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ON THE PYTHAGORAS NUMBER OF FUNCTION FIELDS OF CURVES OVER NUMBER FIELDS

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Abstract. We show that the Pythagoras number $p(K)$ of the function field K of an integral curve over a number field satisfies $p(K) \leqslant 6$. That is, every sum of squares in K is representable as a sum of at most six squares in K.

1. Introduction

Let F be an arbitrary field, and $\Sigma(F^{\cdot 2}) \subset F$ be the set of finite sums of squares $\sum_i a_i^2$ of elements $a_i \in F$. For $a \in \Sigma(F^{\cdot 2})$, let $\ell(a)$ denote the least number n such that a is a sum of n squares, and set $\ell(a) = \infty$ if a is not a sum of squares of elements of F. It turns out that $\Sigma(F^{\cdot 2})$ plays a fundamental role in the arithmetic of F and beyond. For instance, Hilbert Problem 17 is about $\Sigma(F^{\cdot 2})$ in the case $F = \mathbb{R}(t_1, \dots, t_d)$ is the field of rational functions in d variables $t_1, \dots, t_d$ over the field of reals numbers $\mathbb{R}$, and asks whether every $f \in F$ which is semi-positive definite lies in $\Sigma(F^{\cdot 2})$. Hilbert Problem 17 was solved by Artin by introducing and studying the *formally real fields,* i.e., fields F which allow some total ordering. In the process, Artin showed that a field F allows a total ordering if and only if $-1 \notin \Sigma(F^{\cdot 2})$, that is, $\ell(-1) = \infty$, and in particular, a formally real field F has $\mathrm{char}(F) = 0$. Moreover, $\Sigma(F^{\cdot 2}) \subset F$ is precisely the set of *totally positive elements* $x \in F$, i.e., $x \in F^\times$ which are positive in all total field orderings of F. See e.g. [Pf2, Ch. 6] for details about this.

Related to $\Sigma(F^{\cdot 2})$, one of the intriguing fine invariants of general fields F is the so called Pythagoras number $p(F)$ of F, which is defined by

$$p(F) := \sup \{\ell(a) \mid a \in \Sigma(F^{\cdot 2})\},$$

that is, if $p(F)$ is finite, then every $a \in \Sigma(F^{\cdot 2})$ is a sum of at most $p(F)$ squares of elements of F. The Pythagoras number plays an important role in number theory, the theory of quadratic forms, the semi-algebraic geometry, e.g. the real Nullstellensatz, and the study of the real spectrum in algebra, model theory, etc., and the research in this direction is classical and extensive, see e.g. [Ar, Ca, Gr, Pf1, Pf2, Pf3, La, Pr1, Sch, Wi], where much more on this matter can be found. One should notice right away that the question about $p(F)$ is significant/interesting only over fields of characteristic zero, because for $\mathrm{char}(F) = p > 2$, one has $-1 = a^2 + b^2$ in $\mathbb{F}_p$, hence the sum of three squares $q_3 = x_1^2 + x_1^2 + x_2^2$ represents 0, thus represents everything. Therefore, if not explicitly otherwise stated, all the fields F, K, etc., we will consider below have characteristic zero. Among the classical facts about the

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Pythagoras number one has the following (which is just a very tiny portion of what would be a comprehensive list), see e.g. [Pf2, Ch. 7, especially §1] for more details about this.

- If K is a number field, then $p(K) \leqslant 4$ by [Si], cf. [Pf2, Ch. 7, 1.4, (2), (3)].
- If $K = \mathbb{R}(t_1, \dots, t_d)$, then $d+1 \leqslant p(K) \leqslant 2^d$ by [Pf1], cf. [Pf2, Ch. 7, 1.4, (5)]. Moreover, $p(K) = 2^d$ for $d \leqslant 2$, by [CEP, CT1], not known whether $p(K) = 2^d$ for $d > 2$.
- If $K = \mathbb{Q}(t)$, then $p(K) \leqslant 8$ by [La]. More precisely, if $K = k(t)$ with k a number field, then $p(K) \leqslant 5$ by [Po, H-J], cf. [Pf2, Ch. 7, Thm 1.9].

For general fields F one has the following:

- If F is not formally real, then $p(F)$ is either 2^n or $2^n + 1$, cf. [Pf2, Ch. 7, Lemma 1.3].
- For every $n > 0$ there is some (formally real) F with $p(F) = n$, see [Ho].

Note that the fields F above are in general rather "far away" from being arithmetical fields, e.g. function fields over global and/or local fields.

A major progress on studying $p(K)$ for K *finitely generated* with $\mathrm{char}(K) = 0$ and absolute transcendence degree $d = \mathrm{td}(K)$ the transcendence degree of K over $\mathbb{Q}$ was achieved using, first, the Milnor Conjecture [Mn], proved by Orlov–Vishik–Voevodsky [OVV] with input from Voevodsky, Rost, see [Kh, Pf3], and second, higher Hasse local-global principles type results à la Kato, as initiated by Kato in his seminal paper [Ka], see Colliot-Thélène [CT2] for $d = 1$, Colliot-Thélène–Jannsen [CT-J] for $d = 2$, and Jannsen [Jn] for $d > 2$. Namely, as corollaries of the named results one has the following.

Theorem A (See [CT2, CT-J], [Jn, Cor. 0.7]). *Let K be a finitely generated field with* $\mathrm{char}(K) = 0$. *Then $p(K) \leqslant 7$ if* $\mathrm{td}(K) = 1$, *respectively $p(K) \leqslant 2^{d+1}$ if $d = \mathrm{td}(K) > 1$.*

The proof is based on both the (proved) Milnor Conjecture and one of the higher dimensional Hasse local-global principles, conjectured by Kato [Ka, Conjecture 0.4], proved in [Ka, Thm 0.6] in the case $\mathrm{td}(K) = 1$, respectively in Jannsen [Jn, Thm 0.4] in the general case. The weaker upper bound $p(K) \leqslant 2^{d+2}$ in case $\mathrm{td}(K) = d$ can be obtained by more elementary means, still using the Milnor Conjecture, cf. Pfister [Pf3, 4., (b)].

The bounds for the Pythagoras number $p(K)$ in the context of Theorem A above are not sharp, at least not sharp in all cases, e.g. $p\big(\mathbb{Q}(t)\big) = 5$ by Pourchet [Po]. The aim of this short note is to give a sharpening of Theorem A above in the case $d = \mathrm{td}(K) = 1$.

Theorem B (See [P]). *In the above notation, suppose that* $\mathrm{td}(K) = 1$. *Then $p(K) \leqslant 6$.*

We will prove actually a stronger result, see Theorem 2.5 in Section 3, which suggests that Pfister's Conjecture [Pf2, Ch. 7, 1.10] asserting that $p(K) \leqslant 5$ for $\mathrm{td}(K) = 1$ is plausible.

Short historical note. The result above that $p(K) \leqslant 6$ for $\mathrm{char}(K) = 0$, $\mathrm{td}(K) = 1$ was first announced in 1991, but was never "officially" published, and not very much publicized. It was supposed to be presented at the special program in algebra, number theory and model theory in Tel Aviv, but the author did not participate. I would like to thank several experts on the matter, especially J.-L. Colliot-Thélène, Uwe Jannsen, Albrecht Pfister, and many others among whom Moshe Jarden, Alex Prestel, Peter Roquette whom I consulted with once upon a time while producing the first version of this manuscript.

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2. Proof of (Refinements of) Theorem B

Let K be finitely generated with $\mathrm{char}(K) = 0$ and $\mathrm{td}(K) = 1$. Letting $k \subset K$ be the field of constants of K, i.e., the relative algebraic closure of $\mathbb{Q}$ in K, one has: First, k is a number field, and second, $K = k(X)$ is the function field of a projective smooth geometrically integral k-curve X. Let $|X| \subset X$ be the set of closed points, hence the Weil divisor group $\mathrm{Div}(X)$ is the free abelian group on $|X|$. For $P \in |X|$, let $\mathcal{O}_P, \mathfrak{m}_P$ be its local ring and $\kappa_P = \mathcal{O}_P/\mathfrak{m}_P$ be its residue field. Recall that $|X|$ is in bijection with k-valuations rings $\mathcal{O}_w$ of $K|k$ via $(\mathcal{O}_w, \mathfrak{m}_w) = (\mathcal{O}_P, \mathfrak{m}_P)$, and $\kappa_P = \mathcal{O}_P/\mathfrak{m}_P = \mathcal{O}_w/\mathfrak{m}_w$ is finite over k, hence a number field.

Let $\mathbb{P}(k)$ be the set of places of k, and k_v be the completion of k at $v \in \mathbb{P}(k)$. We denote by $\mathbb{P}_p(k) \subset \mathbb{P}(k)$ the set of places above places p of $\mathbb{Q}$, including $p = \infty$. In particular, $v \in \mathbb{P}_p(k)$ are p-adic places if p is a prime number, respectively $v \in \mathbb{P}_\infty(k)$ are archimedean places. Recall that $v \in \mathbb{P}_\infty(k)$ is either real, i.e., $k_v = \mathbb{R}$, or complex, i.e., $k_v = \mathbb{C}$.

This being said, we say that $P \in |X|$ is real, if κ_P has a real place, i.e., there is a field embedding $\kappa_P \hookrightarrow \mathbb{R}$ of κ_P in $\mathbb{R}$; if so, any real place of κ_P defines a real place of k. Let $X^{\mathrm{real}} \subset |X|$ be the set of real closed points of X, and notice that X^{real} might be empty. For $D = \sum_P n_P P \in \mathrm{Div}(X)$, we set $D^{\mathrm{real}} = \sum_{P \in X^{\mathrm{real}}} n_P P$ (possibly trivial). Finally, if $f \in K$ is a non-constant function, let $(f)^{\mathrm{real}} \in \mathrm{Div}(X)$ be the real part of the Weil divisor $(f) \in \mathrm{Div}(X)$ of f. Then one has the following more precise assertion about $p(K)$.

Theorem 2.1 (See [P]). *In the above notation, for every non-constant $f \in \Sigma(K^{\cdot 2})$ one has:*

1) *If $(f)^{\mathrm{real}}$ is trivial, then f is a sum of at most five squares.*
2) *In general, f is a sum of at most six squares.*

We will prove actually a stronger but more technical result, see Theorem 2.5 below, which might serve as a technical tool towards tackling Pfister's Conjecture [Pf2, Ch. 7, 1.10].

We begin by introducing notation and proving Lamma 2.2 and Lemma 2.3 below and summarizing consequences of work by Witt [Wi] in Fact 2.4. Everything we say here should/might be well known to experts.

Namely, in the above notation, for $P \in |X|$, let $w_P : K \to \mathbb{Z}$ be the discrete k-valuation of K with valuation ring $\mathcal{O}_{w_P} = \mathcal{O}_P$. We denote by K_P the w_P-completion of K and notice the following: First, the relative algebraic closure of k in K_P is κ_P, and second, given any uniformizing parameter $t_P \in \mathfrak{m}_P$ at P, one has a (canonical) k-isomorphism $K_P = \kappa_P((t_P))$.

For $v \in \mathbb{P}(k)$, let $X_v := X \times_k k_v$ be the base change of X under $k_v|k$. Then X_v is a projective smooth geometrically integral k_v-curve, because X was geometrically integral over k. Further, $K_v = k_v(X_v) = k_v(X)$ is nothing but the (free) compositum $K_v = k_v K$ of k_v and K over k. Finally, the closed points $P_v \in X_v$ are in bijection with the k_v-valuation rings $\mathcal{O}_{w_v}$ of K_v via $\mathcal{O}_{P_v} = \mathcal{O}_{w_v}$ (and if needed, write w_{P_v} for the valuation defined by P_v).

We next recall a few facts about K_v for $v \in \mathbb{P}_\infty(k)$ the archimedean places of k.

Lemma 2.2. *In the above notation, the following are equivalent:*

(i) *$K = k(X)$ has a total ordering.*
(ii) *There is a real place $v \in \mathbb{P}(k)$ such that $X(k_v) = X_v(k_v)$ is non-empty.*
(iii) *X^{real} is non-empty.*

Proof. (i) ⇒ (ii): Let $\leqslant$ be a total ordering of K and $\mathfrak{p}_K : K \to \mathbb{R}$ be its k-place. Then $\mathfrak{p}_K$ being trivial on k gives rise to a field embedding $k \hookrightarrow \mathbb{R}$, hence to a real place $v \in \mathbb{P}(k)$. If so, $k_v = \mathbb{R}$ and the k-place $\mathfrak{p}_K$ defines a k_v-rational point of X, hence $X(k_v)$ is non-empty.

(ii) ⇒ (iii): Let v be a real place of k, hence $k_v = \mathbb{R}$, such that $X(k_v)$ is non-empty. Recall that the field of real algebraic numbers $k_v^{\rm abs} \subset k_v$ is real closed, and in particular, $k_v|k_v^{\rm abs}$ is an elementary k-extension of real closed fields. Hence since X is defined over k and $X(k_v)$ is non-empty, one has that $X(k_v^{\rm abs})$ is non-empty, see e.g. [P-R, Introduction, Thm 5] or [Pr2, §5, 5.2].[1] On the other hand, every $x \in X(k_v^{\rm abs})$ is given by a closed point $P \in X$ together with a k-embedding $\kappa_P \hookrightarrow k_v^{\rm abs} \hookrightarrow k_v$. Hence $P \in X^{\sf real}$ by mere definition.

(iii) ⇒ (i): Let $\mathfrak{p}_P : K \to \kappa_P$ be the k-place of K at a given $P \in X^{\sf real}$ and $\imath_P : \kappa_P \hookrightarrow \mathbb{R}$ be a k-embedding defining P as a real point of X. Then the k-embedding $\imath_P \circ \mathfrak{p}_P : K \to \mathbb{R}$ gives rise to real place of K, hence to a total ordering of K. □

Next, recall that for closed points $P \in |X|$ and any fixed uniformizing parameter $t_P \in \mathfrak{m}_P$, we identify the w_P-completion K_P of $K = k(X)$ with $\kappa_P((t_P))$. In particular, every non-zero $f \in K$ has a unique representation in $K_P = \kappa_P((t_P))$ of the form

$$f = a_P t_P^n + f_\bullet,\ \text{where}\ f_\bullet := \textstyle\sum_{i>n} a_i t_P^i \in \kappa_P((t_P)) \quad \text{with} \quad a_P \neq 0,\ n = w_P(f)\,.$$

The coefficient $a_P \in \kappa_P$ is called the t_P-leading coefficient of f at P. We notice that the t_P-leading coefficient has the properties:

a) If $f \in \mathcal{O}_P^\times$ is a w_P-unit, the t_P-leading coefficient $a_P \in \kappa_P$ of f is nothing but the image of f in κ_P under $\mathcal{O}_P \twoheadrightarrow \kappa_P$, hence independent of the uniformizing parameter t_P.

b) Let $t'_P = at + \sum_{i>1} a_i t_P^i$ be another uniformizing paramater at P. If a_P, respectively a'_P are the corresponding leading coefficients of f at P, then $a'_P = a^{-w_P(f)} a_P$.

c) The t_P-leading coefficient is multiplicative in the following sense: If $f = gh$ in $K^\times$ and $a_P, b_P, c_P \in \kappa_P$ are the corresponding t_P-leading coefficients at P, then $a_P = b_P\, c_P$.

To complete the list of notations, for effective divisors $A_i \in {\rm Div}(X)$, $i \in I$ finite, we denote:

$$\mathsf{gcd}(A_i)_i = \textstyle\sum_{P\in X} \min_i w_P(A_i)P, \qquad \mathsf{lcm}(A_i)_i = \sum_{P\in X} \max_i w_P(A_i)P$$

Lemma 2.3. *For $P \in X^{\sf real}$, let $f, f_i \in K^\times$ have t_P-leading coefficients $a, a_i \in \kappa_P$ and set $f = at^n + f_\bullet, f_i = a_i t^{n_i} + f_{i\bullet}$ with $n = w_P(f), n_i = w_P(f_i)$. If $f = \sum_i f_i^2$, one has:*

1) *Let $n_0 := 2\min_i n_i$ and set $\Sigma_P = \{i \,|\, 2n_i = n_0\}$. Then $n = n_0$ and $a = \sum_{i\in\Sigma_P} a_i^2$.*
   In particular, $a \in \kappa_P$ is totally positive, i.e., $a > 0$ for all field embeddings $\kappa_P \hookrightarrow \mathbb{R}$.
2) *$(f)_0^{\sf real}$ and $(f)_\infty^{\sf real}$ satisfy: $(f)_0^{\sf real} = 2\,\mathsf{gcd}\big((f_i)_0^{\sf real}\big)_i$ and $(f)_\infty^{\sf real} = 2\,\mathsf{lcm}\big((f_i)_\infty^{\sf real}\big)_i$.*
   In particular, $(f)^{\sf real}$ is of the form $(f)^{\sf real} = 2(A - B)$ with $A, B \geqslant 0$, $\mathsf{gcd}(A, B) = 0$.

Proof. To 1): Given $P \in X^{\sf real}$ and f_i one has: If $i \notin \Sigma_P$, then $w_P(f_i^2) > n_0$ and if $i \in \Sigma_P$, $w_P(f_i^2) = n_0$. Hence $f = \sum_i f_i^2 = \sum_{i\in\Sigma_P}(a_i^2 t_P^{n_0} + f_{i\bullet}) + \sum_{i\notin\Sigma} f_i^2 = (\sum_{i\in\Sigma_P} a_i^2) t_P^{n_0} + f_\circ$ with $w_P(f_\circ) > n_0$. Since $a_i \in \kappa_P$ and κ_P has real embeddings, it follows that the sum of squares $a := \sum_{i\in\Sigma} a_i^2 \in \kappa_P$ is positive for all embeddings $\kappa_P \hookrightarrow \mathbb{R}$, hence $f_\circ = f_\bullet$ and $n = n_0$.

To 2): This follows instantly from assertion 1), because for all $P \in X^{\sf real}$ one has: If P is a zero of f of order n, then $w_P(f) = n = n_0 = \min_i 2w_P(f_i)$, hence $(f)_0^{\sf real} = 2\mathsf{gcd}\big((f_i)_0^{\sf real}\big)_i$ by

[1] See e.g. [F-J, Pr2, P-R] for basics of model theory (of real closed fields). For those who "do not use logic" in their proofs, the same can be achieved using the Implicit Function Thm, invoking that X is smooth, etc.

mere definition. Next let P be a pole of f of order n. Then $-n = w_P(f) = \min_i w_P(f_i)$, hence $n = -w_P(f) = -\min_i w_P(f_i) = \max_i\big(-w_P(f_i)\big)$. Thus one has $(f)^{\text{real}}_\infty = 2\mathsf{lcm}\big((f_i)^{\text{real}}_\infty\big)_i$. □

We finally recall special cases of some (more extensive) results of Witt [Wi].

First, let $v \in \mathbb{P}(k)$ be given. Then k_v is locally compact in the v-topology, hence if $X(k_v) = X_v(k_v)$ is non-empty, it is a compact topological space in the v-topology. Second, every rational function $f_v \in K_v$ defines a continuous map $f_v : X(k_v) \to k_v \cup \infty$, which might have zeros and/or poles in $X_v(k_v)$. In particular, if $x_v \in X(k_v)$ is defined by a closed point $P_v \in X_v$ and w_{P_v} is the corresponding k_v-valuation of K_v, one has:

(i) $f_v(x_v) = 0$ iff $w_{P_v}(f_v) > 0$; (ii) $f_v(x_v) \in k_v^\times$ iff $w_{P_v}(f) = 0$; (iii) $f_v(x_v) = \infty$ iff $w_{P_v}(f) < 0$.

In particular, if v is real and $X(k_v)$ is non-empty, every $f_v \in K_v$ defines a continuous function $f_v : X(k_v) \to \mathbb{R} \cup \infty$, which might be (semi)positive/negative definite of indefinite.

This being said, the special cases of results by Witt [Wi] we will need are as follows.

Fact 2.4 ([Wi, **I**, page 4; **I**′, page 5])**.** *In the above notation, let $v \in \mathbb{P}(k)$ be archimedean. Then $f_v \in K_v^\times$ is a sum of two squares in K_v provided the hypothesis (†) below is satisfied:*

(†) *If $k_v = \mathbb{R}$ and $X(k_v)$ is non-empty, then $f_v(x_v) \geqslant 0$ for all $x_v \in X_v(k_v)$.*

Proof. First, let (†) be satisfied, i.e., $k_v = \mathbb{R}$ and $X(k_v)$ is non-empty and $f_v(x_v) \geqslant 0$ for all $x_v \in X_v(k_v)$. Then by Witt [Wi, **I**, page 4], it follows that f_v is a sum of two squares in K_v.

Second, suppose that (†) is not satisfied, i.e., either $k_v = \mathbb{C}$ or $k_v = \mathbb{R}$ and $X(k_v)$ is empty. Then for every $P_v \in X_v$ one has $\kappa_P \cong_{k_v} \mathbb{C}$, hence $K_{P_v} \cong_{k_v} \mathbb{C}((t_{P_v}))$ for any uniformizing parameter t_{P_v} at P_v, and $f_v = (\frac{1}{2}f_v + \frac{1}{2})^2 + (\frac{i}{2}f_v - \frac{i}{2})^2$ is a sum of two squares in K_{P_v}. Equivalently, the cyclic algebra $\boldsymbol{\alpha} = (-1, f_v)$ is trivial over K_{P_v} for all closed points $P_v \in X_v$. Hence the local-global principle of Witt for the Brauer group of K_v, see [Wi, **I**′, page 5], implies that $\boldsymbol{\alpha} = (-1, f_v)$ is trivial over K_v, thus $f_v \in K_v^\times$ is a sum of two squares in K_v. □

We now announce the stronger assertion from which Theorem 2.1 above—hence Theorem B from Introduction—immediately follows using Lemma 2.3 above.

Theorem 2.5. *Let $K = k(X)$ be the function field of a projective smooth geometrically integral curve X over a number field k. Let $f \in K^\times$ have totally positive t_P-leading coefficients $a_P \in \kappa_P$ at all $P \in X^{\text{real}}$ for some choice of uniformizing parameters t_P. Then one has:*

1) *If $(f)^{\text{real}} \in 4 \cdot \mathrm{Div}(X)$, then f is a sum of five squares in K.*
2) *If $(f)^{\text{real}} \in 2 \cdot \mathrm{Div}(X)$, then f is a sum of six squares in K.*

Proof. Set $(f)^{\text{real}} = 2m(A - B)$ with $1 \leqslant m \leqslant 2$ and $A, B \in \mathrm{Div}(X)$ satisfying $A, B \geqslant 0$, $\mathsf{gcd}(A, B) = 0$; in particular, $A^{\text{real}} = A$ and $B^{\text{real}} = B$, maybe trivial. Let $D > 0$ be a divisor with trivial real part $D^{\text{real}} = 0$ and degree $\deg(D) \gg 0$, e.g. $\deg(D) > 2g_X + 2$. By Riemann-Roch, there are $g_1, h_1 \in K$ with $(g_1)_\infty = A + D$, $(h_1)_\infty = B + D$. Hence $g := g_1^2 + 1$ and $h := h_1^2 + 1$ have $(g)_0^{\text{real}} = 0 = (h)_0^{\text{real}}$ and $(g)_\infty = 2A + 2D$, $(h)_\infty = 2B + 2D$, thus $(g)^{\text{real}}_\infty = 2A$, $(h)^{\text{real}}_\infty = 2B$. Hence $u^m := (h/g)^m = h^m/g^m$ satisfies:

$$(u^m)^{\text{real}} = -(g^m)^{\text{real}} + (h^m)^{\text{real}} = 2mA - 2mB = (f)^{\text{real}}$$

implying that $f_0 := f/u^m$ has $(f_0)^{\text{real}} = 0$. Further, since g and h are sums of squares, by Lemma 2.3, 1) above, it follows that their t_P-leading coefficients at every $P \in X^{\text{real}}$ are totally positive in κ_P. Hence recalling that the t_P-leading coefficient is multiplicative (see item b) before Lemma 2.3 above), the t_P-leading coefficient of $f_0 = f/u$ at every $P \in X^{\text{real}}$ is totally positive in κ_P, because those of f and u are so.

To 1): We first reduce the general case $(f)^{\text{real}} = 4(A - B)$ to the special case when $(f)^{\text{real}} = 0$. Namely in the above notation, one has $(f)^{\text{real}} = 2m(A - B)$ with $m = 2$, and further, $f_0 = f/u^m = f/u^2$ has $(f_0)^{\text{real}} = 0$ and totally positive t_P-leading coefficient in κ_P at every $P \in X^{\text{real}}$. Hence if $f_0 = \sum_{i=1}^5 f_i^2$ in K, then $f = u^2 f_0 = \sum_{i=1}^5 (uf_i)^2$ in K as well.

Therefore it is left to prove assertion 1) in the case $(f)^{\text{real}} = 0$. In order to do so, choose any $a \in \mathbb{Q}$ having ***small*** absolute value $|a|$ and ***large*** 2-adic value $|a|_2$, e.g. $a = \frac{1}{2^n}$ with $n \gg 0$. For the given $f \in K$, consider the function:

$$(*) \qquad \tilde{f} := f - (af + a)^2 = -\big[\big(af + a - \tfrac{1}{2a}\big)^2 + 4a^2(1 - \tfrac{1}{4a^2})\big] \in K\,.$$

Notation/Remark. Let $q_{(\boldsymbol{n})}$ be the n-fold Pfister form $q_{\text{-1},\dots,\text{-1}}$, i.e., the sum of 2^n squares. Then $q_{(\boldsymbol{n})}$ being a Pfister form, its the image $q_{(\boldsymbol{n})}(K^{2^n})$ is closed under multiplication, see e.g. [Pf2, Ch. 2, 2.2]. This extends the classical facts — which follow from Euler's identity — that the set of sums of two, respectively four squares are closed under multiplication.

Recalling that $\mathbb{P}_p(k) \subset \mathbb{P}(k)$ is the set of places of k above p, including $p = \infty$, one has:

(I) If $v \in \mathbb{P}_p(k)$ is a p-adic place, $p \neq 2$, then $q_{(\boldsymbol{2})}$ represents 0 over $\mathbb{Q}_p \subset K_v$. Therefore:

$(*)_p$ *$q_{(\boldsymbol{2})}$ represents $\tilde{f}$ over K_v for any p-adic place $v \in \mathbb{P}_p(k)$, $p \neq 2$.*

(II) If $v \in \mathbb{P}_2(k)$ is dyadic, $1 - \frac{1}{4a^2} = b^2$ is a square in $\mathbb{Q}_2$, hence $-\tilde{f} = \big(af + a - \frac{1}{2a}\big)^2 + (2ab)^2$ is a sum of two squares in K_v, hence a sum of four squares as well. Hence the Pfister form $q_{(\boldsymbol{2})}$ represents both $-\tilde{f}$ and -1 over K_v, thus $q_{(\boldsymbol{2})}$ represents $\tilde{f} = (-\tilde{f})(-1)$ over K_v as well, because the image $q_{(\boldsymbol{2})}(K_v)$ is closed under multiplication. Hence we conclude:

$(*)_2$ *$q_{(\boldsymbol{2})}$ represents $\tilde{f}$ over K_v for any dyadic place $v \in \mathbb{P}_2(k)$.*

(III) If $v \in \mathbb{P}_\infty(k)$, hence an archimedean place, we have the following case discussion.

Case 1. $k_v = \mathbb{R}$ and $X_v(k_v)$ is empty, or $k_v = \mathbb{C}$.

Then by Fact 2.4 one has that every $f_v \in K_v$ is a sum of two squares in K_v, therefore:

$(*)'_\infty$ *$q_{(\boldsymbol{2})}$ represents $\tilde{f}$ over K_v, if either $k_v = \mathbb{R}$ and $X_v(k_v)$ is empty, or $k_v = \mathbb{C}$.*

Case 2. $k_v = \mathbb{R}$ and $X(k_v)$ is non-empty.

Claim. *There are positive constants $c'_v, c''_v > 0$ such that $c'_v < f(x_v) < c''_v$ for all $x_v \in X(k_v)$.*

Proof of the Claim. We first prove that f has no zeros (poles) in $X(k_v)$. By contradiction, suppose that $x_v \in X_v(k_v)$ is a zero (pole) of f, and recall that $k_v^{\text{abs}} \subset k_v$ is a k-extension of real closed fields. Since X and $f \in k(X)$ are defined over k, it follows by [P-R, Introduction, Thm 5] or [Pr2, §5, 5.2], that there is $x_0 \in X(k^{\text{abs}})$ which is a zero (pole) of f.[2] Hence by Lemma 2.2, x_0 defines a zero (pole) $P \in X^{\text{real}}$ of f, implying that $(f)^{\text{real}}$ is non-trivial, contradiction! Thus we conclude that f is defined and has no zeros (poles) on $X_v(k_v)$. Finally, reasoning as above, it follows that if $f(x_v) < 0$ for some $x_v \in X_v(k_v) = X(k_v)$, then

[2] See e.g. [F-J, Pr2, P-R] for basics of model theory (of real closed fields).

there are points $x_0 \in X(k_v^{\rm abs})$ such that $f(x_0) < 0$. In particular, if $P_0 \in X$ is the closed point defined by x_0 (i.e., x_0 is defined by a k-embedding $\kappa_{P_0} \hookrightarrow k_v$), then $P_0 \in X^{\rm real}$ and the leading coefficient $a_P \in \kappa_{P_0}^\times$ of f at P_0 is nothing but $f(x_0)$, hence negative, contradiction!

Next recall that $X(k_v)$ is compact in the v-topology, and the map $f_v : X(k_v) \to k_v$, $x_v \mapsto f(x_v)$ defined by the rational function $f \in K$ is continuous. Hence since $f(x_v) > 0$ for all $x_v \in X(k_v)$, there are positive constants c'_v, c''_v such that one has $c'_v < f(x_v) < c''_v$ for all $x_v \in X(k_v)$. The proof of the Claim is complete.

Next let $\Sigma \subset \mathbb{P}_\infty(k)$ be the set of all real places $v \in \mathbb{P}(k)$ with $X(k_v)$ non-empty. Then Σ is obviously finite, and if Σ is non-empty, set $c' := \min_{v\in\Sigma} c'_v$, $c'' := \max_{v\in\Sigma} c''_v$, where c'_v, c''_v are as defined/introduced in the Claim. Further choose $a \in \mathbb{Q}^\times$ with $|a|$ sufficiently ***small***, to be precise such that $a^2(c''+1)^2 < c'$. Then $\tilde f := f - a^2(f+1)^2 \in K$ satisfies:

For all real $v \in \mathbb{P}_\infty(k)$ with $X(k_v)$ non-empty one has: $\tilde f(x_v) > 0$ for all $x_v \in X(k_v)$.

In particular, $f_v := \tilde f$ satisfies the hypothesis (†) from Fact 2.4 for all real places $v \in \mathbb{P}(k)$ with $X_v(k_v)$ non-empty, hence $\tilde f$ is a sum of two squares over K_v. Therefore, if $|a|$ is sufficiently small as indicated above, the following holds:

$(*)''_\infty$ *$q_{(\mathbf{2})}$ represents $\tilde f$ over K_v for all real $v \in \mathbb{P}_\infty(k)$ with $X(k_v)$ non-empty.*

Finally, we conclude that for $a \in \mathbb{Q}$ having sufficiently ***small*** absolute value $|a|$ and sufficiently ***large*** 2-adic value $|a|_2$, e.g. $a = \frac{1}{2^n}$ with $n \gg 0$, the Pfister form $q_{(\mathbf{2})}$ represents $\tilde f = f - (af+a)^2$ over K_v for all $v \in \mathbb{P}(k)$. Hence by Kato [Ka, Thm 0.8, (2)],[3] one has:

$$q_{(\mathbf{2})} \text{ represents } \tilde f \text{ over } K, \text{ i.e., } \tilde f = (\text{sum of four squares}) \text{ in } K.$$

Therefore, since $\tilde f = f - (af+a)^2$ in K, one has:

$$f = (af+a)^2 + \tilde f = (\text{sum of five squares}) \text{ in } K.$$

This concludes the proof of assertion 1) of Theorem 2.5.

To 2): Since $(f^{\rm real}) = 2m(A-B)$ with $A, B \geqslant 0$ relatively prime and $m = 1$, in the notation at the beginning of the proof of Theorem 2.5, one has that that $f_0 = f/u$ has $(f_0)^{\rm real} = 0$ and totally positive t_P-leading coefficients in κ_P for all $P \in X^{\rm real}$. Hence by assertion 1), it follows f_0 is a sum of five squares in K, say $f_0 = \sum_{i=1}^5 f_i^2$. On the other hand, recalling that $g = g_1^2 + 1$ and $h = h_1^2 + 1$, one has that $u = h/g = (h_1^2+1)/(g_1^2+1) = u_1^2 + u_2^2$ for some $u_1, u_2 \in K$ (which are explicit in terms of g_1, h_1 above). Hence finally one has:

$$f = f_0 u = (f_1^2 + f_2^2)(u_1^2+u_2^2) + (f_3^2+f_4^2)(u_1^2+u_2^2) + f_5^2(u_1^2+u_2^2),$$

hence a sum of six squares in K by the fact that $q_{(\mathbf{1})}(K^2)$ is closed under multiplication.

This concludes the proof of assertion 2), hence the proof of Theorem 2.5 is complete. □

3. Concluding Remarks / open Questions

Naturally, the "elephant in the room" is the question whether Pfister's Conjecture [Pf2, Ch. 7, 1.10] is true for $\mathrm{td}(K) = 1$, i.e., whether $p(K) \leqslant 5$ for finitely generated fields K with $\mathrm{char}(K) = 0$ and $\mathrm{td}(K) = 1$. Note that the (more technical) hypothesis on f in Theorem 2.5

[3] See also the comments about this in loc.cit. at the top of p.146.

is apparently weaker than the hypothesis that f is a sum of squares in K. This suggests (and the author tends to believe) that there is room for improving the bound $p(K) \leqslant 6$, but for the moment it is unclear how to proceed to sharpen the result.

Another interesting question / open problem is about the right generalization of the hypothesis of Theorem 2.5 to higher dimensions. The hope is that finding the "right" generalization of the hypothesis under discussion might be useful in giving better bounds for $p(K)$ in the case $\mathrm{td}(K) > 1$. Beyond finding better/sharp(er) bounds for $p(K)$, I think that relating $\ell(f)$ to other arithmetical/geometrical properties of functions $f \in K$ would be very interesting.

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