University *of* Western Ontario

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New Developments in Anabelian Geometry

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Lecture I: **Galois via Topology**

- Theme: *Non-tautological description of*

$$\mathrm{Gal}_{\mathbb{Q}} = \mathrm{Aut}(\overline{\mathbb{Q}})$$

§ 1. From Topology to Numbers

– Recall RET: *There exists equiv of categories*

Topology&Geometry:

compact Riemann surfaces $\mathcal{X}$

Compl. algebraic curves:

projective smooth complex curves X

Function fields:

function fields $\mathcal{F}$ in one variable over $\mathbb{C}$

$$\mathcal{X} \quad \longleftrightarrow \quad \mathfrak{M}(\mathcal{C}) = \mathcal{F} = \mathbb{C}(X) \quad \longleftrightarrow \quad X$$

- **Basic Question** (Grothendieck):

  Which $\mathcal{X}$, hence which X, hence which $\mathcal{F}$,

  are defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$, hence number fields?

Theorem (Grothendieck/Belyi).

X is defined over $\overline{\mathbb{Q}}$ if and only if

$\exists\ X \twoheadrightarrow \mathbb{P}^1_{\mathbb{C}}$ ramified only at $0, 1, \infty$.

Proof:

"$\Rightarrow$" by Belyi.

"$\Leftarrow$" by Groth. (étale fundam. groups).

Comments:

- This is the origin of Grothendieck's "Designs d'enfants".
- A cover $X \to \mathbb{P}^1_{\mathbb{C}}$ as in Theorem is a *Belyi map*.
- Study the action of $\mathrm{Gal}_{\mathbb{Q}}$ on the space of "Designs"

  (many many people: Malle, Klüners–M., Schneps,

  Lochak–Sch., Zapponi, math-physicists, etc. etc. etc...)

- **<u>Not yet reached Goal</u>**: Topol./combin. description of $\mathrm{Gal}_{\mathbb{Q}}$...

Interesting open Question/Problem

Higher dim extensions of the above Theorem.

Two possible ways:

- First, $X \longrightarrow \mathbb{P}^1_C$ has at most n ram points.
- Replace curves by higher dim varieties, e.g., surfaces (?!?).
- Several partial results...

Theorem (Ronkine 2004; unpublished).

The birat. class of complex proj. surface of general type is defined over $\overline{\mathbb{Q}}$ iff $\exists$ smooth fibration $X_0 \twoheadrightarrow \mathbb{P}^1_{\mathbb{C}} \backslash \{0, 1, \infty\}$.

Another idea (Grothendieck)

- Study $\mathrm{Gal}_{\mathbb{Q}}$ via its action on the "algebraic fundamental group" of $\mathbb{P}^1_{\mathbb{C}} \backslash \{0, 1, \infty\}$:
  - That is $\widehat{F}_{<\tau_0, \tau_1>} =: \widehat{F}_2$ free on loops τ_0, τ_1.
  - And $\exists$ can embedd $\mathrm{Gal}_{\mathbb{Q}} \hookrightarrow \mathrm{Aut}(\widehat{F}_2)$.
  - Etc...

§ 2. Warm-up: π_1^{top} and $\mathrm{Gal}_{\mathbb{R}}$

– $\mathfrak{Var}_{\mathbb{R}}$ category of $\mathbb{R}$-varieties X. Consider:

a) $X^{\text{an}} := X(\mathbb{C})$ and $\widetilde{X^{\text{an}}} \to X^{\text{an}}$ univ. cover.

b) $\pi_1^{\text{top}}(X) := \pi_1^{\text{top}}(X^{\text{an}}) = \mathrm{Aut}_{X^{\text{an}}}(\widetilde{X^{\text{an}}})$.

– $\mathrm{Gal}_{\mathbb{R}} \cong \{\pm 1\}$ acts on X^{an} and $\pi_1^{\text{top}}(X^{\text{an}})$.

• Get exact sequence

$$1 \to \pi_1^{\text{top}}(X) \to ??? \to \mathrm{Gal}_{\mathbb{R}} \to 1$$

and repres $\rho_X : \mathrm{Gal}_{\mathbb{R}} \to \mathrm{Out}\big(\pi_1^{\text{top}}(X)\big)$.

– View $\pi_1^{\text{top}} : \mathfrak{Var}_{\mathbb{R}} \to \mathfrak{Groups}$ as functor.

• Then $(\rho_X)_X$ gives rise to a morphism:

$$\rho_{\mathbb{R}} : \mathrm{Gal}_{\mathbb{R}} \to \mathrm{Aut}(\pi_1^{\text{top}})$$

Theorem. *$\rho_{\mathbb{R}}$ is an isomorphism.*

Comments:

- New non-tautological description of $\mathrm{Gal}_{\mathbb{R}}$.
- What about other base fields $K \subset \mathbb{C}$?

§ 3. Étale/algebr. fundam. groups

– $K \subseteq \mathbb{C}$ base field, e.g. $\mathbb{Q}$, $\mathbb{R}$.

– $\mathfrak{Var}_K$ category of K-varieties X.

Idea: Play same game with K instead of $\mathbb{R}$ **!!!**

- `Bad News:`
  - Gal$_K$ does not act on X^{an} and/or $\pi_1^{\text{top}}(X^{\text{an}})$;
  - No "nice" $\rho_X : \text{Gal}_K \to \text{Out}\big(\pi_1^{\text{top}}(X)\big)$.
- *Good News*:
  - Finite covers $\mathcal{X} \to X^{\text{an}}$ are algebraic (Serre's GAGA).
  - Gal$_K$ acts on the set of all such $\mathcal{X} \to X^{\text{an}}$.

– For each $X \in \mathfrak{Var}_K$, consider:

a) $\widehat{X} \to X$ proj.limit of all $\mathcal{X} \to X^{\text{an}}$, the "algebraic universal cover" of X.

b) $\pi_1^{\text{alg}}(X) := \text{Aut}_X(\widehat{X}) = \widehat{\pi_1^{\text{top}}}(X^{\text{an}})$, the "algebraic fundamental group" of X.

Theory of étale fundam. groups

(as developed by Grothendieck) gives:

– $\exists$ canonical exact sequence:

$$1 \to \pi_1^{\text{alg}}(X) \to \pi_1^{\text{et}}(X) \to \text{Gal}_K \to 1.$$

– Represent $\rho_X : \text{Gal}_K \to \text{Out}\big(\pi_1^{\text{alg}}(X)\big)$.

– $\pi_1^{\text{alg}} : \mathfrak{Var}_K \to \mathfrak{prof.Groups}$ is a functor

(where morphisms of $\mathfrak{prof.Groups}$

are all the continuous outer hom's).

– Finally $(\rho_X)_X$ gives rise to a morphism:

$$\rho_K : \text{Gal}_K \to \text{Aut}(\pi_1^{\text{alg}})$$

Grothendieck's philosophy:

- Study Gal_K via the representation ρ_K.
- Same problem for the representations $\rho_{\mathcal{V}}$ arising from "well understood" sub-categories $\mathcal{V} \subset \mathfrak{Var}_K$.

§ 4. Studying Gal_K via ρ_K

Question/Problem:

1) Find categories $\mathcal{V}$ for which $\mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{alg}})$ has "nice" topological/combinatorial description.

2) Find such categories $\mathcal{V}$ for which $\rho_{\mathcal{V}} : \mathrm{Gal}_K \to \mathrm{Aut}(\pi_{\mathcal{V}}^{\mathrm{alg}})$ is isomorphism.

• *This would give a new description of* Gal_K**!!!**

Example: Teichmüller modular tower

– $\mathcal{M}_{g,n}$ moduli space of curves (genus g, with n marked pts.)

– "Connecting" morphisms ("boundary" embeddings, gluings, etc.)

– $\mathcal{T} = \{\mathcal{M}_{g,n} \mid g, n\}$ category of varieties over $\mathbb{Q}$, the *Teichmüller modular tower.*

– Note: $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

$$\mathcal{M}_{0,n} \cong (\mathcal{M}_{0,4})^{n-3} \setminus \{\text{fat diagonal}\}$$

Facts (to Question/Problem 1):

– $\widehat{GT} = \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{V}_0})$ is the famous

Grothendieck–Teichmüller group.

– Intensively studied by Drinfel'd, Ihara, Deligne,

Schneps, Sch.–Lochak, Sch.–Nakamura,

Sch.–Harbater, Ihara–Matsumoto, etc., etc.

– $\widehat{GT} = \{(\lambda, f) \mid \lambda \in \widehat{\mathbb{Z}}^{\times}; \; f \in [\widehat{F}_2, \widehat{F}_2]$, rel. I, II, III$\}$

– Several variants ${}^{I}GT$, ${}^{II}GT$, ${}^{IV}GT$, etc. of $\widehat{GT}$.

– Actually: rel. I, II, III, are not independent

(Schneps, Sch.–Lochak; Furusho: III suffices)

– Boggi–Lochak (to be thoroughly checked):

- $\exists$ variant ${}^{\mathrm{new}}GT$ of $\widehat{GT}$ such that

$${}^{\mathrm{new}}GT = \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{T}}).$$

Conclusion: Question/Problem 1 has quite

satisfactory answer(s) for $\mathcal{V} = \mathcal{T}$

and its subcategories.

Facts (to Question/Problem 2):

– Belyi: Let $K|\mathbb{Q}$ be number field.

If $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1\infty\} \in \mathcal{V}$, *then*

$\rho_{\mathcal{V}} \colon \mathrm{Gal}_K \to \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{V}})$ *injective.*

– What about surjectivity?

• **I/OM** (Ihara/Oda–Matsumoto Conj).

$\rho_{\mathbb{Q}} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathfrak{Var}_{\mathbb{Q}}})$ *is isomorphism.*

Theorem. I/OM *has positive answer.*

• Actually: Given $X \in \mathfrak{Var}_{\mathbb{Q}}$, $\dim(X) > 1$, set:

- $\mathcal{V}_X = \{$Zariski opens $V \subset X,\ U \subset \mathbb{P}^1\}$, and canonical inclusions $V' \hookrightarrow V''$ and projections $V \to U$, as morphisms.

-*Then* $\rho_{\mathcal{V}_X} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(\pi^{\mathrm{alg}}_{\mathcal{V}_X})$ *is isom.*

Note: With $X = \mathbb{P}^2$, this gives (in principle) a pure topol./combin. construction of $\mathrm{Gal}_{\mathbb{Q}}$**!**

LECT. II: **Grothendieck & beyond**

§ 5. Anabelian phenomena (Exmpl)

a) $\mathcal{X}$ compact Riemann surface, genus g.

Theorem.

$\pi_1^{\mathrm{top}}(\mathcal{X}, x) \cong < \sigma_1, \tau_1, \ldots, \sigma_g, \tau_g \mid \prod_i [\sigma_i, \tau_i] = 1 >$.

Comment...

b) K field, $\mathrm{Gal}_K = \mathrm{Aut}(K^{\mathrm{sep}}|K)$

absolute Galois group of K.

Theorem (Artin–Schreier, 1927).

Suppose that Gal_K *is finite* $\neq \{1\}$.

Then $\mathrm{Gal}_K \cong \{\pm 1\}$, *and* K *is real closed.*

Comments...

Questions:

- What about the isomorphy type of $\mathcal{X}$?
- What about the isomorphy type of K?

c) **Example: Global fields**

Global fields are the finite extensions of:

- $\mathbb{Q}$, e.g., $K = \mathbb{Q}[\zeta_n]$, $\zeta_n = e^{\frac{2\pi i}{n}}$, etc.
- $\mathbb{F}_p(x)$, e.g., $K = \mathbb{F}_p(x, y)$, $y^2 = x^3 + x + 1$, etc.

Theorem (Neukirch, Uchida, Iwasawa, 1970's).

Let L and K be global fields. Then one has:

$\mathrm{Gal}_K \cong \mathrm{Gal}_L$ *as prof.groups* $\Rightarrow$ $K \cong L$ *as fields.*

Comments:

- Actually, every isomorphism $\mathrm{Gal}_K \cong \mathrm{Gal}_L$ originates functorially from a unique field isomorphism $L \cong K$.
- It is though not clear how to "recover" the field isomorphism type of K from Gal_K.

§ 6. Grothendieck's Anab Geom

– $X \in \mathfrak{Var}_K$, more general $X \in \mathfrak{Sch}_K$, then:

$$1 \to \pi_1^{\mathrm{alg}}(X) \to \pi_1^{\mathrm{et}}(X) \to \mathrm{Gal}_K \to 1.$$

• Grothendieck's idea is that under certain

$***$ *"anabelian" hypothesis on schemes X* $***$

geometry/arithmetic of X should be encoded functorially in the homotopy sequence

$$\pi_1^{\mathrm{et}}(X) \to \mathrm{Gal}_K.$$

Comment: "Geometry/arithmetic" means isomorphy type, morphisms between anabelian varieties, rational points, etc.

Conjecture: The following are anab. schemes:

- Finitely generated infinite fields.
- Hyperbolic curves over such fields.

• Not clear which higher dim varieties should be.

Some results

I) **Finitely generated (infinite) fields**

– Finitely generated field are of the form:

$K = \mathbb{Q}(x_1, \ldots, x_n)$ or $K = \mathbb{F}_p(x_1, \ldots, x_n)$.

– They are obvious generalization of global fields.

– They are exactly the *function fields* of (irreducible) varieties over $\mathbb{Q}$ or $\mathbb{F}_p$ (all p).

Theorem (P 1994).

1) *The isomorphism type of fin. gen. infinite fields K is functorially encoded in* Gal_K.

2) *Every open embedding* $\mathrm{Gal}_K \hookrightarrow \mathrm{Gal}_L$ *arises functorially from some finite* $L \hookrightarrow K$.

Comments:

- Thm above generalizes the famous Neukirch–Unchida Thm.
- Gives a "Galois characterization of finitely generated fields".

II) **Hyperbolic curves**

– $X \to K$ is *hyperbolic* curve, if it has:

a) smooth geom. irred completion X^c.

b) $\chi_X = 2 - 2g - r < 0$, $r = |\,\overline{X}^c \setminus \overline{X}\,|$

– And recall the canonical exact sequence:

$$1 \to \pi_1^{\mathrm{alg}}(X) \to \pi_1^{\mathrm{et}}(X) \to \mathrm{Gal}_K \to 1.$$

Examples:

- $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- $X = E \setminus \{\mathrm{pt}\}$, E elliptic curve.
- X smooth complete, genus > 1.

Theorem (Tamagawa, 1995).

1) *$K \subset \mathbb{C}$ fin. gen. subfield. Then the isom type of an* <u>affine</u> *hyperbolic curve $X \to K$ is functorially encoded in $\pi_1^{\mathrm{et}}(X)$.*

2) *Moreover, every open embedd $\pi_1^{\mathrm{et}}(X) \hookrightarrow \pi_1^{\mathrm{et}}(Y)$ arises functorially from some étale $X \twoheadrightarrow Y$.*

Further results:

– Tamagawa: *Similar holds over finite fields K.*

- **Actually**: This last result is the **Key Fact**!

  ...proceed using "specialization techniques".

– Mochizuki (1996): *Same holds for* complete *hyperbolic curves over fin. gen. fields $K \subset \mathbb{C}$.*

Comment: Tool kit includes Tamagawa's **Key Fact**.

– Stix (2000): *Similar holds for* all *hyperbolic "non-constant" curves over fin. gen. fields.*

Comment: Tool kit includes Mochizuki's methods, and a result by Raynaud, Pop–Saidi, Tamagawa.

Remark: $K \subset \mathbb{C}$ fin.gen, $X \to K$ hyperbolic.

- $\pi_1^{\text{top}}(X)$ tells only whether X^{an} open or not.
- OTOH, $\pi_1^{\text{et}}(X)$ encodes isom type of X.
- Isom types of X^{an} and of X are the same (GAGA).

Conclude: $\pi_1^{\text{et}}(X)$ *encodes isom type of X^{an} !*

§ 7. Beyond Grothendieck's...

- "Yoga" of Grothendieck's anabelian geometry

  is the presence of a reach arithmetical action...

- During the 1990's one realized that there are

  unexpected anabelian phenomena in

  total absence of arithmetical action!

1) Bogomolov (1990). Consider:

- $K|k$ function field, td.deg > 1, $k = \overline{k}$.
- $\mathrm{Gal}''_K \to \mathrm{Gal}'_K$ maximal *abelian-by-central*

  pro-ℓ quotient of Gal_K, $\ell \neq$ char.

Conjecture (Bogomolov's Program, 1990):

$K|k$ can be recovered from Gal''_K.

Comment:

- $\mathrm{tr.deg}(K|k) > 1$ is necessary, because...
- This goes far beyond Grothendieck's anabelian idea:
  - a) First, no Galois action, because $\mathrm{Gal}_k = \{1\}$.
  - b) Gal''_K carries only minimal Galois information.

2) Tamagawa (1990's). Consider:

- $K = \overline{K}$, char > 0, $X \to K$ affine hyperbolic.
- $p^{\rm tm} : \pi_1^{\rm alg}(X) \to \pi_1^{\rm tm}(X)$ the tame quotient.

Theorem (Tamagawa).

a) *The tame quotient is encoded in* $\pi_1^{\rm alg}(X)$.

b) *Isom type of* $X \subset \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ *encoded in* $\pi_1^{\rm tm}(X)$.

3) Raynaud, P–Saidi, Tamagawa ($2000 \pm \epsilon$):

- $K = \overline{K}$, $\mathcal{M}_g(K)$ moduli space to genus $g > 1$.
- $\pi_g : \mathcal{M}_g(K) \to \mathfrak{Prof.groups}$, $C \mapsto \pi_1^{\rm alg}(C)$.

- If char$(K) = 0$, then π_g is constant:

$$\pi_g(X) = \pi_1^{\rm alg}(X) = \widehat{\Pi}_g.$$

- But if $K = \overline{\mathbb{F}}_p$, then π_g is "interesting":

Theorem (Raynaud, P–Saidi, Tamagawa).

$\pi_g : \mathcal{M}_g(\overline{\mathbb{F}}_p) \to \mathfrak{Prof.groups}$ *has finite fibers.*

Comments:...

§ 8. Bogomolov's Program

Recall:

- $K|k$ function field, td.deg > 1, $k = \overline{k}$.
- $\mathrm{Gal}''_K \to \mathrm{Gal}'_K$ maximal *abelian-by-central* pro-ℓ quotient of Gal_K, $\ell \neq$ char.

Conjecture (Bogomolov's Program, 1990):
$K|k$ can be recovered functorially from Gal''_K.

Comment:
- $\mathrm{tr.deg}(K|k) > 1$ is necessary, because...
- This goes far beyond Grothendieck's anabelian idea:
  a) First, no Galois action, because $\mathrm{Gal}_k = \{1\}$.
  b) Gal''_K carries only minimal Galois information.

Evidence:

– Bogomolov (1990), B–Tschinkel (2002):
Theory of commuting liftable pairs.

Comment: Recovering valuations of K...

Theorem (P 1999/2003/2007).
Bogomolov's Program okay over $k = \overline{\mathbb{F}}_p$.

Comment: B.–Tsch. special case for $\mathrm{tr.deg}(K|k) = 2$.

Strategy of proof (P):

Main Idea: Consider $\mathcal{P}(K,+) := K^\times/k^\times$
the "projectivization" of the k-v.s. $(K,+)$.

- Then $(K,+,\cdot)$ can be recovered from
$\mathcal{P}(K,+)$ endowed with its collineations,
via Artin's *Fundam. Thm. Proj. Geometry.*

NOW:

- Kummer Theory: $\widehat{K^\times} = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}'_K, \mathbb{Z}_\ell)$.

- And $\mathcal{P}(K,+) = K^\times/k^\times \hookrightarrow \widehat{K^\times}$.

Hence to do list: Given $\mathrm{Gal}''_K \twoheadrightarrow \mathrm{Gal}'_K$,

1) Recover $K^\times/k^\times \hookrightarrow \widehat{K^\times}$.
2) Recover the collineations inside $K^\times/k^\times$.
3) Check compatibility with Galois Theory.

HOW TO DO THAT:

- Local Theory, i.e., recover:
  - primes of $K|k$; divisorial sets D_X of primes.
- Global Theory, i.e., recover:
  - Div(X), then $K^\times/k^\times$, then collineations; and finally check Galois compatibility.

Local Theory (few words):

- primes of $K|k$: DVR R_v with $k \subset R_v \subset K$ such that $\text{tr.deg}(Kv\,|\,k) = \text{tr.deg}(K|k) - 1$.

- $D = \{v_i\}_i$ geometric, if $\exists$ normal model $X \to k$ such that $D = D_X := \{v \mid \text{Weil prime div. of } X\}$.

• Recovering the primes:

- 1st Method: Use B.-Tsch. "commuting pairs"...

- 2nd Method: Use Mináč et al...

Comment: This is very very technical stuff...

Lecture III: The p-adic world

§ 9. A result by Mochizuki

- $k|\mathbb{Q}_p$ finite field extension, $X \in \mathfrak{Var}_k$.

- $\pi_1^{\mathrm{alg}}(X) \to \Pi_X^{\mathrm{alg}}$ maximal pro-p quotient.

• $\exists$ canonical exact sequence:

$$1 \to \Pi_X^{\mathrm{alg}} \to \Pi_X \to \mathrm{Gal}_k \to 1.$$

- $X \mapsto \Pi_X$ functor from $\mathfrak{Var}_k$ to Gal_k–groups.

Theorem (Mochizuki 1999).
Let $X, C \in \mathfrak{Var}_k$ with C hyperbolic curve. Then every open Gal_k–hom $\Pi_X \to \Pi_C$ corresponds functorially to a dominant k–hom $X \to C$.

Comments:

- The proof is very very technical.
- Main technical tools: p-adic Hodge Theory,
  and Faltings theory of almost étale covers.

Comments:

- The above theorem generalizes all the previous anabelian results for fin. gen fields $K \subset \mathbb{C}$ and hyperbolic curves over such fields.

- The above theorem goes beyond Grothendieck's anabelian geometry, as it uses p-adic arithmetic information only, and not global one.

- As an application, Mochizuki proves that k–surfaces which are "Artin neighborhoods" are anabelian. But it is not clear what to do / how to proceed in higher dimensions.

- Corry-P 2007: Using Thm above one shows:

Theorem (Corry–P 2007).

Let $K|k, L|k$ be function fields. Then every open group Gal_k*–hom* $\Pi_K \to \Pi_L$ *originates functorially from a field k–embed $L \hookrightarrow K$.*

§ 10. On the section Conjecture

- It concerns the rational points of curves.

– k base field, $X \to k$ hyperbolic curve,

$X_0 \supseteq X$ smooth completion; $S = X_0 \backslash X$.

– $\widetilde{X} \to X$ algebraic univ cover,

$pr_X : \pi_1^{\text{et}}(X) \to \text{Gal}_k$ canonical projection.

- For $x \in X_0(k)$, $\tilde{x} \in \widetilde{X}$ preimage,

let $D_{\tilde{x}|x} \subset \pi_1^{\text{et}}(X)$ decomposition group.

<u>One has</u>:

1) If $x \in X$, then $pr_X : D_{\tilde{x}|x} \to \text{Gal}_k$ isom.

Hence $\exists$ conjugacy class of sections

$s_x : \text{Gal}_k \to \pi_1^{\text{et}}(X)$ defined by x.

2) If $x \in S$, i.e., "cuspidal point" of X.

Then $\exists$ "bouquet" of conjugacy classes of

sections $s_x : \text{Gal}_k \to \pi_1^{\text{et}}(X)$ defined by x.

Comments/Examples: Tangential base pts, etc...

• Let k be finitely generated infinite field,

$X \to k$ hyperbolic curve over k,

$X_0 \to k$ its smooth completion.

Section Conjecture.

Let $X \to k$ non-constant. Then every section

$$s : \mathrm{Gal}_k \to \pi_1^{\mathrm{et}}(X) \ \textit{ of } \ pr_X : \pi_1^{\mathrm{et}}(X) \to \mathrm{Gal}_k$$

arises from some $x \in X_0(k)$ as indicated above.

Birational section Conjecture.

Let $K := k(X) = k(X_0)$. Then every section

$$s : \mathrm{Gal}_k \to \mathrm{Gal}_K \ \textit{ of } \ pr_K : \mathrm{Gal}_K \to \mathrm{Gal}_k$$

arises from some $x \in X_0(k)$ as indicated above.

Comments:

- Initial motivation: Method to prove Mordell's Conj.
- Unfortunately: Relation to Mordell's Conj unclear yet.
- Unfortunately: Still completely mysterious/unknown.

Variants:

- The corresponding p-adic conjectures:
  - obtained for k finite extension of $\mathbb{Q}_p$.
- The corresp truncated (p-adic) conjectures:
  - obtained by replacing $\pi_1^{\text{et}}(X)$, resp Gal_K, by corresponding "verbal" quotients.
  - E.g., $\pi_1^{\text{et}}(X)$ replaced by Π_X, etc...
- The motivic variant: Replace $\pi_1^{\text{et}}(X)$ by the "motivic fundamental group".

Evidence:

- Nakamura (1990's): k number field, $X \subset \mathbb{P}^1$. Then "cuspidal" k-rational points of X are in bijection with "cuspidal" sections.

- Tamagawa's "conditional" section Conj (1990's).

- Mochizuki's p-adic "cuspidal" sections (2005/06/07).

Results:

Theorem (Koenigsmann 2004).

The birational p-adic section conjecture holds.

- Actually, one can do <u>much better</u>, as follows:

- $k|\mathbb{Q}_p$ finite with $\mu_p \subset k$, and $X \to k$,

and $X_0 \to k$, and $K = k(X)$ as above.

- $k''|k \hookrightarrow K''|K$ max. $\mathbb{Z}/p$ meta-abelian ext.

- $\overline{\mathrm{pr}}_K : \overline{\mathrm{Gal}}''_K \to \overline{\mathrm{Gal}}''_k$ canonical projection.

Remarks:

- $\overline{\mathrm{Gal}}''_k$ is a finite well known meta-abelian p-group

(by local class field theory).

- $\overline{\mathrm{Gal}}''_K$ can be effectiv constructed/computed.

Theorem (P 2007). *Every section*

$$s : \overline{\mathrm{Gal}}''_k \to \overline{\mathrm{Gal}}''_K \ \ \textit{of}\ \ \overline{\mathrm{pr}}_K : \overline{\mathrm{Gal}}''_K \to \overline{\mathrm{Gal}}''_k$$

arises from some $x \in X_0(k)$ as indicated above.

Theorem (M. Kim 2005):

Motivic section Conj holds for $X = E \backslash \{pt\}$,

and gives new proof of Siegel's Theorem.

Hopes:

Minhyong Kim:

- Using non-abelian p-adic Hodge Theory:

Section Conj + "minimal" Conj imply a

(p-adically) effective Mordell's Conjecture!

Sh. Mochizuki:

- Using p-adic anabelian ideas:

The "right" p-adic uniformization would

imply the ABC Conjecture!

Others?

Short list of open Problems:

1) Prove/disprove: $\mathbb{Q} \hookrightarrow \widehat{GT}$ is isomorphism.

2) Prove pro-ℓ/truncated variants of I/OM.

3) Prove such variants of I/OM for
"generalized" Drinfel'd upper half-planes.

4) Relation between Problem 2 and the
representations of $\mathrm{Gal}_{\mathbb{Q}}$, respectively $\mathrm{Gal}_{\mathbb{Q}_p}$
(global/local Langlans Philosophy).

5) Prove the hom-form of the anab. conjectures:

a) If K, L fin. gen infinite fields, then
every open homomorphism $\mathrm{Gal}_K \to \mathrm{Gal}_L$
originates from a field embedding $L \hookrightarrow K$.

b) If $X \to K$, $Y \to L$ are hyperbolic curves,
then every open hom $\pi_1^{\mathrm{et}}(X) \to \pi_1^{\mathrm{et}}(Y)$
originates from dominant morph $X \to Y$.

6) Prove pro-ℓ/truncated variants of Problem 5.

7) Generalize Belyi's Theorem.

8) What are the higher dim anabelian varieties?

9) Relation between the section Conjecture and effective Mordell's Conjecture.

10) Prove/disprove the global/p-adic section Conjecture.

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