

# 2nd International Valuation Conf

— V a l l a d o l i d —

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## L a r g e F i e l d s

— O l d & N e w —

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# 1) INTRODUCTION/MOTIVATION

\* *Inverse Galois Problem / Embedding Problems*

O l d (20<sup>th</sup> Century)

- $K$  field,  $\text{Gal}_K$  abs Galois group.
- *Embedding problem* for  $\text{Gal}_K$  is any diagram

$$\text{EP :} \quad \begin{array}{ccc} & & \text{Gal}_K \\ & & \downarrow \gamma \\ & & A \\ B & \xrightarrow{\alpha} & \end{array}$$

- Denote:  $K_{\text{EP}} := \left( \text{fixed field of } \ker(\gamma) \right)$ .
- EP is called finite, if  $B$  finite,  
and split if  $\alpha$  has sections.
- A (proper) solution of EP is any (surjective)  
 $\beta : \text{Gal}_K \rightarrow B$  with  $\gamma = \alpha \circ \beta$ .

- $K(t)$  rational function field over  $K$
- $pr : \text{Gal}_{K(t)} \rightarrow \text{Gal}_K$  canonical projection.
- Set  $\gamma_t = \gamma \circ pr : G_{K(t)} \rightarrow A$ , and get:
 
$$\text{EP}_t = \left( G_{K(t)} \xrightarrow{\gamma_t} A, B \xrightarrow{\alpha} A \right)$$
 induced embedding problem for  $G_{K(t)}$ .
- For  $\beta_t$  proper solution of  $\text{EP}_t$  denote:
 
$$K(t)_{\beta_t} := \left( \text{fixed field of } \ker(\beta_t) \right).$$
- $\beta_t$  *regular* for  $\text{EP} / \text{EP}_t$  if  $K(t)_{\beta_t} \cap \overline{K} = K_{\text{EP}}$ .

**Remarks:**

- 1)  $K$  Hilbertian, then proper  $\beta_t$  produces  
 “many” proper  $\beta$  via specializations  $t \mapsto a \in K$ .
- 2) Suppose  $A = \{1\}$ . Then every proper (regular)  
 solution of  $\text{EP}$  realizes  $B$  (regularly) over  $K$ .

## **Fundam Problem Prove/disprove:**

$K$  arbitrary field, EP finite split for  $G_K$ .

Then  $EP_t$  has proper regular solutions.

## **Partial (positive) results:**

- Ha, Liu, F–V, P, etc.: Case  $A = \{1\}$ ,  $K...$
- F–V for  $K$  PAC of char zero; P general PAC.
- F–H–V for  $K$  PRC; P for  $K$  PSC.

All the above are special cases of:

**Main Thm.** *If  $K$  large field, and EP finite split for  $G_K$ . Then  $EP_t$  has proper regular solutions.*

## **Corollaries:**

- $K$  Hilbertian large. Then every finite split EP for  $G_K$  has proper solutions.
- The Inverse Galois Problem is solvable over Hilbertian large fields.

## II) Large fields

What is a name... **epais, fertile, anti-Mordellic, weite Körper, AMPLE, ample, pop-fields...**

-  $K$  is called **large**, if  $\forall X$  smooth  $K$ -curve:

$$|X(K)| > 0 \Rightarrow |X(K)| > 1.$$

**Proposition.** 1) *TFAE*:

i)  $K$  large ii)  $K \prec_{\exists} K(u)^h$  iii)  $K \prec_{\exists} K((u))$

iv)  $\forall X$  smooth  $K$ -variety one has:

$$X(K) \neq \emptyset \Rightarrow X(K) \text{ Zariski dense.}$$

2)  $L|K$  alg extension. Then  $K$  large  $\Rightarrow L$  large.

**Examples (classical/obvious):**

- Separably closed fields; PAC fields.
- $K = \mathbb{R}$ ,  $K = \mathbb{Q}_p$ ; complete fields  
(by the Implicit Function Theorem).
- Henselian fields w.r.t. non-trivial valuation  
(by the Higher dim. Hensel's Lemma).
- In partic: PAC, PRC, PpC, PSC,  $p$ -fields...

### III) Large vs Fundam Problem...

N e w (21<sup>th</sup> Century)

-Harbater–Stevenson (2005): Let  $K = k((t, u))$ .

*Every non-trivial finite split EP for  $\text{Gal}_K$   
has  $|K|$  distinct proper solutions.*

- Paran (2009):  $R$  Noeth compl ring...  $K = \text{Quot}(R)$ .

*Every split EP for  $G_K$  has proper solutions.*

**Remark:**

1)  $K = k((t, u)), \text{Quot}(R)$  has no Henselian val!

2) Generally believed  $K$  as above are not large.

Question: Are  $k((t, u)), \text{Quot}(R)$  large?...

Recall:  $R$  is *Henselian* w.r.t.  $\mathfrak{a} \neq (0)$ , if all polynomials  $f(X) \in R[X]$  satisfy: Simple roots of  $f(X) \equiv 0 \pmod{\mathfrak{a}}$  lift to simple roots of  $f(X)$ .

## Examples:

- Henselian valuation rings  $\mathcal{O}_v, \mathfrak{m}_v \dots$
- $k[[t, u]], A[[x_1, \dots, x_n]], \text{ etc...}$
- Complete Noetherian  $R$  rings, etc...

## Theorem 1

*Let  $R$  be a Henselian domain w.r.t.  $\mathfrak{a} \neq (0)$ . Then  $K = \text{Quot}(R)$  is large field. Hence every finite split EP for  $\text{Gal}_K$  has proper regular solutions.*

**Examples:**  $k((t, u)), \text{Quot}(R)$ , etc., are large...

## Theorem 2

*Let  $R$  be Noetherian complete w.r.t.  $\mathfrak{a} \neq (0)$ , and  $K = \text{Quot}(R)$ . If  $\text{Krull.dim}(R) > 1$ , then every finite split EP for  $G_K$  has  $|K|$  proper solutions.*

**Comment:**  $\exists$  (more gen) sufficient cond's for having  $|K|$  proper solutions, like...

## IV) Proofs

### Proof of Thm 1:

- $C$  a smooth  $K$ -curve with  $x \in C(K)$ .
- W.l.o.g.  $C \subset \mathbb{A}_K^2$ , and  $x = (0, 0)$ .
- And  $C$  defined by  $f(X_1, X_2) = 0$  over  $R$ .
- W.l.o.g.  $f(X_1, X_2) = \delta X_2 + \tilde{f}$ , with  $\delta \neq 0$ , and...
- “Change of variables”:  $X_1 = \delta Y_1, X_2 = \delta Y_2$ .
- Get:  $g(Y_1, Y_2) = f(\delta Y_1, \delta Y_2) = \delta^2 [Y_2 + \tilde{g}(Y_1, Y_2)]$ .
- Then  $h(Y_1, Y_2) := Y_2 + \tilde{g}(Y_1, Y_2)$  defines  $C$  too.
- For  $a \in \mathfrak{a}$ , look at  $h_a(X) := h(a, X)$ .
- $\bar{0} \in R/\mathfrak{a}$  is a simple root of  $h_a(X) \pmod{\mathfrak{a}}$ .
- Conclude that  $\exists b \in \mathfrak{a}$  such that  $h_a(b) = 0$ .
- \* Equiv,  $h(a, b) = 0$ , i.e.,  $(a, b) \in C(K)$ , etc.

## Proof of Thm 2:

- $R$  Noetherian, complete,  $\text{Krull.dim}(R) > 1$ .
- Denote  $\mathcal{V} := \{ \mathfrak{p} \mid \mathfrak{p} \text{ minimal } \neq (0) \}$ .
- For  $L|K$  fin Galois, let  $\mathcal{V}_{L|K} \subseteq \mathcal{V}$   
be the totally decomposed primes in  $L|K$ .

**Lemma.** a)  $\text{supp}_{\mathcal{V}}(a)$  is finite  $\forall a \in K^{\times}$

b)  $|\mathcal{V}_{L|K}| = |K|$

*Proof.* Quite technical (don't give it as an exercise to your students, *es sei denn...*)

Finally apply:

### **Theorem 3.**

*Let  $K$  be Hilbertian large field,  $X$  a set of DVR's of  $K$  satisfying Lemma. Then every non-trivial finite split EP for  $G_K$  has  $|K|$  proper totally ramified disjoint solutions.*

**Proof:** ...quite involved... Rough idea:

- By contrad:  $\exists$  EP non-trivial s.t. its distinct proper solutions  $\beta_i, i \in I$ , satisfy  $|I| < |K|$ .
- Let  $L \subset K$  be rel.alg.closed subfield s.t. all the  $\beta_i$  are “defined” over  $L$ .
- Then  $|L| = |I| < |\mathcal{V}| = |K|$  (WHY?).
- Hence  $\exists v \in \mathcal{V}$  trivial on  $L$  (WHY?).
- Using the *v-adic density* of Hilbertian sets, and the Main Thm, construct a solution  $\beta$  of EP s.t.  $v$  is not totally inert in  $K_\beta|K$ .
- Hence  $\beta$  independent of all  $\beta_i$ , Contradiction!

# F u t u r e S t u f f :

It's

your

turn,

very young

generation! . . .