmetical lattices for $\mathscr{G}_{\mathscr{D}_K}$ endowed with $\mathfrak{A}$, respectively for $\mathscr{H}_{\mathscr{D}_L}$ endowed with $\mathfrak{B}$. Now by Proposition 35, it follows that $\hat{\phi}(L^\times_{(\ell)})$ is contained in a unique arithmetical lattice of $\mathscr{G}_{\mathscr{D}_K}$. Since the arithmetical lattices of $\mathscr{G}_{\mathscr{D}_K}$ are ℓ-adically equivalent to $K^\times_{(\ell)}$, there exists an ℓ-adic unit ε such that

$$\hat{\phi}(L^\times_{(\ell)}) \subseteq \varepsilon \cdot K^\times_{(\ell)} \,.$$

Therefore, after replacing Φ by $\varepsilon \cdot \Phi$, without loss of generality we can make the following hypothesis:

Hypothesis I. *$\hat{\phi}$ maps $L^{\times}_{(\ell)}$ isomorphically into $K^{\times}_{(\ell)}$.*

We further recall that $j_K(K^\times) = K^\times/k^\times$ and $j_L(L^\times) = L^\times/l^\times$ are true lattices in $K^{\times}_{(\ell)}$, respectively $L^{\times}_{(\ell)}$. In order to simplify notation, we denote by

$$\mathsf{x} = j_K(x) = k^\times x, \quad \mathsf{y} = j_L(y) = l^\times y$$

the image of $x \in K^\times$ under j_K, respectively that of $y \in L^\times$ under j_L. Further, we will always denote elements of $K^{\times}_{(\ell)}$, respectively of $L^{\times}_{(\ell)}$, in boldface:

$$\boldsymbol{x} \in K^{\times}_{(\ell)}, \quad \boldsymbol{y} \in L^{\times}_{(\ell)}\,.$$

Next we want to understand the following: Let $\Phi_{\kappa_x} \in \mathfrak{A}$ correspond to some $\Phi_{\kappa_y} \in \mathfrak{B}$, and let $\kappa_{x,(\ell)} \subset \widehat{\kappa}_x$ and $\kappa_{y,(\ell)} \subset \widehat{\kappa}_y$ be the unique divisorial lattices such that $\hat{\phi}_{\kappa_x}(\kappa_{x,(\ell)}) \subset K^{\times}_{(\ell)}$, respectively $\hat{\phi}_{\kappa_y}(\kappa_{y,(\ell)}) \subset L^{\times}_{(\ell)}$. Then by Proposition 35 (2), we get $\hat{\phi}_{\kappa_x\kappa_y}(\kappa_{y,(\ell)}) = \kappa_{x,(\ell)}$, $\hat{\phi} \circ \hat{\phi}_{\kappa_y}(\kappa_{y,(\ell)}) = \hat{\phi}_{\kappa_x}(\kappa_{x,(\ell)})$. Hence taking into account the identifications from Notations 42, i.e., $\kappa_{x,(\ell)} = K^{\times}_{(\ell)} \cap \hat{\phi}_{\kappa_x}(\widehat{\kappa}_x)$ inside $\widehat{K} = \widehat{\Lambda}_{\mathscr{G}_{\mathscr{D}_K}}$, and $\kappa_{y,(\ell)} = L^{\times}_{(\ell)} \cap \hat{\phi}_{\kappa_y}(\widehat{\kappa}_y)$ inside $\widehat{L} = \widehat{\Lambda}_{\mathscr{H}_{\mathscr{D}_L}}$, the above assertion is equivalent to the fact that if $\Phi_{\kappa_x} \in \mathfrak{A}$ corresponds to $\Phi_{\kappa_y} \in \mathfrak{B}$, then

$$\hat{\phi}(\kappa_{y,(\ell)}) = \kappa_{x,(\ell)}\,.$$

Hence we have $j_L(\kappa_y^\times) \subset \kappa_{y,(\ell)}$ and $j_K(\kappa_x^\times) \subset \kappa_{x,(\ell)}$, and the task now is to understand the precise relation between $\hat{\phi} \circ j_L(\kappa_y{}^\times)$ and $j_K(\kappa_x^\times)$ inside $\kappa_{x,(\ell)}$.

Lemma 46 *Let $\Phi_{\kappa_x} \in \mathfrak{A}$ correspond to some $\Phi_{\kappa_y} \in \mathfrak{B}$. Then there exist unique relatively prime and prime to ℓ integers $m,n > 0$ such that the following hold:*

(1) $\hat{\phi}\big(n \cdot j_L(ly+l)^\times\big) = m \cdot j_K(kx+k)^\times$ *and* $\hat{\phi}\big(n \cdot j_L(\kappa_y^\times)\big) = m \cdot j_K(\kappa_x^\times)$ *in* $\kappa_{x,(\ell)}$.

(2) $\hat{\phi}\big(n \cdot j_L(y)\big) = m \cdot j_K(x)$*, provided* $\mathbb{Z}_{(\ell)} \cdot j_L(y)$ *is mapped by* $\hat{\phi}$ *into* $\mathbb{Z}_{(\ell)} \cdot j_K(x)$*, and such a choice of a generator* x *for* κ_x *is always possible.*

Proof. We begin by considering the systems of *arithmetical inertia generators* $\mathfrak{T}_{\kappa_y}$ of $\mathscr{G}_{\kappa_y}$, respectively $\mathfrak{T}_{\kappa_x}$ of $\mathscr{G}_{\kappa_x}$, as introduced at Definition/Remark 19 (2). We let $(y) = w' - w$ be the divisor of $y \in \kappa_y = l(y)$, and $(x) = v' - v$ be the divisor of $x \in \kappa_x = k(x)$. Then $\kappa_x = k(x)$, and in the notations from Definition/Remark 31 (1), we have:

(a) $\mathsf{y} := j_L(y) = \varphi_{w'} - \varphi_w$, and $\mathscr{P}_{\mathsf{y}} := \mathscr{P}_w = j_L(ly+l)^\times \subset \kappa_{y,(\ell)}$ is the generating set at w with respect to $\mathfrak{T}_{\kappa_y}$.

(b) $\mathsf{x} := j_K(x) = \varphi_{v'} - \varphi_v$, and $\mathscr{P}_{\mathsf{x}} := \mathscr{P}_v = j_K(kx+k)^\times \subset \kappa_{x,(\ell)}$ is the generating set at v with respect to $\mathfrak{T}_{\kappa_y}$.

Moreover, we can choose x from the beginning in such a way that v', v are the preimages of w', w under $\Phi_{\kappa_x\kappa_y}$. Equivalently, we have $\hat{\phi}(\mathbb{Z}_{(\ell)} \cdot \mathsf{y}) = \mathbb{Z}_{(\ell)} \cdot \mathsf{x}$. Therefore there exist unique relatively prime integers $m,n > 0$ such that

$$(*) \qquad \hat{\phi}(n \cdot \mathsf{y}) = m \cdot \mathsf{x}.$$

On the other hand, the image $\Phi_{\kappa_x\kappa_y}(\mathfrak{T}_{\kappa_y})$ of $\mathfrak{T}_{\kappa_y}$ under the isomorphism $\Phi_{\kappa_x\kappa_y}$ is a distinguished system of inertia generators for κ_x such that $\hat{\phi}(\mathscr{P}_w)$ is the generating set at v with respect to $\Phi_{\kappa_x\kappa_y}(\mathfrak{T}_{\kappa_y})$. By the uniqueness up to ℓ-adic equivalence of the distinguished systems of inertia generators we have $\Phi_{\kappa_x\kappa_y}(\mathfrak{T}_{\kappa_y}) = \mathfrak{T}^{\varepsilon}_{\kappa_x}$ for a unique ℓ-adic unit $\varepsilon \in \mathbb{Z}_\ell^\times$. Hence $\hat{\phi}(\mathscr{P}_w) = \varepsilon^{-1} \cdot \mathscr{P}_v$, and in particular, $\hat{\phi}(\mathsf{y}) = \varepsilon^{-1} \cdot \mathsf{x}$ inside $\kappa_{x,(\ell)}$. Then by the fact $(*)$ above, it follows that $\varepsilon = n/m$; hence both m,n are relatively prime to ℓ. Finally, we get

$$(*)' \qquad \hat{\phi}(n \cdot \mathscr{P}_{\mathsf{y}}) = m \cdot \mathscr{P}_{\mathsf{x}}.$$

Clearly, if m',n' are relatively prime integers such that $\hat{\phi}(n' \cdot \mathscr{P}_{\mathsf{y}}) = m' \cdot \mathscr{P}_{\mathsf{x}}$, then we must have $\hat{\phi}(n' \cdot \mathsf{y}) = m' \cdot \mathsf{x}$. Therefore, $(m,n) = (m',n')$ by the uniqueness of m,n.

Finally, since $\mathscr{P}_{\mathsf{y}}$ and $\mathscr{P}_{\mathsf{x}}$ generate $\kappa_y^\times/l^\times$ inside $\kappa_{y,(\ell)}$, respectively $\kappa_x^\times/k^\times$ inside $\kappa_{x,(\ell)}$, we deduce that for the unique m,n above, one has

$$(*)'' \qquad \hat{\phi}\big(n \cdot j_L(\kappa_y^\times)\big) = m \cdot j_K(\kappa_x^\times).$$

This completes the proof of the lemma. □

Norming 47 In the context of Lemma 46 above, suppose that $\mathbb{Z}_{(\ell)} \cdot j_L(y)$ is mapped by $\hat{\phi}$ into $\mathbb{Z}_{(\ell)} \cdot j_K(x)$. Then we will say that $\hat{\phi}$ is y-normed if $\hat{\phi} \circ j_L(y) = j_K(x)$.

Clearly, a priori, $\hat{\phi}$ might not be normed with respect to any $\Phi_{\kappa_y} \in \mathfrak{B}$ and the corresponding $\Phi_{\kappa_x} \in \mathfrak{A}$. Nevertheless, we can "artificially" remedy this as follows: With the notation from Lemma 46 above, suppose that we have chosen the generator x such that $\mathbb{Z}_{(\ell)} \cdot j_L(y)$ is mapped by $\hat{\phi}$ into $\mathbb{Z}_{(\ell)} \cdot j_K(x)$. Hence we have $\hat{\phi} \circ j_L(y) = (m/n) \cdot j_K(x)$. Further, notice that $\eta_{\hat{\phi}} := m/n$ is an ℓ-adic unit. And replacing the morphism $\Phi : \Pi_K \to \Pi_L$ by its $\eta_{\hat{\phi}}$-multiple $\Phi' := \eta_{\hat{\phi}} \cdot \Phi$ amounts to replacing $\hat{\phi}$ by its $(1/\eta_{\hat{\phi}})$-multiple $\hat{\phi}' := (1/\eta_{\hat{\phi}}) \cdot \hat{\phi}$. In particular, we have $\eta_{\hat{\phi}'} = 1$; hence $\hat{\phi}'$ is y-normed.

Hence we have the following: Let $\kappa_y \in \mathfrak{B}$ and its corresponding $\kappa_x \in \mathfrak{A}$ be given such that $\mathbb{Z}_{(\ell)} \cdot j_L(y)$ is mapped by $\hat{\phi}$ into $\mathbb{Z}_{(\ell)} \cdot j_K(x)$. Then after replacing Φ by a properly chosen multiple $\eta \cdot \Phi$ with $\eta \in \mathbb{Z}_{(\ell)}$, the resulting Kummer homomorphism $(1/\eta) \cdot \hat{\phi}$ is y-normed. Hence mutatis mutandis, we can suppose that $\hat{\phi}$ satisfies the following norming hypothesis:

Hypothesis II. *$\kappa_x \in \mathfrak{A}$ corresponds to $\kappa_y \in \mathfrak{B}$, and $\hat{\phi} \circ j_L(y) = j_K(x)$, hence $\hat{\phi}$ is y-normed.*

Remark/Notation 48 If $\hat{\phi}$ is y-normed, then by Lemma 46, $\hat{\phi}$ defines bijections

$$(\dagger) \qquad \hat{\phi} : j_L(ly+l)^\times \to j_L(ky+k)^\times, \quad \hat{\phi} : j_L(\kappa_y^\times) \to j_L(\kappa_x^\times).$$

We set $M_K := \hat{\phi}\big(j_L(L^\times)\big) \cap j_K(K^\times)$, and let $M_L \subseteq j_L(L^\times)$ be the preimage of M_K under $\hat{\phi}$. Then $j_L(\kappa_y^\times) \subset M_L$ and $j_K(\kappa_x^\times) \subset M_K$ by the fact $(\dagger)$ above, and notice that

$$\hat{\phi} : M_L \to M_K$$

is an isomorphism which maps $j_L(\kappa_y^\times)$ isomorphically onto $j_K(\kappa_x^\times)$. We will say that $u \in L^\times$ and $t \in K^\times$ correspond to each other if the following hold:

$$j_L(u) \in M_L, \quad j_K(t) \in M_K, \quad \text{and} \quad \hat{\phi} \circ j_L(y) = j_K(x)\,.$$

Finally we notice that $M_L \otimes \mathbb{Z}_{(\ell)} = L^\times_{(\ell)}$ inside $\widehat{L}$.

Lemma 49 *Suppose that $t \in K$ and $u \in L$ correspond to each other via $\hat{\phi}$. Then $\mathscr{P}_t := (kt+k)^\times/k^\times = j_K(kt+k)^\times \subset M_K$, $\mathscr{P}_u := (lu+l)^\times/l^\times = j_L(lu+l)^\times \subset M_L$.*

Proof. Case 1: $u \in \kappa_y$. Then $t \in \kappa_x$, and we are in the situation of Lemma 46 above with $m = n = 1$, from which the assertion follows.

Case 2: $u \notin \kappa_y$. Since $\kappa_y = l(y)$ is relatively algebraically closed in L, it follows that u, y are algebraically independent over l. Correspondingly, the same is true for t, x, i.e., t, x are algebraically independent over k. Then by the Fact 43 (1), we have:

(i) $t_{a',a} := t/(a'x+a)$ is a general element of K for almost all $a', a \in k$.

(ii) $u_{b',b} := u/(b'x+b)$ is a general element of L for almost all $b', b \in l$.

Hence by condition (†) of Remark/Notation 48, we conclude the following: For a', a as at (i), let $y_{a',a} \in (ly+l)^\times$ be such that $\hat{\phi} \circ j_L(y_{a',a}) = j_K(a'x+a)$. Then by (ii), $u_{a',a} := u/y_{a',a}$ is a general element of L for almost all $a', a \in k$. And note that

$$\hat{\phi} \circ j_L(u_{a',a}) = \hat{\phi} \circ j_L(u/y_{a',a}) = j_K\big(t/(a'x+a)\big) = j_K(t_{a',a}).$$

In particular, since $\mathfrak{A}$ and $\mathfrak{B}$ contain some Bertini-type subsets, we can suppose that $\kappa_{t_{a',a}} \in \mathfrak{A}$ and $\kappa_{u_{a',a}} \in \mathfrak{B}$, and $\kappa_{t_{a',a}}$ corresponds to $\kappa_{u_{a',a}}$ under Φ. On the other hand, since by hypothesis we have $j_K(t) \in M_K$ and $j_L(u) \in M_L$, and by Remarks/Notation 48 above, $j_K(kx+k) \subset M_K$ and $j_L(ly+l) \subset M_L$, it follows that for almost $a, a' \in k$, the following hold:

(a) $t_{a',a} \in K$ and $u_{a',a} \in L$, respectively $t_{a',a+1} \in K$ and $u_{a',a+1} \in L$, are general elements which correspond to each other under $\hat{\phi}$.

(a)$'$ Hence $\hat{\phi}$ is normed with respect to both $u_{a',a}$ and $u_{a',a+1}$.

For b, b' as at (ii), let $x_{b',b} \in kx+k$ be such that $\hat{\phi}\big(j_L(b'y+b)\big) = j_K(x_{b',b})$. Then by (i), for all b and almost all b', the element $t_{b',b} := t/x_{b',b}$ is general, and note that

$$j_K(t_{b',b}) = j_K(t/x_{b',b}) = \hat{\phi} \circ j_L\big(u/(b'y+b)\big) = \hat{\phi} \circ j_L(u_{b',b}).$$

In particular, $\kappa_{t_{b',b}} \in \mathfrak{A}$ and $\kappa_{u_{b',b}} \in \mathfrak{B}$, and $\kappa_{t_{b',b}}$ corresponds to $\kappa_{u_{b',b}}$ under Φ, and $\hat{\phi}$ is normed with respect $u_{b',b}$. Reasoning as above, for almost $b', b \in l$ one has:

(b) $t_{b',b} \in K^\times$ and $u_{b',b} \in L$, respectively $t_{b',b+1} \in K$ and $u_{b',b+1} \in L$, are general elements which correspond to each other under $\hat{\phi}$.

(b)$'$ Hence $\hat{\phi}$ is normed with respect to both $u_{b',b}$ and $u_{b',b+1}$.

But then by the fact (†) from Remark/Notation 48 applied to the functions $t_{a',a} \in K^\times$ and $u_{a',a} \in L$, it follows that $j_K(\kappa^\times_{t_{a',a}}) = \hat{\phi} \circ j_L(\kappa^\times_{u_{a',a}}) \subset \hat{\phi} \circ j_L(L^\times)$, and therefore we also have $j_K(\kappa^\times_{t_{a',a}}) \subset \hat{\phi} \circ j_L(L^\times) \cap j_K(K^\times) = M_K$; and the same holds correspondingly for the other three pairs of functions which correspond to each other under $\hat{\phi}$. Thus finally we get

$$j_K(\kappa^\times_{t_{a',a}}) \subset M_K, \;\; j_K(\kappa^\times_{t_{a',a+1}}) \subset M_K, \quad j_L(\kappa^\times_{u_{b',b}}) \subset M_L, \;\; j_K(\kappa^\times_{u_{b',b+1}}) \subset M_L\,.$$

Finally, for $a, a', a'' \in k$, consider the functions

$$t_{a'',a',a} = (a''t + a'x + a + 1)/(t + a'x + a)\,.$$

Then by Fact 43 (1), it follows that for all a'', and almost all a', a, the function $t_{a'',a',a}$ is a general element of K too. On the other hand, a direct computation shows that

$$t_{a'',a',a} = \frac{a'x + a + 1}{a'x + a} \cdot \frac{a''t_{a',a+1} + 1}{t_{a',a} + 1}\,.$$

Since the images via j_K of both the denominators and the numerators of the fractions above lie in M_K, we get $j_K(t_{a'',a',a}) \in M_K$. Reasoning as previously in the case of $t_{a',a}$, we find general elements $u_{a'',a',a} \in L$ such that $\kappa_{t_{a'',a',a}}$ corresponds to $\kappa_{u_{a'',a',a}}$, etc. And we further define correspondingly functions

$$u_{b'',b',b} = \frac{b'x + b + 1}{b'x + b} \cdot \frac{b''u_{b',b+1} + 1}{u_{b',b} + 1}\,,$$

and find functions $t_{b'',b',b} \in K$, etc. Finally one gets $j_K(\kappa_{t_{a'',a',a}}) \subset M_K$ for all $a'' \in k$, and almost all $a', a \in k$. And correspondingly $j_L(\kappa_{u_{b'',b',b}}) \subset M_L$ for all $b'' \in l$, and almost all $b', b \in l$.

Now we conclude the proof of the fact that $j_K(kt + k)^\times \subseteq M_K$ as follows: First, since $j_K(\kappa^\times_{t_{a'',a',a}}) \subset M_K$, we have $j_K(t_{a'',a',a} - 1) \in M_K$. On the other hand,

$$t_{a'',a',a} - 1 = [(a'' - 1)t + 1]/(t + a'x + a)\,.$$

Now observe that $t + a'x + a = (a'x + a + 1)/t_{0,a',a}$. Hence $j_K(t + a'x + a) \in M_K$, as $j_K(t_{0,a',a}), j_K(a'x + a + 1) \in M_K$. Thus we finally deduce that $j_K\big((a'' - 1)t + 1\big) \in M_K$ for all $a'' \in k$. Hence $\mathscr{P}_t = j_K(kt + k)^\times \subset M_K$, as $j_K(t) \in M_K$ by hypothesis.

In a completely similar way, one concludes that $\mathscr{P}_u = j_L(lu + l)^\times \subset M_L$. $\square$

Lemma 50 *Let $K_0 = j_K^{-1}(M_K) \cup \{0\} \subseteq K$ and $L_0 = j_L^{-1}(M_L) \cup \{0\} \subseteq L$ be the preimages of M_K, respectively M_L, in K, respectively L, together with 0 added. Then $K_0 \subseteq K$ and $L_0 \subseteq L$ are function subfields.*

Proof. Indeed, since M_K is a subgroup of $\widehat{K}$, its preimage $j_K^{-1}(M_K)$ in $K^\times$ is a subgroup too. We check that K_0 is closed with respect to addition: For $t, t' \in K_0$ nonzero, $t'' = t'/t \in K_0$, and $t + t' = t(t'' + 1)$. On the other hand, by Lemma 49 we have $t'' + 1 \in K_0$. Hence finally we get $t + t' = t(t'' + 1) \in K_0$. The proof of the assertion concerning L_0 is similar, and we omit it. $\square$

Next we observe that $M_K = j_K(K_0^\times) = K_0^\times/k^\times$ can be viewed in a canonical way as the projectivization $\mathscr{P}(K_0) := K_0^\times/k^\times$ of the infinite-dimensional k-vector space $(K_0,+)$. And correspondingly, $M_L = j_L(L_0^\times) = L_0^\times/l^\times =: \mathscr{P}(L_0)$ is the projectivization of the infinite-dimensional l-vector space $(L_0,+)$. And since the Kummer homomorphism $\hat{\phi} : \widehat{L} \to \widehat{K}$ maps M_L bijectively onto M_K, the restriction of $\hat{\phi}$ defines a bijection:

$$\phi := \hat{\phi}|_{\mathscr{P}(L_0)} : \mathscr{P}(L_0) = M_L \to M_K = \mathscr{P}(K_0).$$

Notice that the lines in $\mathscr{P}(K_0)$ are subsets of the form $\mathfrak{l}_{t_0,t_1} := (kt_0 + kt_1)^\times/k^\times$ with t_0,t_1 k-linearly independent functions in K_0. In particular, setting $t := t_1/t_0$, we see that $\mathfrak{l}_{t_0,t_1} = t_0 \cdot \mathscr{P}_t$, where $\mathscr{P}_t := (kt+k)^\times/k^\times = j_K(kt+k)^\times$. Further note that $\mathfrak{l}_{t_0,t_1}$ depends only on $\mathsf{t}_0 = j_K(t_0)$ and $\mathsf{t}_1 := j_K(t_1)$, and not on the functions t_0,t_1 themselves. We will therefore also write $\mathfrak{l}_{\mathsf{t}_0,\mathsf{t}_1}$ for the line $\mathfrak{l}_{t_0,t_1}$, and $\mathscr{P}_{\mathsf{t}}$ for $\mathscr{P}_t$.
Correspondingly, the same holds for lines in $\mathscr{P}(L_0)$.

Lemma 51 *The morphism $\phi : \mathscr{P}(L_0) \to \mathscr{P}(K_0)$ respects colineations; more precisely, ϕ maps each line $\mathfrak{l}_{u_0,u_1} \subset \mathscr{P}(L_0)$ bijectively onto $\mathfrak{l}_{t_0,t_1} \subset \mathscr{P}(K_0)$, where $t_0 = \phi(u_0)$, $t_1 = \phi(u_1)$.*

Proof. Setting $\mathsf{t} = \phi(\mathsf{u})$, we get $\mathfrak{l}_{\mathsf{t}_0,\mathsf{t}_1} = \mathsf{t}_0 \cdot \mathfrak{l}_{\mathsf{t}}$ and $\mathfrak{l}_{\mathsf{u}_0,\mathsf{u}_1} = \mathsf{u}_0 \cdot \mathfrak{l}_{\mathsf{u}}$. Hence taking into account that ϕ respects the multiplication, it follows that it is sufficient to show that ϕ maps $\mathscr{P}_{\mathsf{u}}$ bijectively onto $\mathscr{P}_{\mathsf{t}}$, provided $\mathsf{t} := \phi(\mathsf{u})$.
Recall that $\hat{\phi}$ is y-normed, and $\hat{\phi}(\mathsf{y}) = \mathsf{x}$, where $\mathsf{y} = j_L(y)$, $\mathsf{x} = j_K(x)$, for x and y corresponding to each other under $\hat{\phi}$. Moreover, by fact (†) from Remark/Notations 48, $\hat{\phi}$ maps $\mathscr{P}_{\mathsf{y}} = \mathscr{P}_y$ bijectively onto $\mathscr{P}_{\mathsf{x}} = \mathscr{P}_x$. Recall that for every 1-index v of $\mathscr{G}_{\mathscr{D}_K}$, and the corresponding 1-index w of $\mathscr{H}_{\mathscr{D}_L}$, one has commutative diagrams of the form, see Remark 26 (3), and (4)

$$\begin{array}{ccc} \widehat{U}_w & \xrightarrow{j_w} & \widehat{Lw} \\ \downarrow{\scriptstyle\hat{\phi}} & & \downarrow{\scriptstyle\hat{\phi}_v} \\ \widehat{U}_v & \xrightarrow{j_v} & \widehat{Kv} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widehat{L} & \xrightarrow{j^w} & \mathbb{Z}_\ell \varphi_w \\ \downarrow{\scriptstyle\hat{\phi}} & & \downarrow{\scriptstyle a_{vw}} \\ \widehat{K} & \xrightarrow{j^v} & \mathbb{Z}_\ell \varphi_v \end{array}$$

Let κ_t be the relative algebraic closure of $k(t)$ in K_0. We claim that $\phi(\mathscr{P}_u) \subset j_K(\kappa_t)$. Indeed, let v be such that $v(\mathsf{t}') \neq 0$ for some $\mathsf{t}' = \phi(\mathsf{u}')$ with $\mathsf{u}' \in \mathscr{P}_{\mathsf{u}}$. Then by the commutativity of the second diagram above we get $w(\mathsf{u}') \neq 0$. But then it follows that j_w is trivial on $l(u)^\times \cap U_w$. Hence by the commutativity of the first diagram above, it follows that j_v is trivial on $\phi \circ j_L\big(l(u)^\times\big)$, in particular on $\phi(\mathscr{P}_{\mathsf{u}})$. By contradiction, suppose that $\phi(\mathscr{P}_u) \not\subset j_K(\kappa_t)$. Then $\exists$ $\mathsf{u}_1 \in \mathscr{P}_u$ and $t_1 \in K_0$ such that t and t_1 are algebraically independent over k, and $j_K(t_1) =: \mathsf{t}_1 = \phi(\mathsf{u}_1)$. On the other hand, since t,t_1 are algebraically independent over k, there exist "many" v satisfying the following: v is not trivial on $k(t)$ and t is a v-unit, and v is trivial on $k(t_1)$. Note that v being non-trivial on $k(t)$ and t being a v-unit implies that the residue of t at v lies in k; hence $j_v(\mathsf{t}) = 0$. Now let w correspond to v under Φ. Then by the commutativity of the above diagrams, $\mathsf{u} = j_L(u)$ is a w-unit, and $j_w(\mathsf{u}) = 0$. Therefore, w is non-trivial on $l(u)$. Further, $\mathsf{u}_1 = j_L(u_1)$ is a w-unit, and $j_w(\mathsf{u}_1) \neq 0$.

Hence w satisfies both that w is non-trivial on $l(u)$ and that j_w is non-trivial on $U_w \cap l(u)^\times$. Contradiction!

Now choose a prime divisor v of $K|k$ such that the following are satisfied:

(i) v is trivial on κ_x, and t is a v-unit.

(ii) x and t have equal residues in Kv; hence $j_v(x) = j_v(t)$.

Note that (ii) implies that v is trivial on κ_t too, hence j_v maps both $\kappa_x^\times$ and $\kappa_t^\times$ injectively into the residue field Kv.

Now let w correspond to v under the proper morphism $\Phi : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_L}$. Then reasoning as in the proof of Proposition 35, it follows that the following hold:

(j) w is trivial on κ_y.

(jj) y and u have equal residues in Lw; hence $j_w(u) = j_w(y)$.

Further, since j_w and j_v respect addition and multiplication, the following hold:

$$j_v(\mathscr{P}_t) = \mathscr{P}_{j_v(t)},\ \ j_v(\mathscr{P}_x) = \mathscr{P}_{j_v(x)},\ \text{ and }\ j_w(\mathscr{P}_u) = \mathscr{P}_{j_w(u)},\ \ j_w(\mathscr{P}_y) = \mathscr{P}_{j_w(y)}\,.$$

Hence by (i), (ii), respectively (j), (jj), we get $\mathscr{P}_{j_v(t)} = \mathscr{P}_{j_v(x)}$ and $\mathscr{P}_{j_w(u)} = \mathscr{P}_{j_w(y)}$. Since $\hat{\phi}(\mathscr{P}_y) = \mathscr{P}_x$ by the choice of x and y, it follows from $\hat{\phi}_v \circ j_w = j_v \circ \hat{\phi}$ that

$$\hat{\phi}_v(\mathscr{P}_{j_w(y)}) = \hat{\phi}_v\big(j_w(\mathscr{P}_y)\big) = j_v\big(\hat{\phi}(\mathscr{P}_y)\big) = j_v(\mathscr{P}_x) = \mathscr{P}_{j_v(x)}\,.$$

Since $j_v \circ \hat{\phi} = \hat{\phi}_v \circ j_w$, from the equalities above we finally get:

$$j_v\big(\hat{\phi}(\mathscr{P}_u)\big) = \hat{\phi}_v\big(j_w(\mathscr{P}_u)\big) = \hat{\phi}_v\big(\mathscr{P}_{j_w(u)}\big) = \hat{\phi}_v(\mathscr{P}_{j_w(y)}) = \mathscr{P}_{j_v(x)} = \mathscr{P}_{j_v(t)} = j_v(\mathscr{P}_t)\,.$$

Hence $j_v\big(\hat{\phi}(\mathscr{P}_u)\big) = j_v(\mathscr{P}_t)$. Since both $\hat{\phi}(\mathscr{P}_u)$ and $\mathscr{P}_t$ are subsets of $j_K(\kappa_t^\times)$, and j_v is injective on $j_K(\kappa_t^\times)$, we get $\hat{\phi}(\mathscr{P}_u) = \mathscr{P}_t$, as claimed. □

In order to conclude the first part of the proof of Theorem 45 we proceed as follows:

By Lemma 50 above, ϕ respects colineations. Therefore, by the *fundamental theorem of projective geometries,* see e.g. Artin [1], ϕ is the projectivization $\phi = \mathscr{P}(\phi')$ of some linear ι_0-isomorphism $\phi' : (L_0,+) \to (K_0,+)$, i.e., there exists a field isomorphism $\iota_0 : l \to k$, and ϕ' is an isomorphism of abelian groups, such that $\phi'(au) = \iota_0(a)\phi'(u)$ for all $a \in l$ and $u \in L_0$. Moreover, ϕ' is unique up to composition by homotheties of the form $l_a \circ \phi' \circ l_b$ (all $a \in k$, $b \in l$). Further, since $k^\times = \ker(j_K)$ and $l^\times = \ker(j_L)$, it follows that $\phi'(l) = k$. We set

$$\phi_0 := \big(1/\phi'(1)\big)\,\phi'\,,$$

and claim that ϕ_0 is a field isomorphism which maps l isomorphically onto k. Indeed, for a fixed $y \in L_0$, consider $\phi_y : L_0 \to K_0$ defined by $\phi_y(u) := \phi_0(yu)$. Then ϕ_y is a linear ι_0-isomorphism. Set $x = \phi_0(y)$. Then considering projectivizations, and using the fact that $\phi = \mathscr{P}(\phi_0)$ is multiplicative, it follows that for all $u \in L_0$ we have

$$\mathscr{P}(\phi_y)(u) = \mathscr{P}(\phi_0)(yu) = \mathscr{P}(l_x) \circ \mathscr{P}(\phi_0)(u)\,,$$

where l_x is the multiplication by x on K_0. Therefore, there exist $a \in k^\times$ and $b \in l^\times$ such that $l_b \circ \phi_y \circ l_b = l_x \circ \phi_0$. In other words, $a\,\phi_0(ybu) = x\,\phi_0(u)$ for all $u \in L_0$. Setting $u = 1$, and taking into account that $\phi_0(1) = 1$, we have $a\,\imath_0(b)\,x = x$. Thus $a\,\imath_0(b) = 1$, and hence the effects of l_a and l_b cancel each other. Hence we have

$$\phi_0(yu) = x\,\phi_0(u) = \phi_0(y)\,\phi_0(u), \quad (\text{all } u, y \in L_0),$$

hence ϕ_0 is a field morphism. And since $\phi_0(L_0) = K_0$ and $\phi_0(l) = k$, it follows that $\phi_0 : L_0|l \to K_0|k$ is an isomorphism of field extension, as claimed.

Finally, in order to conclude the proof of Theorem 45 we prove the following:

Lemma 52 *$L|L_0$ is a purely inseparable field extension.*

Proof. First, recall that by the last assertion mentioned at Remark/Notations 48, we have $M_L \otimes \mathbb{Z}_{(\ell)} = L^\times_{(\ell)}$ inside $\widehat{L}$. Equivalently, for every $u \in L^\times$ there exists a prime to ℓ integer $n_u > 0$ such that $u^{n_u} \in L_0$. This means in particular that $L|L_0$ is an algebraic extension. Since $L|l$ is a function field over the (algebraically closed) field l, it follows that $L_0|l$ is so, and $L|L_0$ is actually a finite field extension, of degree $[L:L_0] = n > 0$. From this we deduce that if n_u is minimal such that $u^{n_u} \in L$, then $n_u|n$. In particular, all the n^{th} powers u^n, $u \in L$, are contained in L_0. But then $n = [L:L_0]$ must be a power of the characteristic, as claimed. □

For the uniqueness of $\imath$ up to Frobenius twists, one uses the last lemma above, and applies Proposition 38.

This completes the proof of Theorem 45. □

7 Appendix

Here we recall a few facts concerning the pro-ℓ abelian quotient of the fundamental group, and facts concerning the structure of the divisor class group as an abstract group. All these seem to be folklore and might be well be known to the experts, but I cannot give a quick reference. Throughout this section, $K|k$ is a function field over an algebraically closed field k of characteristic $p \geq 0$.

7.1 The Kummer interpretation of ${}_n\mathfrak{Cl}(X)$

Let $K|k$ be a function field, and $X \to k$ a normal model of $K|k$. Recall that D_X is the set of prime divisors of $K|k$ which are defined by the Weil prime divisors of X, and $\mathrm{div}_X : K^\times \to \mathrm{Div}(X)$ the corresponding divisor map. Then denoting by $\mathscr{U}_X := \Gamma(X, \mathscr{O}_X)^\times$ the invertible global functions on X, one has $\ker(\mathrm{div}_X) = \mathscr{U}_X$, and $\mathscr{U}_X/k^\times$ is a finitely generated free abelian group. Finally, one has an exact sequence of the form

$$1 \to \mathscr{U}_X \to K^\times/k^\times \to \mathscr{H}_X(K) \to 0,$$

where $\mathscr{H}_X(K) := \mathrm{div}_X(K)$ is the group of principal (Weil) divisors of X, and the exact sequence defining the divisor class group is

$$1 \to \mathscr{H}_X(K) \to \mathrm{Div}(X) \to \mathfrak{Cl}(X) \to 0.$$

We now want to recall the Kummer interpretation of the prime to the characteristic torsion of $\mathfrak{Cl}(X)$, which is as follows: Let $\mathrm{char}(k) = p \geq 0$ be the characteristic of k, and n a positive integer not divisible by p. Then tensoring the last exact sequence with $\mathbb{Z}/n$, we get the exact sequence

$$0 \to {}_n\mathfrak{Cl}(X) \hookrightarrow \mathscr{H}_X(K)/n \to \mathrm{Div}(X)/n \to \mathfrak{Cl}(X)/n \to 0,$$

where ${}_n\mathfrak{Cl}(X)$ is the n-torsion of $\mathfrak{Cl}(X)$. In particular, if we denote by Δ_n the preimage of ${}_n\mathfrak{Cl}(X) \subset \mathscr{H}_X(F)/n$ in $\mathscr{H}_X(K)$, it follows that we have a canonical exact sequence

$$0 \to \Delta_n \hookrightarrow K^\times/n \to \mathrm{Div}(X)/n \to \mathfrak{Cl}(X)/n \to 0,$$

because $K^\times/n \cong (K^\times/k^\times)/n$ by the fact that k being algebraically closed. Notice also that Δ_n fits into an exact sequence of the form $1 \to \mathscr{U}_X/n \to \Delta_n \to {}_n\mathfrak{Cl}(X) \to 0$; thus it is a quotient of $(\mathbb{Z}/n)^I$ for some index set I.

Fact 53 *In the above context, set $K_n := K[\sqrt[n]{\Delta_n}]$. Then $K_n|K$ is an n-elementary abelian extension of K satisfying:*

(1) $\mathrm{Gal}(K_n|K) \cong \mathrm{Hom}(\Delta_n, \mu_n)$ *canonically.*

(2) *$K_n|K$ is the maximal n-elementary abelian extension of K in which all $v \in D_X$ are unramified.*

(3) *In particular, for $\ell \neq \mathrm{char}(k)$, let $K_{\ell^\infty}|K$ be the maximal pro-ℓ abelian extension in which all $v \in D_X$ are unramified. Then*

$$\mathrm{Gal}(K_{\ell^\infty}|K) = \Pi_{1,D_X} = \mathrm{Hom}(\Delta_\infty, \mu_{\ell^\infty}),$$

where $\Delta_\infty = \varinjlim_n \Delta_n$ fits canonically in $0 \to \mathscr{U}_X \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \Delta_\infty \to {}_{\ell^\infty}\mathfrak{Cl}(X) \to 0$.

(4) *Finally, ${}_n\mathfrak{Cl}(X)$ and $K_n|K$ are finite by Fact 54 below.*

Proof. The first assertion is clear by Kummer theory. For the second assertion, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
1 \to & \Delta_n & \to & K^\times/n & \to & \mathrm{Div}(X)/n & \to & \mathfrak{Cl}(X)/n & \to 0 \\
 & \downarrow & & \downarrow & & \| & & \| & \\
0 \to & {}_n\mathfrak{Cl}(X) & \to & \mathscr{H}_X(K) & \to & \mathrm{Div}(X)/n & \to & \mathfrak{Cl}(X)/n & \to 0
\end{array}$$

We first show that all $v \in D_X$ are unramified in $K_n|K$. Equivalently, we have to prove that for all $f \in K^\times$ whose images lie in Δ_n, all $v \in D_X$ are unramified in $K(\sqrt[n]{f}\,)|K$. Now if the image of $f \in K^\times$ lies in Δ_n, then the image of f in $\mathscr{H}_X(K)$ lies actually in ${}_n\mathfrak{Cl}(X)$. Therefore, the divisor (f) of f has trivial image in $\mathrm{Div}(X)/n$; in other words, there exists some divisor $P \in \mathrm{Div}(X)$ such that $nP = (f)$, i.e., $v(f) \in n \cdot vK$ for all $v \in D_X$. But then by Hilbert decomposition theory, it follows that every $v \in D_X$

is unramified in $K(\sqrt[n]{f}\,)|K$. Since f was arbitrary, it follows that all $v \in D_X$ are unramified in $K_n|K$.

Conversely, for $f \in K^\times$, suppose that all $v \in D_X$ are unramified in $K(\sqrt[n]{f}\,)|K$. Equivalently, $v(f) \in n \cdot vK$ for all $v \in D_X$; hence $\mathrm{div}_X(f) \in n \cdot \mathrm{Div}(X)$. Thus there exists some divisor $P \in \mathrm{Div}(X)$ such that $\mathrm{div}_X(f) = nP$ in $\mathrm{Div}(X)$. But then $\mathrm{div}_X(f) \in \mathscr{H}_X(K)$ has a trivial image in $\mathrm{Div}(X)/n$, and therefore, the image of f in $K^\times/n$ lies actually in Δ_n, as claimed.

Finally, the proof of assertion (3) follows from assertion (2) by "taking limits." □

7.2 *On the Weil divisor class group* $\mathfrak{Cl}(X)$

We next want to say a few words about the divisor class group $\mathfrak{Cl}(X) := \mathrm{Div}(X)/\mathscr{H}_X(K)$ (as an abstract group) of a normal model $X \to k$ of $K|k$ in more detail.

Fact 54 *Let k be an algebraically closed field, and $X \to k$ be an integral normal variety. Then the divisor class group $\mathfrak{Cl}(X)$ can be written as a (direct) sum:*

$$(*) \qquad \mathfrak{Cl}(X) \cong A_0(X) + A_1(X),$$

where $A_0(X)$ is the maximal divisible subgroup of $\mathfrak{Cl}(X)$, and $A_1(X)$ is a finitely generated abelian group. Further:

(1) *The maximal torsion divisible subgroup $A_t(X) \subseteq A_0(X)$ of $\mathfrak{Cl}(X)$ is a quotient of $(\mathbb{Q}/\mathbb{Z})^r$ for some $r \geq 0$.*

(2) *If k is an algebraic closure of a finite field, then $A_0(X) = A_t(X)$.*

Proof. First, since $A_0(X) \subseteq \mathfrak{Cl}(X)$ is the maximal divisible group, it follows that it has complements $A(X)$; hence $\mathfrak{Cl}(X) = A_0(X) + A(X)$ as a direct sum. Let K be the function field of X. Let $Y \to k$ and $X \to k$ be normal models of $K|k$ such that $D_X \subseteq D_Y$. We first claim the following:

Claim 1. $\mathfrak{Cl}(X)$ satisfies $(*)$ iff $\mathfrak{Cl}(Y)$ satisfies $(*)$.

Indeed, let $A_0(X) \subset \mathfrak{Cl}(X)$ and $A_0(Y) \subset \mathfrak{Cl}(Y)$ be the (unique) maximal divisible subgroups. Setting $S := D_Y \backslash D_X$, we have that S is finite, $D_Y = D_X \cup S$, and the canonical projection map $\mathrm{pr} : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ has as kernel $\Delta_S := \sum_{v \in S} \mathbb{Z}v$. Thus the canonical projection $\mathrm{pr} : \mathrm{div}_Y(K^\times) \to \mathrm{div}_X(K^\times)$ has kernel $\mathscr{U}_S := \mathrm{div}_Y(K^\times) \cap \Delta_S$, which a finitely generated group, and the canonical map $\overline{\mathrm{pr}} : \mathfrak{Cl}(Y) \to \mathfrak{Cl}(X)$ has kernel $\Delta_S/\mathscr{U}_S$. Further note that $\overline{\mathrm{pr}}\big(A_0(Y)\big) \subset A_0(X)$, because the former group is divisible, thus contained in the unique maximal divisible subgroup $A_0(X)$ of $\mathfrak{Cl}(X)$, and $\overline{\mathrm{pr}} : A_0(Y) \to A_0(X)$ has kernel $\Delta_{S,0} := (\Delta_S/\mathscr{U}_S) \cap A_0(Y)$, which is a finitely generated group, as $\Delta_S/\mathscr{U}_S$ is so. Hence setting $A_S := (\Delta_S/\mathscr{U}_S)/\Delta_{S,0}$, we finally get an exact sequence of the form:

$$(\dagger) \qquad 0 \to A_S \to A(Y) \to A(X) \to 0.$$

Since A_S is a finitely generated group, it follows that $A(Y)$ if finitely generated iff $A(X)$ is so. This concludes the proof of Claim 1.

We next notice that the assertion of the above Claim 1 holds in the same form for the Cartier divisor class group $\mathfrak{CaCl}(X)$, and the proof of this fact is word-by-word the same as in the case of $\mathfrak{Cl}(X)$. Further, if X is smooth (enough factorial), then $\mathfrak{CaCl}(X) = \mathfrak{Cl}(X)$.

We conclude the proof of Fact 54 as follows: Let $X \to k$ be an arbitrary normal integral variety. Further, let $\tilde{X} \to k$ be a projective normal model of the function field $K := \kappa(X)$ of $X \to k$. Then $\tilde{X}$ and X are birational, and hence they have isomorphic open subsets $U \subset X$ and $\tilde{U} \subset \tilde{X}$. Moreover, we can suppose that $U \cong \tilde{U}$ are actually smooth over k. By Claim 1 and its Cartier form, it follows that, first, $\mathfrak{Cl}(X)$ has the structure $(*)$ iff $\mathfrak{Cl}(U)$ does, and second, $\mathfrak{CaCl}(\tilde{X})$ has the structure $(*)$ iff $\mathfrak{CaCl}(\tilde{U})$ does. On the other hand, since $U \cong \tilde{U}$ is smooth over k, hence factorial, we have $\mathfrak{Cl}(U) \cong \mathfrak{CaCl}(\tilde{U})$. Thus finally, it is sufficient to show that $\mathfrak{CaCl}(\tilde{X})$ has the structure $(*)$. In order to conclude, recall that for $\tilde{X} \to k$ projective integral normal, the structure of $\mathfrak{CaCl}(\tilde{X})$ is known: Indeed, by Kleiman [14], Theorem 4.8, Theorem 5.4, Corollary 6.17, Remark 6.19, it follows that $\mathfrak{CaCl}(\tilde{X}) = \mathrm{Pic}(\tilde{X})(k)$ has the structure $(*)$.

This concludes the proof of Fact 54. □

Fact 55 *In the above context and notation, let $X \to k$ and $Y \to k$ be normal models of $K|k$ such that $D_X \subseteq D_Y$. In the notations from the proof of Claim 1, let $A_{\mathrm{tor}}(X) \subseteq A_0(X)$ be the torsion subgroup, whence $A_{\mathrm{tor}}(X)$ is divisible too; and let $A_\tau(X) \subseteq \mathfrak{Cl}(X)$ be the preimage of the torsion group of $\mathfrak{Cl}(X)/A_0(X)$, hence $A_\tau/A_0(X)$ is finite, and $\mathfrak{Cl}(X)/A_\tau(X)$ is finitely generated free abelian. The projection $\overline{\mathrm{pr}} : \mathfrak{Cl}(Y) \to \mathfrak{Cl}(X)$ gives rise to homomorphisms $\mathrm{pr}_\tau : A_\tau(Y) \to A_\tau(X)$, $\mathrm{pr}_0 : A_0(Y) \to A_0(X)$, $\mathrm{pr}_{\mathrm{tor}} : A_{\mathrm{tor}}(Y) \to A_{\mathrm{tor}}(X)$, and the following hold:*

(1) $\mathrm{pr}_{\mathrm{tor}}$ *has finite kernel and cokernel isomorphic to $(\mathbb{Q}/\mathbb{Z})^r$, where r the rational rank of $(\Delta_S/\mathscr{U}_S) \cap A_0(Y)$. Therefore, $A_{\mathrm{tor}}(X) \cong A_{\mathrm{tor}}(Y) \oplus (\mathbb{Q}/\mathbb{Z})^r$ as abstract groups.*

(2) *The rational ranks of $A_1(Y)$, $A_1(X)$ satisfy $\mathrm{rr}\big(A_1(Y)\big) = \mathrm{rr}\big(A_1(X)\big) + |S| - r$. Hence one has $\mathrm{rr}\big(A_1(Y)\big) = \mathrm{rr}\big(A_1(X)\big) + |S|$ iff $\mathscr{U}_S$ is trivial iff $\mathscr{U}_Y = \mathscr{U}_X$.*

(3) *If the equivalent conditions from (2) above are satisfied, then $\mathrm{pr}_{\mathrm{tor}}$ and pr_0 are isomorphisms, and pr_τ maps $A_\tau(Y)/A_0(Y)$ injectively into $A_\tau(X)/A_0(X)$.*

(4) *Suppose that X has/satisfies the following equivalent properties:*
   - i) *$A_\tau(X)/A_0(X)$ and $A_{\mathrm{tor}}(X)$ are minimal.*
   - ii) *$A_1(Y) \cong A_1(X) \oplus \mathbb{Z}^{|D_Y \setminus D_X|}$ for all normal models $Y \to k$ of $K|k$ with $D_X \subset Y$.*
   - iii) *$|A_\tau(X)/A_0(X)|$ is minimal and $\mathrm{rr}\big(A_1(Y)\big) = \mathrm{rr}\big(A_1(X)\big) + |D_Y \setminus D_X|$ for all Y as above.*

Then $A_{\mathrm{tor}} \subseteq A_0(X) \subseteq A_\tau(X)$ are birational invariants of the function field $k(X)$.

Proof. Everything follows immediately from the proof of Claim 1. For assertions (2), (3) and (4), use the exact sequence (†). In particular, note that if the equivalent conditions from assertions (2) are satisfied, then $\mathscr{U}_S$ trivial, and therefore $\Delta_{S,0} = \Delta \cap A_0(X)$ is a torsion subgroup of the free abelian group Δ_S. Hence $\Delta_{S,0}$ is trivial, and $A_S = \Delta_S$ is a finite free $\mathbb{Z}$-module. Thus the exact sequence $(\dagger)$ becomes $0 \to \Delta_S \to A(Y) \to A(X) \to 0$, etc. □

7.3 On the (pro-ℓ abelian) "birational" fundamental group $\Pi_{1,K}$

Let $K|k$ be a function field. It is well known that if $K|k$ has regular complete models $X \to k$, then the "usual" fundamental group $\pi_1(X)$ whose open subgroups parametrize all the étale connected covers of X is a birational invariant of $K|k$, in the sense that $\pi_1(X)$ does not depend on the particular regular complete model $X \to k$.

In case $K|k$ does admit complete regular models, one can consider the following replacement for the fundamental group of complete regular models: Let $D_{K|k}$ be the set of prime divisors of $K|k$. For every $v \in D_{K|k}$, let $\overline{v}$ be prolongations of v to an algebraic closure $\overline{K}$ of K, and $T_{\overline{v}} \subset G_K$ the inertia group of $\overline{v}$ in the separable closure $K^{\mathrm{s}}|K$ of K in $\overline{K}$. Then the prolongations $\overline{v}$ are conjugated under G_K; thus the set of all the inertia groups $T_{\overline{v}}$ is closed under G_K-conjugation. Therefore, the closed subgroup $T_K \subset G_K$ generated by all the $T_{\overline{v}}$ for all $v \in D_{K|k}$ and all their prolongations $\overline{v}$ to $\overline{K}$ is a normal subgroup of G_K. Moreover, setting

$$\pi_{1,K} := G_K/T_K,$$

we see by the functoriality of Hilbert decomposition theory that the fixed field $\tilde{K}$ of T_K in K^{s} is the maximal field extension of K in which all the prime divisors v of $K|k$ are not ramified. We will call $\pi_{1,K}$ the birational fundamental group for $K|k$.

More generally, let $D \subset D_{K|k}$ be any set of prime divisors of K, e.g., $D = D_X$ is the set of prime divisors defined by the Weil prime divisors of a normal model $X \to k$ of $K|k$. Then we denote by $T_D \subset G_K$ the subgroup generated by all the $T_{\overline{v}}$ with $v \in D$ and $\overline{v}$ all the prolongations of v to $\overline{K}$. As above, T_D is normal in G_K, and the quotient

$$\pi_{1,D} := G_K/T_D$$

will be called the fundamental group for D. We notice that open subgroup of $\pi_{1,D}$ parametrize all the finite extensions $L|K$ of K in which all $v \in D$ are not ramified.

In the same way, we introduce/define the pro-ℓ abelianizations of the fundamental groups introduced above,

$$\Pi_{1,K} := \Pi_K/T_{D_{K|k}} \quad \text{and} \quad \Pi_{1,D} := \Pi_K/T_D,$$

and call them the pro-ℓ abelian birational fundamental group for $K|k$, respectively for D.

Remarks 56 Let $X \to k$ be a normal model of $K|k$, and $\pi_1(X) \to \Pi_1(X) := \pi_1^{\ell,\mathrm{ab}}(X)$ the canonical projection. In the above context and notation, the following hold:

(a) For every $D \subseteq D_K$ there are canonical surjective projections $\pi_{1,D} \to \pi_{1,K}$, and $\pi_{1,D_X} \to \pi_1(X)$, which by the functoriality of Hilbert decomposition theory give rise to surjective projections $\Pi_{1,D} \to \Pi_{1,K}$ and $\Pi_{1,D_X} \to \Pi_1(X)$.
(b) Nevertheless, if X is regular, then $\pi_{1,D_X} = \pi_1(X)$, thus also $\Pi_{1,D_X} = \Pi_1(X)$, by the *purity of the branch locus.*
(c) And if X is complete and regular, then $\pi_{1,K} = \pi_1(X)$, thus also $\Pi_{1,K} = \Pi_1(X)$, by the *purity of the branch locus.*

Fact 57 *In the above context, the following hold:*

(1) *Let $X \to k$ be a complete normal model of $K|k$, and $U \subseteq X$ is a regular open subvariety. Then there are canonical surjective projections $\pi_1(U) \to \pi_{1,K} \to \pi_1(X)$.*
(2) *There exists a geometric set D_X such that $\Pi_{1,D_X} \to \Pi_{1,K}$ is an isomorphism. Hence $\Pi_{1,D_{X'}} \to \Pi_{1,K}$ is an isomorphism, provided $D_X \subseteq D_{X'}$.*
(3) *Let X be an affine normal curve with $\Pi_{1,D_X} = \Pi_{1,K}$. Then either $X \cong \mathbb{A}^1_k$, or X is isomorphic to $E\backslash\{\mathrm{pt}\}$ with E a complete curve of genus one.*
(4) *If X is a normal model of $K|k$ such that $\Pi_{1,D_X} = \Pi_{1,K}$, then the group of global invertible sections on X is $\mathscr{U}_X := k^\times$.*

Proof. To (1): The existence and surjectivity of $\pi_1(U) \to \pi_{1,K}$ follow from (a) and (b) above. For the existence and surjectivity of $\pi_{1,K} \to \pi_1(X)$, we notice first that since $X \to k$ is a complete variety, for every $v \in D^1_K$, there exists a dominant canonical k-morphism $\mathrm{Spec}\,\mathscr{O}_v \to X$. On the other hand, since a base change of an étale cover is étale, we have that if $Y \to X$ is some finite connected étale cover defined by some finite quotient of $\pi_1(X)$, then Y is integral, and $Y \times_X \mathrm{Spec}\,\mathscr{O}_v$ is étale over $\mathscr{O}_v$. Equivalently, v is unramified in the field extension $k(X) \hookrightarrow k(Y)$; hence the image of the inertia group T_v in $\pi_1(X)$ is trivial, etc.

For assertion (2), first consider a small enough affine open subset $X_0 \subset X'$ such that X_0 is regular. Then $X_0 \to k$ is a quasi-projective regular model for $K|k$; hence $\pi_{1,D_{X_0}} = \pi_1(X_0)$, by the purity of the branch locus. Therefore, $\pi_{1,D_{X_0}}$ is finitely generated, hence a finite module as $\mathbb{Z}_\ell$, as it is an abelian pro-ℓ group. Since $\Pi_{1,D_{X_0}} \to \Pi_{1,K}$ is surjective, $\Pi_{1,K}$ is a finite $\mathbb{Z}_\ell$-module too. But then

$$\Delta = \ker(\Pi_{1,D_{X_0}} \to \Pi_{1,K})$$

is also a finite $\mathbb{Z}_\ell$-module. Finally, by the definition of $\Pi_{1,K}$, it follows that for every $g \in \Delta$ there exists some $v \in D^1_K$ such that $g \in T_v$. Since Δ is finitely generated, there exists a finite set $\Sigma \subset D^1_K$ such that the images of T_v (all $v \in \Sigma$) in $\Pi_{1,D_{X_0}}$ generate Δ. In order to conclude, consider any quasi-projective normal model $X \to k$ such that $D_{X'}, \Sigma \subseteq D_X$ (hence in particular, $D_{X_0} \subseteq D_X$ too).

Assertion (3) follows immediately from the structure theorem for (the abelian pro-ℓ quotient of the) fundamental groups of a normal curve.

Finally, assertion (4) follows from the following: By contradiction, let $f \in U_X$ be a non-constant global invertible section. For every $n = \ell^e$ with $e \geq 0$, consider the normalization $X_n \to X$ of X in the finite subextension $K_n := K[\sqrt[n]{f}\,]$ of $K \hookrightarrow K'$. Then since f is a v-unit for all $v \in D_X$, it follows that v is unramified in $X_n \to X$, and therefore, $\mathrm{Gal}(K_n|K)$ is a quotient of Π_{1,D_X}. On the other hand, if w is a prime divisor of $K|k$ with $w(f) > 0$, then w is ramified in $K_n|K$ for $n \gg 0$; hence $\mathrm{Gal}(K_n|K)$ is not a quotient of $\Pi_{1,K}$. Contradiction! □

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