

ON THE BIRATIONAL p -ADIC SECTION CONJECTURE

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ABSTRACT. In this manuscript we introduce/prove a \mathbb{Z}/p *meta-abelian* form of the birational p -adic Section Conjecture for curves. This is a much stronger result than the usual p -adic birational Section Conjecture for curves, and makes an *effective* p -adic Section Conjecture for curves quite plausible.

1. INTRODUCTION

Let $X \rightarrow k$ be a complete geometrically integral smooth curve over a field k . Recall that Grothendieck’s Section Conjecture, as it evolved from GROTHENDIECK’s *Esquisse d’un Programme* [G1] and his *Letter to Faltings* [G2], predicts that under certain “anabelian hypotheses” π_1 gives rise to a bijection between the k -rational points of X , which are actually the sections of $X \rightarrow k$, and the (conjugacy classes) of sections of $\pi_1(X) \rightarrow \pi_1(k)$.

The aim of this note is to formulate and prove a very “minimalistic” birational variant of this conjecture in the case where k is a finite field extension of \mathbb{Q}_p .

- For the beginning, let k be an arbitrary base field, and $K|k$ the function field of a complete geometrically integral smooth curve $X \rightarrow k$. Let $\tilde{K}|K$ be some Galois extension, and let $\text{Gal}(\tilde{K}|K)$ denote its Galois group. Further let $\tilde{k} := \bar{k} \cap \tilde{K}$ be the “constants” of \tilde{K} , and consider the resulting canonical exact sequence:

$$1 \rightarrow \text{Gal}(\tilde{K}|K\tilde{k}) \longrightarrow \text{Gal}(\tilde{K}|K) \xrightarrow{\tilde{p}r_K} \text{Gal}(\tilde{k}|k) \rightarrow 1.$$

Let $\tilde{X} \rightarrow X$ be the normalization of X in the field extension $K \hookrightarrow \tilde{K}$. For $x \in X$ and a preimage $\tilde{x} \in \tilde{X}$, let $T_x \subseteq Z_x$ be the inertia, respectively decomposition, groups of $\tilde{x}|x$, and $G_x := \text{Aut}(\kappa(\tilde{x})|\kappa(x))$ be the residual automorphism group. By decomposition theory, one has a canonical exact sequence

$$(*) \quad 1 \rightarrow T_x \rightarrow Z_x \rightarrow G_x \rightarrow 1.$$

Now let x be k -rational, i.e., $\kappa(x) = k$. Since $\tilde{k} \subset \kappa(\tilde{x})$, the projection $Z_x \xrightarrow{\tilde{p}r_K} \text{Gal}(\tilde{k}|k)$ gives rise to a canonical surjective homomorphism $G_x \rightarrow \text{Gal}(\tilde{k}|k)$, which in general is not injective. Nevertheless, if $\tilde{k} = \kappa(\tilde{x})$, then $G_x \rightarrow \text{Gal}(\tilde{k}|k)$ is an isomorphism. Hence if the exact sequence $(*)$ splits, then $\tilde{p}r_K$ has sections $\tilde{s}_x : \text{Gal}(\tilde{k}|k) \rightarrow Z_x \subset \text{Gal}(\tilde{K}|K)$,

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called **sections above** x ; and notice that the conjugacy classes of sections \tilde{s}_x above x build a “bouquet”, which is in canonical bijection with the non-commutative continuous cohomology pointed set $H_{\text{cont}}^1(\text{Gal}(\tilde{k}|k), T_x)$ defined via the split exact sequence $(*)$.

Note that if $\text{char}(k) = 0$, then T_x is $\text{Gal}(\tilde{k}|k)$ -isomorphic to a quotient of $\widehat{\mathbb{Z}}(1)$, thus Abelian, hence $H_{\text{cont}}^1(\text{Gal}(\tilde{k}|k), T_x)$ is a group. Further, if $\tilde{K} = K^s$ and $\tilde{k} = k^s$ are separable closures of K and k , then $G_x = \text{Gal}(k^s|k)$, and $(*)$ is split, thus sections above x exist; and if $\text{char}(k) = 0$, then $T_x \cong \widehat{\mathbb{Z}}(1)$ as G_k -modules, hence $H_{\text{cont}}^1(G_k, T_x) \cong \widehat{k}^\times$ via Kummer Theory.

Finally, if v is an arbitrary valuation of K , and \tilde{v} is a prolongation of v to \tilde{K} , then we denote by $T_v \subseteq Z_v$ the inertia/decomposition groups of $\tilde{v}|v$, and by $G_v = Z_v/T_v$ the residual automorphism group; and if $\tilde{s}_v : \text{Gal}(\tilde{k}|k) \rightarrow Z_v \subseteq \text{Gal}(\tilde{K}|K)$ is a section of $\tilde{p}r_K$, then we say that \tilde{s}_v is a **section above** v .

- Next let p be a fixed prime number. We denote by $K'|K$ a maximal \mathbb{Z}/p elementary abelian extension of K , and by K'' a maximal \mathbb{Z}/p elementary abelian extension of K' . Then $K''|K$ is a Galois extension, which we call the maximal \mathbb{Z}/p **elementary meta-abelian extension** of K . Note that $k' := \bar{k} \cap K'$, and $k'' := \bar{k} \cap K''$ are the maximal \mathbb{Z}/p elementary abelian extension, respectively the maximal \mathbb{Z}/p elementary meta-abelian extension of k . Further we consider the canonical surjective projections:

$$pr'_K : \text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k), \quad pr''_K : \text{Gal}(K''|K) \rightarrow \text{Gal}(k''|k).$$

We will say that a section $s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ of pr'_K is **liftable**, if there exists a section $s'' : \text{Gal}(k''|k) \rightarrow \text{Gal}(K''|K)$ of pr''_K which lifts s' to $\text{Gal}(k''|k)$.

Note that if the p^{th} roots of unity μ_p are contained in k , hence in K , then by Kummer Theory we have: $K' = K[\sqrt[p]{K}]$, and $K'' = K'[\sqrt[p]{K'}]$, and similarly for k .

- From now on, in the above context suppose that k is a finite extension of \mathbb{Q}_p . Then the promised “minimalistic” form of the birational p -adic Section Conjecture is the following:

Theorem A. *In the above notations, suppose that $\mu_p \subset k$. Then the following hold:*

1) *Every k -rational point $x \in X$ gives rise to a bouquet of conjugacy classes of liftable sections $s'_x : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ above x , which is in bijection with $H^1(\text{Gal}(k'|k), \mathbb{Z}/p(1))$.*

2) *Let $s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ be a liftable section. Then there exists a unique k -rational point $x \in X$ such that s' equals one of the sections s'_x as defined above.*

Actually one can reformulate the question addressed by Theorem A in terms of p -adic valuation, and get the following stronger result, see Section 2, H), for notations, definitions, and a few facts on p -adically closed fields and p -adic valuations v , e.g., the p -adic rank d_v of v , respectively AX-KOCHEN [A-K], and PRESTEL-ROQUETTE [P-R] for proofs:

Theorem B. *Let k be a p -adically closed field with p -adic valuation v , and suppose that $\mu_p \subset k$. Let $K|k$ be a field extension with $\text{tr.deg}(K|k) = 1$. Then the following hold:*

1) *Let w be a p -adic valuation of K with $d_w = d_v$. Then w prolongs v to K , and gives rise to a bouquet of conjugacy classes of liftable sections $s'_w : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ above w .*

2) *Let $s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ be a liftable section. Then there exists a unique p -adic valuation w of K such that $d_w = d_v$, and $s' = s'_w$ for some s'_w as above.*

Remark.

1) First, we notice that the above assertions do not hold if $\mu_p \not\subset k$. Indeed, if $\mu_p \not\subset k$, then the maximal pro- p quotient $G_k(p)$ of G_k is a pro- p free group on $[k : \mathbb{Q}_p] + 1$ generators, see e.g. [NSW], Theorem 7.5.11. From this it follows that all the sections $s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ of pr'_K are liftable; thus not all the sections originate from k -rational points $x \in X$.

2) Nevertheless, in the case μ_p is not contained in the base field, assertions similar to Theorems A and B hold in the following form: Let $l|\mathbb{Q}_p$ be some finite extension, $Y \rightarrow l$ a complete geometrically integral smooth curve with function field $L = \kappa(Y)$. Let $k|l$ be a finite Galois extension with $\mu_p \subset k$. Setting $K := Lk$, consider the field extensions $K'|K \hookrightarrow K''|K$ and $k'|k \hookrightarrow k''|k$ as above. Then $k' = K' \cap \bar{l}$ and $k'' = K'' \cap \bar{l}$, and $K'|L$ and $K''|L$, as well as $k'|l$ and $k''|l$ are Galois extensions too, and one gets surjective canonical projections

$$pr'_L : \text{Gal}(K'|L) \rightarrow \text{Gal}(k'|l), \quad pr''_L : \text{Gal}(K''|L) \rightarrow \text{Gal}(k''|l).$$

As above, we will say that a section $s'_L : \text{Gal}(k'|l) \rightarrow \text{Gal}(K'|L)$ of pr'_L is **liftable**, if there exists a section $s''_L : \text{Gal}(k''|l) \rightarrow \text{Gal}(K''|L)$ of pr''_L which lifts s'_L . Then one has the following extensions of Theorem A and Theorem B:

Theorem A⁰. *In the above notations and hypothesis, the following hold:*

1) *Every l -rational point $y \in Y$ gives rise to a bouquet of conjugacy classes of liftable sections $s'_y : \text{Gal}(k'|l) \rightarrow \text{Gal}(K'|L)$ above y , which is in bijection with $H^1(\text{Gal}(k'|l), \mathbb{Z}/p(1))$.*

2) *Let $s'_L : \text{Gal}(k'|l) \rightarrow \text{Gal}(K'|L)$ be a liftable section. Then there exists a unique l -rational point $y \in Y$ such that s'_L equals one of the sections s'_y as defined above.*

Theorem B⁰. *Let l be a p -adically closed field with p -adic valuation v , and let $L|l$ be a field extension with $\text{tr.deg}(L|l) = 1$. Then in the above notations the following hold:*

1) *Let w be a p -adic valuation of L with $d_w = d_v$. Then w prolongs v to L , and gives rise to a bouquet of conjugacy classes of liftable sections $s'_w : \text{Gal}(k'|l) \rightarrow \text{Gal}(K'|L)$ above w .*

2) *Let $s'_L : \text{Gal}(k'|l) \rightarrow \text{Gal}(K'|L)$ be a liftable section. Then there exists a unique p -adic valuation w of L such that $d_w = d_v$, and s'_L equals one of the sections s'_w as above.*

We notice that Theorem A⁰ obviously implies the full Galois birational p -adic Section Conjecture, but not vice-versa; see KOENIGSMANN [Ko], for a proof of the latter (among other things), and Remark 2.7 of the present manuscript.

Indeed, for given $Y \rightarrow l$ with function field $L = \kappa(Y)$ as above, let $s : G_l \rightarrow G_L$ be a section of the canonical projection $G_L \rightarrow G_l$. Consider:

a) Finite field extensions $L_i|L$ with $\text{im}(s) \subset G_{L_i}$, and let $Y_i \rightarrow l$ be a complete smooth curve with function field $L_i = \kappa(Y_i)$; and notice that $Y_i \rightarrow l$ is geometrically integral.

b) Finite Galois extensions $k_i|l$ with $\mu_p \subset k_i$, and set $K_i := L_i k_i$. Let $\phi'_i : G_l \rightarrow \text{Gal}(k'_i|l)$ and $\psi'_i : G_{L_i} \rightarrow \text{Gal}(K'_i|L_i)$ be the canonical projections.

Then s gives rise functorially (in L_i and k_i) to liftable sections $s'_i : \text{Gal}(k'_i|l) \rightarrow \text{Gal}(K'_i|L_i)$ of the canonical projection $pr'_i : \text{Gal}(K'_i|L_i) \rightarrow \text{Gal}(k'_i|l)$, such that for $k_i \subseteq k_j$ and $L_i \subseteq L_j$, thus $K_i \subseteq K_j$, one has: $s'_i = pr_{j_i} \circ s'_j$, where $pr_{j_i} : \text{Gal}(K'_j|L_j) \rightarrow \text{Gal}(K'_i|L_i)$ is the canonical projection. By Theorem A⁰ there exists a unique l -rational point $y_i \in Y_i(l)$ such that $s'_i = s'_{y_i}$ in the usual way; and since $s'_i = pr_{j_i} \circ s'_j$, the uniqueness of $y_i \in Y_i(l)$ implies that the canonical

morphism $Y_j \rightarrow Y_i$ maps $y_j \in Y_j(l)$ to $y_i \in Y_i(l)$, and $s'_{y_i} = pr_{ji} \circ s'_{y_j}$. Conclude from this that if $y \in Y(l)$ is the common image of all the points $y_i \in Y_i(l)$ in $Y(l)$, then one has the following: $s = \varinjlim_i s'_i = \varinjlim_i s'_{y_i} = s_y$.

As an application of the results and techniques developed here, one can prove the following fact concerning the p -adic Section Conjecture for curves: Let $k|\mathbb{Q}_p$ be a finite extension, and $X \rightarrow k$ a hyperbolic curve. Then there exists a finite *effectively computable* family of finite geometrically \mathbb{Z}/p elementary abelian (ramified) covers $\varphi_i : X_i \rightarrow X$, $i \in I$, satisfying:

i) $\cup_i \varphi_i(X_i(k)) = X(k)$, i.e., every k -rational point of X “survives” in at least one of the covers $X_i \rightarrow X$.

ii) A section $s : G_k \rightarrow \pi_1(X)$ can be lifted to a section $s_i : G_k \rightarrow \pi_1(X_i)$ for some $i \in I$ if and only if s arises from a k -rational point $x \in X(k)$ in the way described above.

The details of the proof will be given subsequently.

Concerning the proof of the above theorems: The main technical point is a generalization of the Tate–Roquette–Lichtenbaum Local-Global Principle for Brauer groups of function fields of curves over p -adically closed fields, as introduced and studied in Pop [P]. As a result of that, one is lead to analyze the cohomological behavior of \mathbb{Z}/p -elementary abelian extension of Henselizations of the function fields in discussion.

2. GENERALITIES

A) \mathbb{Z}/p derived series and quotients

Let G be a profinite group. We denote by G^i the derived \mathbb{Z}/p series of G , hence we have by definition $G^1 := G$, and $G^{i+1} := [G^i, G^i](G^i)^p$ for $i > 0$. We will further set $\overline{G}^i := G^1/G^{i+1}$ for $i > 0$. Hence in particular we have: $\overline{G}^1 := G^1/G^2$ is the maximal \mathbb{Z}/p -elementary quotient of G , and $\overline{G}'' := G^1/G^3$ is the maximal \mathbb{Z}/p elementary meta-abelian quotient of G , i.e., the maximal quotient of G which is an extension of \overline{G}^1 by some \mathbb{Z}/p -elementary abelian extension.

One checks without any difficulty that mapping every profinite group G to \overline{G}^i , $i > 0$, defines a functor from the category of all profinite groups onto the category of all pro- p groups whose derived \mathbb{Z}/p series has length $\leq i$. In particular, if $pr : G \rightarrow H$ is a (surjective) morphism of profinite groups, then the following hold:

- 1) pr gives rise canonically to a (surjective) morphism $pr^i : \overline{G}^i \rightarrow \overline{H}^i$.
- 2) Every section $s : H \rightarrow G$ of $pr : G \rightarrow H$, gives rise to a section $s^i : \overline{H}^i \rightarrow \overline{G}^i$ of pr^i .

Finally, in the context above, we say that a section $s' : \overline{H}^1 \rightarrow \overline{G}^1$ of pr^1 is *liftable*, if there exists a section $s'' : \overline{H}'' \rightarrow \overline{G}''$ of pr'' which reduces to s' , or equivalently, which lifts s' .

B) Cohomology and sections

Let G be a profinite group. We endow \mathbb{Z}/p with the trivial G -action, and let $H^n(G, \mathbb{Z}/p)$ be the cohomology groups of G with values in \mathbb{Z}/p . Then in the notations of the previous sub-section, for all $i > 0$ we have

$$H^1(G, \mathbb{Z}/p) = \text{Hom}(G, \mathbb{Z}/p) = \text{Hom}(\overline{G}^i, \mathbb{Z}/p) = H^1(\overline{G}^i, \mathbb{Z}/p),$$

and for every i , the cup product gives rise to a canonical pairing:

$$\mathrm{Hom}(\overline{G}^i, \mathbb{Z}/p) \times \mathrm{Hom}(\overline{G}^i, \mathbb{Z}/p) = \mathrm{H}^1(\overline{G}^i, \mathbb{Z}/p) \times \mathrm{H}^1(\overline{G}^i, \mathbb{Z}/p) \xrightarrow{\cup^i} \mathrm{H}^2(\overline{G}^i, \mathbb{Z}/p).$$

Next let $pr : G \rightarrow H$ be a quotient of G , and $pr' : \overline{G}' \rightarrow \overline{H}'$ and $pr'' : \overline{G}'' \rightarrow \overline{H}''$ the corresponding surjective projections as introduced in the previous sub-section.

Lemma 2.1. *In the above notations, let $s' : \overline{H}' \rightarrow \overline{G}'$ be a liftable section of $pr' : \overline{G}' \rightarrow \overline{H}'$, and let $\Gamma \subseteq G$ be the preimage of $s'(\overline{H}') \subseteq \overline{G}'$ under the canonical projection $G \rightarrow \overline{G}'$. Then for characters $\chi_H, \psi_H \in \mathrm{Hom}(H, \mathbb{Z}/p)$ and the induced ones $\chi_\Gamma, \psi_\Gamma \in \mathrm{Hom}(\Gamma, \mathbb{Z}/p)$, the following are equivalent:*

- i) $\chi_H \cup \psi_H = 0$ in $\mathrm{H}^2(\overline{H}'', \mathbb{Z}/p)$.
- ii) $\chi_H \cup \psi_H = 0$ in $\mathrm{H}^2(H, \mathbb{Z}/p)$.
- iii) $\chi_\Gamma \cup \psi_\Gamma = 0$ in $\mathrm{H}^2(\Gamma, \mathbb{Z}/p)$.

Proof. The implications i) \Rightarrow ii) and ii) \Rightarrow iii) follow by taking the inflation maps coming from the surjective group homomorphisms $\Gamma \rightarrow H \rightarrow \overline{H}''$. Finally, the implication iii) \Rightarrow i) is as follows: Suppose that $\chi_\Gamma \cup \psi_\Gamma = \delta(\varphi)$ is the boundary of some map $\varphi : \Gamma \rightarrow \mathbb{Z}/p$. We claim that $\chi_H \cup \psi_H = 0$ in $\mathrm{H}^2(\overline{H}'', \mathbb{Z}/p)$. Indeed, $\chi_\Gamma \cup \psi_\Gamma = \delta(\varphi)$ means that for all $g, h \in \Gamma$ one has:

$$(\chi_\Gamma \cup \psi_\Gamma)(g, h) = g\varphi(h) - \varphi(gh) + \varphi(g) = \varphi(h) - \varphi(gh) + \varphi(g),$$

the last equality taking place by the fact that G , hence Γ , act trivially on \mathbb{Z}/p . Now if g or h lie in $G^2 \subset \Gamma$, then we have $(\chi_\Gamma \cup \psi_\Gamma)(g, h) = 0$. Equivalently, if g or h lie in $G^2 \subset \Gamma$, then $\varphi(h) - \varphi(gh) + \varphi(g) = 0$. In particular, the restriction of φ to G^2 is a group homomorphism to \mathbb{Z}/p . In particular, the restriction of φ to $G^3 = [G^2, G^2](G^2)^p$ is trivial, and finally φ factors through $\Gamma/G^3 \subset \overline{G}''$. Therefore, $\chi_{G^2} \cup \psi_{G^2} = 0$ in $\mathrm{H}^2(\Gamma/G^3, \mathbb{Z}/p)$. Now let $s'' : \overline{H}'' \rightarrow \overline{G}''$ be a lifting of the section s' , and remark that $s''(\overline{H}'') \subseteq \Gamma/G^3$. Then the restriction of $\chi_{G^2} \cup \psi_{G^2} = 0$ to $s''(\overline{H}'') \subseteq \Gamma/G^3$ is trivial too, i.e., $\chi_H \cup \psi_H = 0$ in $\mathrm{H}^2(s''(\overline{H}''), \mathbb{Z}/p)$. Thus finally, $\chi_H \cup \psi_H = 0$ in $\mathrm{H}^2(\overline{H}'', \mathbb{Z}/p)$, as claimed. \square

C) Basics from Galois cohomology

Let K be an arbitrary field of characteristic $\neq p$, and G_K be its absolute Galois group. Further let G_K^i and \overline{G}_K^i be the derived \mathbb{Z}/p series, respectively quotients of G_K . We recall the following basic/fundamental facts:

a) By Kummer Theory, one has a canonical isomorphism $K^\times/p = \mathrm{H}^1(G_K, \mu_p)$. In particular, if $\mu_p \subset K$, then the absolute Galois group G_K acts trivially on μ_p , hence choosing some identification $\iota : \mu_p \rightarrow \mathbb{Z}/p$ of trivial G_K modules, we get:

$$K^\times/p = \mathrm{H}^1(G_K, \mu_p) = \mathrm{Hom}(\mathrm{Gal}(K'|K), \mu_p) \xrightarrow{\iota} \mathrm{Hom}(\mathrm{Gal}(K'|K), \mathbb{Z}/p).$$

b) Let ${}_p\mathrm{Br}(K)$ denote the p -torsion subgroup of $\mathrm{Br}(K)$. Then ${}_p\mathrm{Br}(K) = \mathrm{H}^2(G_K, \mu_p)$ canonically. Hence if $\mu_p \subset K$, then $\iota : \mu_p \rightarrow \mathbb{Z}/p$ gives rise to an isomorphism:

$${}_p\mathrm{Br}(K) = \mathrm{H}^2(G_K, \mu_p) \xrightarrow{\iota} \mathrm{H}^2(G_K, \mathbb{Z}/p).$$

c) Consider the cup product $K^\times/p \otimes K^\times/p \xrightarrow{\cup} H^2(G_K, \mu_p \otimes \mu_p)$, $(a, b) \mapsto \chi_a \cup \chi_b$, which is actually surjective by the Merkurjev–Suslin Theorem. Hence if $\mu_p \subset K$, then the isomorphism $\iota : \mu_p \rightarrow \mathbb{Z}/p$ gives rise to a surjective morphism

$$K^\times/p \otimes K^\times/p \xrightarrow{\cup} H^2(G_K, \mathbb{Z}/p), \quad (a, b) \mapsto \chi_a \cup \chi_b.$$

Combining these observations with the Lemma 2.1 above we get the following: Let $K|k$ be a regular field extension, and suppose that $\text{char}(k) \neq p$, and $\mu_p \subset k$. As in the Introduction, we consider a maximal \mathbb{Z}/p elementary abelian extension $K'|K$ of K , and the corresponding $k' := K' \cap \bar{k}$, etc., and the resulting canonical surjective projections:

$$pr'_K : \text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k), \quad pr''_K : \text{Gal}(K''|K) \rightarrow \text{Gal}(k''|k).$$

Lemma 2.2. *In the above context, let $s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ be a liftable section of $pr'_K : \text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k)$, and let $M \subset K'$ be the fixed field of $s'(\text{Gal}(k'|k))$ in K' . Then for any elements $a, b \in k^\times$, and the corresponding p -cyclic k -algebras $A_k(a, b)$, respectively $A_M(a, b)$, one has: $A_k(a, b)$ is trivial in $\text{Br}(k)$ if and only if $A_M(a, b)$ is trivial in $\text{Br}(M)$.*

D) *Hilbert decomposition in elementary \mathbb{Z}/p abelian extensions*

Let K be a field of characteristic $\neq p$ containing μ_p . Let v be a valuation of K , and v' some prolongation of v to K' , and $V_{v'} \subseteq T_{v'} \subseteq Z_{v'}$ be the ramification, the inertia, and the decomposition, groups of $v'|v$ in $\text{Gal}(K'|K)$, respectively. We remark that because $\text{Gal}(K'|K)$ is commutative, the groups $V_{v'}$, $T_{v'}$, and $Z_{v'}$ depend only on v . Therefore we will simply denote them by V_v , T_v , and Z_v . Finally, we denote by $K^Z \subseteq K^T \subseteq K^V$ the corresponding fixed fields in K' .

Lemma 2.3. *In the above notations, the following hold:*

1) *Let $U^v := 1 + p^2\mathfrak{m}_v$. Then K^Z contains $\sqrt[p]{U^v}$, and $K^Z = K[\sqrt[p]{U^v}]$, provided p is a v -unit. In particular, if w_1 and w_2 are independent valuations of K , then $Z_{w_1} \cap Z_{w_2} = \{1\}$.*

2) *If $p \neq \text{char}(Kv)$, then $V_v = \{1\}$ and $K'v' = (Kv)'$. Hence $G_v := Z_v/T_v = \text{Gal}(K'v'|Kv)$. And if $p = \text{char}(Kv)$, then $V_v = T_v$, and the residue field $K'v'$ contains $(Kv)^{\frac{1}{p}}$ and the maximal \mathbb{Z}/p elementary abelian extension of Kv .*

3) *Let $L := K_v^{\text{h}}$ be the Henselization of K with respect to v . Then $L' = LK'$ is a maximal \mathbb{Z}/p elementary extension of L . Therefore we have $\text{Gal}(L'|L) \cong Z_v$ canonically.*

Proof. To 1): Everything is clear, but maybe the assertion concerning the independent valuations w_1 and w_2 . In order to prove this, consider $x \neq 0$ arbitrary. Since w_1 and w_2 are independent, there exist $y \neq 0$ which are arbitrarily w_1 -close to 1 and arbitrarily w_2 -close to x . Precisely, there exists $y \neq 0$ such that: First, $w_1(1 - y) > 2w_1(p)$; and second, $w_2(x - y) > 2w_2(p) + w_2(x)$, or equivalently, $w_2(1 - y/x) > 2w_2(p)$. But then by the first assertion of the Lemma we have: $\sqrt[p]{y} \in K^{Z_{w_1}}$ and $\sqrt[p]{y/x} \in K^{Z_{w_2}}$, hence $\sqrt[p]{x} \in K^{Z_{w_2}}K^{Z_{w_1}}$. Since $K^{Z_{w_2}}K^{Z_{w_1}} = (K')^{Z_{w_2} \cap Z_{w_1}}$, and $x \in K^\times$ was arbitrary, we thus get $K' \subseteq (K')^{Z_{w_2} \cap Z_{w_1}}$. Hence finally $Z_{w_2} \cap Z_{w_1} = 1$, as claimed.

To 2): If $p \neq \text{char}(Kv)$, then everything is clear by Kummer Theory, and general valuation theory. If $p = \text{char}(Kv)$, and $p \neq \text{char}(K)$, it follows that $\text{char}(K) = 0$. Recall that by Artin–Schreier Theory, the maximal \mathbb{Z}/p elementary abelian extension of Kv is generated by the roots of all the Artin–Schreier polynomials $Y^p - Y - \bar{a}$, with $\bar{a} \in Kv$. We show that every such polynomial has a root in the residue field of some \mathbb{Z}/p cyclic extension $K[\alpha]$ with $\alpha^p = u$

for some $u \in K$. Indeed, by the general non-sense of Kummer Theory versus Artin–Schreier Theory, one has the following:

- Let $X^p - u \in \mathcal{O}_v[X]$ be some Kummer polynomial over K . We note that $\lambda := \zeta_p - 1 \in K$, as $\mu_p \subset K$, and recall that $p = \prod_{0 < \mu < p} (1 - \zeta_p^\mu)$. Since $1 - \zeta_p^\mu = -\lambda(1 + \dots + \zeta_p^{\mu-1})$, thus in particular $(1 + \dots + \zeta_p^{\mu-1}) \equiv \mu \pmod{\lambda}$, and we finally get: $p \equiv \lambda^{p-1}(p-1)! \equiv -\lambda^{p-1} \pmod{\lambda^p}$, because $(p-1)! \equiv -1 \pmod{p}$ by Wilson’s Theorem. Hence setting $X := \lambda X_0 + 1$, and $u := \lambda^p u_0 + 1$, the equation $X^p = u$ is equivalent to the equation $X_0^p - X_0 + \lambda f(X_0) = u_0$, where $f(X_0) \in \mathcal{O}_v[X_0]$ is an explicitly computable polynomial. Therefore, if $\wp = \text{Frob} - \text{id}$ is the Artin–Schreier operator, and $\bar{u}_0 \in K v \setminus \wp(K v)$, then v is totally inert in $K_u := K[\sqrt[p]{u}]$. And if w is the unique prolongation of v to K_u , then the residue field of w is $K_u w = (K v)[\beta]$, with $\beta^p - \beta = \bar{u}_0$.

By reversing the process above, we see that each Artin–Schreier extension of $K v$ is obtained by reducing a properly chosen Kummer \mathbb{Z}/p extension of K .

To 3): First, if v has rank one, then K is dense in $L := K_v^{\text{h}}$. Hence given $\hat{u} \in \mathcal{O}_L$, there exists $u \in \mathcal{O}_K$ such that $\hat{u} = u(1 + \eta)$ in K^{h} with $v^{\text{h}}(\eta) > 2v^{\text{h}}(p)$. But then $1 + \eta$ is a p^{th} power in K^{h} by Hensel’s Lemma, hence the roots of $X^p - u$ and the roots of $X^p - \hat{u}$ generate the same field extension of K^{h} . In order to treat the general case, one makes induction on the rank of the valuation v , and “takes limits”. \square

E) Elementary \mathbb{Z}/p abelian extensions of Henselian fields

In this subsection we will prove a technical result concerning elementary \mathbb{Z}/p abelian extensions of Henselian fields. The context is as follows: Let L be a Henselian field with respect to a valuation w . Suppose that $\text{char}(L) = 0$ and $\text{char}(L w) = p > 0$, and that $\mu_p \subset L$. Further let $L' = L[\sqrt[p]{L^\times}]$ be the maximal elementary \mathbb{Z}/p abelian extension of L , and $\text{Gal}(L'|L) := \text{Gal}(L'|L)$ be its Galois group. Since w is Henselian, w has a unique prolongation to L' , which we again denote by w .

Lemma 2.4. *In the above context, suppose that w is a rank one valuation. Let $\Lambda|L$ be a sub-extension of $L'|L$ such that $L'|\Lambda$ is a finite extension. Then the following hold:*

- 1) $L'w|\Lambda w$ is finite, and Λw contains $(Lw)^{\frac{1}{p}}$.
- 2) If Lw is not finite, or $wL \not\approx \mathbb{Z}$, then for every $u \in L$ there exists $t \in L^\times$ such that $L_t := L[\sqrt[p]{t}] \subseteq \Lambda$ and $w(u) \in p \cdot wL_t \subseteq p \cdot w(\Lambda)$. Hence $wL \subseteq p \cdot w\Lambda$.
- 3) If particular, if $wL \not\subseteq p \cdot w\Lambda$, then $wL \approx \mathbb{Z}$, and Lw is finite.

Proof. The proof is inspired by [P], Korollar 2.7, and uses in an essential way Lemma 2.6 of loc.cit. Let \mathcal{O} and \mathfrak{m} be the valuation ring, respectively the valuation ideal of w . Then by loc.cit., one has exact sequences of the form:

$$(*) \quad 1 \rightarrow \mathcal{O}^\times/p \rightarrow L^\times/p \rightarrow w(L)/p \rightarrow 1 \quad \text{and} \quad 1 \rightarrow (1 + \mathfrak{m})/p \rightarrow \mathcal{O}^\times/p \rightarrow (Lw)^\times/p \rightarrow 1.$$

By Kummer theory (note that $\mu_p \subset L$ by hypothesis), one has $\Lambda = L[\sqrt[p]{\Delta}]$ for a subgroup $\Delta \subset L^\times$ such that Δ contains the p^{th} powers of all the elements of L^\times , and L^\times/Δ is canonically Pontrjagin dual (thus non-canonically isomorphic) to $\text{Gal}(L'|\Lambda)$. In particular, $L^\times/\Delta = (L^\times/p)/(\Delta/p)$ is a finite elementary \mathbb{Z}/p abelian group. Hence by assertions (*), it follows that setting $\Delta_0 := \Delta \cap \mathcal{O}^\times$, and $\Delta_1 := \Delta \cap (1 + \mathfrak{m})$, we have: $(1 + \mathfrak{m})/\Delta_1$ and $\mathcal{O}^\times/\Delta_0$ are finite groups, and if Δw denotes the image of Δ_0 in Lw^\times , then $Lw^\times/\Delta w$ is a finite group.

To 1): First, if Lw is finite, then Lw is perfect, thus there is nothing to prove. Now suppose that Lw is infinite. Then since $Lw^\times/\Delta w$ is finite, it follows that Δw is infinite too. Hence for every $a \in Lw$, there exist $x \neq y$ in Δw such that $a-x, a-y \neq 0$, and $(a-x)\Delta w = (a-y)\Delta w$. Equivalently, $\exists z \in \Delta w$ such that $a-x = z(a-y)$, hence $a = (x-yz)/(1-z)$. On the other hand, since $x, y, z \in \Delta w$, one has $x^{\frac{1}{p}}, y^{\frac{1}{p}}, z^{\frac{1}{p}} \in (\Delta w)^{\frac{1}{p}} \subset \Lambda w$, hence $a^{\frac{1}{p}} \in \Lambda w$. Since a was arbitrary, we get $(Lw)^{\frac{1}{p}} \subseteq \Lambda w$, as claimed.

To 2): By the discussion above, it follows that $(1+\mathfrak{m})/\Delta_1$ is finite, and let $1+a_j, 1 \leq j \leq n$, be representatives for $(1+\mathfrak{m})/\Delta_1$.

Case 1) w is not discrete on L .

Then for every $u \in L^\times$, there exists some $u_1 \in L^\times$ such that $0 < w(uu_1^p) < w(p), w(a_j)$ for all $j = 1, \dots, n$. Since $1+uu_1^p \in 1+\mathfrak{m}$, there exists j and some $t \in \Delta_1$ such that

$$1+uu_1^p = t(1+a_j).$$

Set $t = 1+a$. Since $0 < w(uu_1^p) < w(p), w(a_j)$, it immediately follows by the ultra-metric triangle inequality that $w(uu_1^p) = w(a)$. On the other hand, since $t \in \Delta$, one has $t = \theta^p$ for some $\theta \in \Lambda$, i.e., $L_t := L[\sqrt[p]{t}] = L[\theta] \subseteq \Lambda$. Hence $1+a = \theta^p$, and setting $\theta = 1+b$, one gets $1+a = (1+b)^p$. From this we obtain $w(b) > 0$. And since $w(a) = w(uu_1^p) < w(p)$, and $1+a = (1+b)^p$, the ultra-metric triangle inequality implies that $w(a) = w(b^p)$ in wL_t . Thus finally one has:

$$w(u) + pw(u_1) = w(uu_1^p) = w(a) = p \cdot w(b),$$

hence $w(u) = pw(b) - pw(u_1) \in p \cdot wL_t$, as claimed.

Case 2) w is discrete on L .

Suppose that Lw is not finite. Let $\mathfrak{m} \subset \mathcal{O} \subset L$ be the valuation ideal, respectively valuation ring, of w in L . Since L contains μ_p , and $p \geq 2$, it follows that we have the inclusions $(1+\mathfrak{m})^p \subseteq (1+\mathfrak{m}^p) \subseteq 1+\mathfrak{m}^2$. After choosing a uniformizing parameter π of \mathcal{O} , one gets in the usual way an isomorphism of groups

$$\phi : (1+\mathfrak{m})/(1+\mathfrak{m}^2) \rightarrow Lw^+, \quad 1+x\pi \mapsto x \pmod{\mathfrak{m}}.$$

Hence it follows that $(1+\mathfrak{m})/(1+\mathfrak{m}^p)$ is infinite, as it has as homomorphic image the infinite group $(1+\mathfrak{m})/(1+\mathfrak{m}^2) \cong Lw^+$. Next recall that $(1+\mathfrak{m})/\Delta_1$ is a finite group. Therefore $\phi(1+\mathfrak{m})/\phi(\Delta_1) = Lw^+/\phi(\Delta_1)$ is finite too. Hence there exist (infinitely many) elements $t := 1+a \in \Delta_1$ with $a \in \pi\mathcal{O}^\times$. For any such $t \in \Delta_1$, we have $t = \theta^p$ for some $\theta \in \Lambda$, hence we have as above $L_t = L[\theta]$. And setting $\theta := 1+b$, we have $1+a = (1+b)^p$. Equivalently,

$$a = \sum_{i=1}^{p-1} \binom{p}{i} b^i + b^p = pb\epsilon + b^p$$

for some w -unit $\epsilon \in \Lambda$. Since π divides p in \mathcal{O} , one has $w(pb\epsilon) > w(\pi)$, and therefore $w(\pi) = w(a) = w(b^p) = p \cdot w(b)$ in $w\Lambda$. Since $wL = \mathbb{Z}w(\pi)$, it follows that $w(u) \subseteq p \cdot wL_t$, as claimed. \square

F) Inertial cohomology

In this subsection we recall a well-known result concerning the cohomology of the maximal inert extension of a Henselian field (which goes back to Witt). The situation is as follows: Let L be a Henselian field with respect to a valuation w , and $L_1|L$ be a finite unramified Galois extension, and let $G := \text{Gal}(L_1|L)$ be the Galois group of $L_1|L$. Let $\mathcal{O}_L \subset \mathcal{O}_{L_1}$ and $\mathfrak{m}_L \subset \mathfrak{m}_{L_1}$ be the corresponding valuation rings, respectively valuation ideals. As remarked

in [P], Lemma 2.2, the group of principal units $1 + \mathfrak{m}_{L_1}$ is G -cohomologically trivial, and there exists an exact sequence of cohomology groups:

$$0 \rightarrow H^2(G, L_1 w^\times) \rightarrow H^2(G, L_1^\times) \rightarrow H^1(G, (\mathbb{Q} \otimes wL)/wL) \rightarrow 0,$$

hence we have a split exact sequence of the form:

$$(\dagger) \quad 0 \rightarrow \text{Br}(L_1 w|Lw) \rightarrow \text{Br}(L_1|L) \rightarrow \text{Hom}(G, (\mathbb{Q} \otimes wL)/wL) \rightarrow 0.$$

We also remark that if $M|L$ is some algebraic extension, say linearly disjoint with L_1 , and $M_1 = ML_1$ is the compositum (in some fixed algebraic closure), then the above exact sequence gives rise to a commutative diagram of the form

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Br}(L_1 w|Lw) & \rightarrow & \text{Br}(L_1|L) & \rightarrow & \text{Hom}(G, (\mathbb{Q} \otimes wL)/wL) \rightarrow 0 \\ & & \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res} \\ 0 & \rightarrow & \text{Br}(M_1 w|Mw) & \rightarrow & \text{Br}(M_1|M) & \rightarrow & \text{Hom}(G, (\mathbb{Q} \otimes wM)/wM) \rightarrow 0 \end{array}$$

where the first two vertical maps are the canonical restriction maps, and the last one is induced by the canonical embedding $wL \hookrightarrow wM$. We will use these observations in order to prove the following:

Lemma 2.5. *Let L be Henselian with respect to a rank one valuation w and satisfy: $\text{char}(L) = 0$, and $\mu_p \subset L$, and $\text{char}(Lw) = p > 0$. Let $L_1|L$ be a p -cyclic unramified sub-extension of $L'|L$, hence $G \cong \mathbb{Z}/p$, and $\Lambda|L$ be a sub-extension of $L'|L$ such that $L'|\Lambda$ is finite, and $\Lambda|L$ and $L_1|L$ are linearly disjoint. Suppose that the restriction map*

$$\text{res} : \text{Br}(L_1|L) \rightarrow \text{Br}(\Lambda_1|\Lambda) \subseteq \text{Br}(\Lambda)$$

is non-trivial. Then $wL \approx \mathbb{Z}$ and $Lw|_{\mathbb{F}_p}$ is a finite extension, i.e., L is a discrete valued field with finite residue field of characteristic p .

Proof. By contradiction, suppose that the conclusion of the Lemma 2.5 does not hold.

Since $G = \text{Gal}(L_1|L)$ has order p , it follows that $L_1 = L[\sqrt[p]{a}]$ for some $a \in L$, and $\text{Br}(L_1|L)$ consists of cyclic algebras of index p of the form $A_L(a, u)$ with $u \in L^\times$. In particular, $\text{Br}(L_1|L)$ is a torsion group of exponent p . Further, since $L_1 w|Lw$ is also cyclic of degree p , it follows that $\text{Br}(L_1 w|Lw)$ is generated by cyclic algebras of index p , and moreover, every such algebra from $\text{Br}(L_1 w|Lw)$ is also split by some purely inseparable extension of degree p of Lw . Therefore, the restriction map $\text{Br}(L_1 w|Lw) \xrightarrow{\text{res}} \text{Br}(Lw^{\frac{1}{p}})$ is trivial. On the other hand, by Lemma 2.4, 1), above, we have $Lw^{\frac{1}{p}} \subseteq \Lambda w$. Hence the restriction map

$$(*) \quad \text{Br}(L_1 w|Lw) \xrightarrow{\text{res}} \text{Br}(\Lambda_1 w|\Lambda w) \subseteq \text{Br}(\Lambda w)$$

is trivial. Therefore, if $A_L(a, u) \in \text{Br}(L_1|L)$ has non-trivial image in $\text{Br}(\Lambda_1|\Lambda)$, then by the exact sequence (\dagger) above, and the above diagram applied for $M := \Lambda$, we get: $A_L(a, u)$ does not lie in the image of $\text{Br}(L_1 w|Lw)$ in $\text{Br}(L_1|L)$. Equivalently, $A_L(a, u)$ is ramified, i.e., $w(u)$ is non-trivial in wL/p . Since we supposed —by contradiction— that the conclusion of Lemma 2.5 does not hold, by Lemma 2.4, 2), there exists $L_t := L[\sqrt[p]{t}] \subseteq \Lambda$ with $t \in L^\times$ such that $w(u) \in p \cdot wL_t$. But then by the fundamental (in)equality, we have:

$$p = [L_t : L] \geq [L_t w : Lw] \cdot (wL_t : wL) \geq [L_t w : Lw] \cdot p \geq p.$$

Therefore, the above inequalities are actually equalities, and $[L_t w : Lw] = 1$, i.e., $L_t w = Lw$. And also $L_{t,1} w = L_1 w$, where $L_{t,1} := L_t L_1$ is the compositum of L_t and L_1 inside Λ_1 .

Hence from the above commutative diagram applied to $M := L_t$, it follows that the image $A_{L_t}(a, u)$ of $A_L(a, u)$ in $\text{Br}(L_{t,1}|L_t)$ lies actually in $\text{Br}(L_{t,1}w|L_tw) = \text{Br}(L_1w|Lw)$. But then it follows that the image of $A_{L_t}(a, u)$ in $\text{Br}(\Lambda_1|\Lambda)$ lies actually in the image of $\text{Br}(L_{t,1}w|L_tw) = \text{Br}(L_1w|Lw)$ in $\text{Br}(\Lambda_1w|\Lambda w)$. On the other hand, the image of $\text{Br}(L_1w|Lw)$ in $\text{Br}(\Lambda_1w|\Lambda w)$ is trivial by the discussion at (*) above. Hence finally it follows that $A_\Lambda(a, u)$ is trivial in $\text{Br}(\Lambda_1|\Lambda)$, contradiction! \square

G) *On $\text{Gal}(k'_1|k_1)$ and $\text{Br}(k_1)$*

Let $k|\mathbb{Q}_p$ be a finite extension with $\mu_p \subset k$. Let $k_1|k$ be an arbitrary (not necessarily Galois and not necessarily finite) algebraic extension, and $[k_1 : k]$ denote its degree (as a super-natural number). As usual, let $k'_1|k_1$ be a maximal \mathbb{Z}/p -elementary extension of k_1 , and $\text{Gal}(k'_1|k_1) := \text{Gal}(k'_1|k_1)$ its Galois group.

Lemma 2.6. *In the above context the following hold:*

- 1) *The restriction map ${}_p\text{Br}(k) \rightarrow \text{Br}(k_1)$ is injective iff $[k_1 : k]$ is not divisible by p .*
- 2) *Suppose that $(p, [k_1 : k]) = 1$. Then $\text{Gal}(k'_1|k_1) \cong (\mathbb{Z}/p)^{e_{k_1}+2}$, where $e_{k_1} := [k_1 : \mathbb{Q}_p]$.*

Proof. To 1): After identifying $\text{Br}(k)$ with \mathbb{Q}/\mathbb{Z} via the invariant $\text{inv}_k : \text{Br}(k) \rightarrow \mathbb{Q}/\mathbb{Z}$, the restriction $\text{Br}(k) \rightarrow \text{Br}(k_1)$ becomes the multiplication by $[k_1 : k]$. Hence ${}_p\text{Br}(k) \rightarrow \text{Br}(k_1)$ is injective if and only if $[k_1 : k]$ is not divisible by p .

To 2): If $k_1|k$ is finite, then the assertion follows by local class field theory. Further, the canonical projection $\text{Gal}(k'_1|k_1) \rightarrow \text{Gal}(k'|k)$ is surjective, as $[k_1 : k]$ is prime to p . Finally, by taking limits over all the finite sub-extensions $k_i|k$ of $k_1|k$, the assertion follows. \square

H) *p -adic valuations and formally p -adic fields*

We recall a few basic facts about p -adic valuations and (formally) p -adically closed fields, see AX-KOCHEN [A-K] and PRESTEL-ROQUETTE [P-R] for more details.

1) A valuation v of a field k is called (formally) p -adic, if the residue field kv is a finite field \mathbb{F}_q with $q = p^{f_v}$, and the value group vk has a minimal positive element 1_v such that $v(p) = e_v \cdot 1_v$ for some natural number $e_v > 0$. The number $d_v := e_v f_v$ is called the p -adic rank (or degree) of the p -adic valuation v . Note that a field k carrying a p -adic valuation v must necessarily have $\text{char}(k) = 0$, as $v(p) \neq \infty$, and $\text{char}(kv) = p$.

2) Let v be a p -adic valuation of k with valuation ring \mathcal{O}_v . Then $\mathcal{O}_1 := \mathcal{O}[1/p]$ is the valuation ring of the unique maximal proper coarsening v_1 of v , which is called the **canonical coarsening** of v . Note that setting $k^0 := kv_1$, and $v_0 = v/v_1$ the corresponding valuation on k^0 we have: v_0 is a p -adic valuation of k^0 with $e_{v_0} = e_v$ and $f_{v_0} = f_v$, hence $d_{v_0} = d_v$, and moreover, v_0 is a discrete valuation of k^0 . In particular, the following hold:

a) v has rank one iff v_1 is the trivial valuation iff $v = v_0$.

b) Giving a p -adic valuation v of a field k of p -adic rank $d_v = e_v f_v$ is equivalent to giving a place \mathfrak{p} of k with values in a finite extension l of \mathbb{Q}_p such that the residues field kp of \mathfrak{p} is dense in l , and $l|\mathbb{Q}_p$ has ramification index e_v and residual degree f_v .

c) If $v_i < v$ is a strict coarsening of v , then $v_i \leq v_1$, and the quotient valuation v/v_i on the residue field kv_i is a p -adic valuation with $e_{v/v_i} = e_v$ and $f_{v/v_i} = f_v$, thus $d_{v/v_i} = d_v$. (Actually, $(kv_i)(v_i/v_1) \cong kv_1$ and $(kv_i)(v_i/v) \cong kv$ canonically.)

3) Let v be a p -adic valuation of k , and $l|k$ a finite field extension, and $w|v$ denote the prolongations of v to l . Then all w are p -adic valuations. Moreover, the *fundamental equality* holds: $[l : k] = \sum_{w|v} e(w|v)f(w|v)$, where $e(w|v)$ and $f(w|v)$ are the ramification index, respectively the residual degree of $w|v$. Further, if w_1 is the canonical coarsening of w , and $w_0 = w/w_1$ is the canonical quotient on the residue field lw_1 , then by general decomposition theory of valuations one has: $e(w|v) = e(w_1|v_1)e(w_0|v_0)$ and $f(w|v) = f(w_0|v_0)$; further, $e_w = e_v e(w_0|v_0)$, and $f_w = f_v f(w|v)$, thus $d_w = d_v e(w_0|v_0) f(w|v)$.

4) A field k is called (formally) **p -adically closed**, if k carries a p -adic valuation v such that for every finite extension $l|k$ one has: If v has a prolongation w to l with $d_w = d_v$, then $l = k$. One has the following characterization of the p -adically closed fields: For a field k endowed with a p -adic valuation v , and its canonical coarsening v_1 , the following are equivalent:

- i) k is p -adically closed with respect to v .
- ii) v is Henselian, and $v_1 k$ is divisible (maybe trivial).
- iii) v_1 is Henselian, and $v_1 k$ is divisible (maybe trivial), and the residue field $k^0 := kv_1$ is relatively algebraically closed in its completion $\widehat{k^0}$ (which is itself a finite extension of \mathbb{Q}_p).

We also notice that if k is p -adically closed with respect to some p -adic valuation v , then the valuation ring of v is completely determined by k . In particular, for every field k there exists at most one valuation v (up to equivalence of valuations) such that k is p -adically closed with respect to v .

5) Finally, for every field k endowed with a p -adic valuation v , there exist p -adic closures \tilde{k}, \tilde{v} such that $d_{\tilde{v}} = d_v$. Moreover, the space of the isomorphy classes of p -adic closures of k, v has a concrete description as follows: Let v_1 be the canonical coarsening of v , and $\widehat{k^0}|\mathbb{Q}_p$ the completion of the residue field of $k^0 = kv_1$. Then there exists a canonical exact sequence of the form $1 \rightarrow I_{v_1} \rightarrow D_v \xrightarrow{pr} G_{\widehat{k^0}} \rightarrow 1$, and the space of the isomorphy classes of p -adic closures of k, v is in bijection with the space of sections of pr , thus with $H_{\text{cont}}^1(G_{\widehat{k^0}}, I_{v_1})$.

6) The following hold: If L is p -adically closed with respect to the p -adic valuation w , and $l \subseteq L$ is a subfield which is relatively closed in L , then l is p -adically closed with respect to $v := w|_l$, and v and w have equal p -adic ranks, and L and l are elementary equivalent. Therefore, the elementary equivalence class of a p -adically closed field k is determined by both: the *absolute subfield* $k^{\text{abs}} := k \cap \overline{\mathbb{Q}}$ of k , as well as the completion $\widehat{k^0} = \widehat{k^{\text{abs}}}$. Note that the p -adic valuation of k^{abs} is discrete, and k^{abs} is actually the relative algebraic closure of \mathbb{Q} in $k^0 := kv_1$. Further, $\overline{L} = \overline{Ll} = \overline{L}\overline{\mathbb{Q}}$. Therefore, if $L|l$ is an extension of p -adically closed fields of the same rank, then the canonical projection $G_L \rightarrow G_l$ is an isomorphism.

7) Finally, let $(L, w) | (l, v)$ be an extension of p -adically closed fields with $d_w = d_v$. Let $k|l$ be some Galois extension, and set $K := Lk$. Then in the notations from the Introduction, the following canonical projections are isomorphisms:

$$(\dagger) \quad pr'_L : \text{Gal}(K'|L) \rightarrow \text{Gal}(k'|l), \quad pr''_L : \text{Gal}(K''|L) \rightarrow \text{Gal}(k''|l).$$

Remark 2.7. Let l, v be a finite field extension of \mathbb{Q}_p , and $L = \kappa(Y)$ be the function field of a complete smooth curve $Y \rightarrow l$. Let $s : G_l \rightarrow G_L$ be a section of $pr_L : G_L \rightarrow G_l$, and \tilde{L} be the fixed field of $\text{im}(s)$ in \overline{L} . Then $\overline{L} = \tilde{L}\overline{\mathbb{Q}} = \tilde{L}\overline{\mathbb{Q}}$, and $G_{\tilde{L}} \rightarrow G_l \rightarrow G_{l^{\text{abs}}}$ are all isomorphisms. Hence \tilde{L} is p -adically closed by assertion E.11 of POP [P], and elementary equivalent to l^{abs} , hence to l , by 6) above; and if \tilde{w} is the valuation of \tilde{L} , then $d_{\tilde{w}} = d_v$

and $v = \tilde{w}|_l$. Thus $w := \tilde{w}|_L$ is a p -adic valuation of L with $d_w = d_v$ and $w|_l = v$. Hence by point 2), b), above, the canonical coarsening w_1 of w defines an l -rational place of $L|l$, thus an l -rational point $y \in Y(l)$, such that $\text{im}(s)$ is contained in a decomposition group D_y above y . Hence recalling that distinct decomposition groups above l -places of $L|l$ have trivial intersection (by a Theorem of F. K. Schmidt), it follows that y and D_y are uniquely determined by $\text{im}(s)$. This proves the *birational p -adic Section Conjecture* for $Y \rightarrow l$. See KOENIGSMANN [Ko] for much more about this.

I) *A local-global principle for the Brauer group*

Here we recall the following result, which was proved in [P], Theorem 4.5, and uses in an essential way the mentioned results by TATE, ROQUETTE, LICHTENBAUM:

Fact. *Let k be a p -adically closed field, and let $M|k$ be a field extension of transcendence degree $\text{tr.deg}(M|k) \leq 1$. Further let $w|v$ denote the prolongations of the p -adic valuation v of k to M , and for each w let M_w^h be a Henselization of M with respect to w . Then the canonical exact sequence of Brauer groups below is exact:*

$$0 \rightarrow \text{Br}(M) \rightarrow \prod_{w|v} \text{Br}(M_w^h).$$

We will use a more special form of the above Fact which reads as follows: Let w be a prolongation of v to M , and $\mathcal{O}_w, \mathfrak{m}_w$ be its valuation ring, respectively valuation ideal. Further let $\mathcal{O}_{w_1} := \mathcal{O}_w[1/p]$ be the coarsening of \mathcal{O}_w obtained by inverting the prime number p ; and denote by w_1 the corresponding coarsening of w . Then w_1 is a prolongation to M of the canonical coarsening v_1 of v . Further, setting $M_0 := Mw_1$ and $w_0 := w/w_1$, it follows by general valuation theory that $M_0|k_0$ is a field extension with $\text{tr.deg}(M_0|k_0) \leq 1$, and w_0 is a prolongation of v_0 to M_0 . For every prolongation $w|v$ the following are equivalent:

- i) w_0 is a rank one valuation.
- ii) The minimal prime ideal of \mathcal{O}_w which contains the rational prime number p is the valuation ideal \mathfrak{m}_w .

In particular, for every prolongation $w|v$ of v to M there exists a unique coarsening \tilde{w} such that \tilde{w} is a prolongation of v to M and \tilde{w} satisfies the equivalent conditions i), ii), above. Indeed, for any given $w|v$, let $\tilde{\mathfrak{m}}$ be the minimal prime ideal of \mathcal{O}_w which contains the prime number p . Then by general valuation theory, the localization $\tilde{\mathcal{O}} := (\mathcal{O}_w)_{\tilde{\mathfrak{m}}}$ is a valuation ring with valuation ideal $\tilde{\mathfrak{m}}$, and its valuation \tilde{w} is the unique coarsening of w satisfying the equivalent conditions i), ii), above.

Fact 2.8. *Let k be a p -adically closed field, and let $M|k$ be a field extension of transcendence degree $\text{tr.deg}(M|k) \leq 1$. Let \mathcal{W} be the set of all the prolongations $w|v$ of v to M satisfying the equivalent conditions i), ii), above. Then the canonical exact sequence of Brauer groups below is exact:*

$$0 \rightarrow \text{Br}(M) \rightarrow \prod_{w \in \mathcal{W}} \text{Br}(M_w^h).$$

Proof. For a non-trivial division algebra A over M , let $w|v$ be a prolongation such that denoting by M_w^h the Henselization of M with respect to w , one has: $A_{M_w^h} \neq 0$ in $\text{Br}(M_w^h)$. Now let \tilde{w} be the unique coarsening of w such that $\tilde{w} \in \mathcal{W}$. Then since \tilde{w} is a coarsening of

w , it follows that M_w^h contains a Henselization $M_{\tilde{w}}^h$ of M with respect to \tilde{w} . On the other hand, since $M_{\tilde{w}}^h \subseteq M_w^h$, and $A_{M_{\tilde{w}}^h} \neq 0$ in $\text{Br}(M_{\tilde{w}}^h)$, it follows that $A_{M_w^h} \neq 0$ in $\text{Br}(M_w^h)$. \square

3. PROOF OF THEOREM B

To 1): Let \tilde{K}, \tilde{w} be a p -adic closure of K, w . Let \tilde{k}, \tilde{v} be the relative algebraic closure of k in \tilde{K} endowed with the restriction of \tilde{w} to \tilde{k} . Then $d_{\tilde{v}} = d_{\tilde{w}} = d_w$. Since $d_v = d_w$ by hypothesis, we get $d_{\tilde{v}} = d_v$, hence $\tilde{k} = k$. Conclude by applying relation (†) from subsection H, 7), with $l := k$ and $L := \tilde{K}$, and taking into account that the isomorphisms $\text{Gal}(\tilde{K}''|\tilde{K}) \rightarrow \text{Gal}(k''|k)$ factors through $\text{Gal}(K''|K) \rightarrow \text{Gal}(k''|k)$, thus gives rise to a liftable section of $\text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k)$.

To 2): Let $s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ be a liftable section, and let $M \subset K'$ be the fixed field of $\text{im}(s')$. Consider $a, b \in k$ such that $k_1 := k[\sqrt[p]{a}]$ is the unique unramified extension of degree p of k , and such that the p -cyclic algebra $A_k(a, b)$ is non-trivial in $\text{Br}(k)$, or equivalently, $\chi_a \cup \chi_b \neq 0$ in $\text{H}^2(G_k, \mathbb{Z}/p)$. Then by Lemma 2.2, it follows that $A_M(a, b)$ is non-trivial in $\text{Br}(M)$. Hence by Fact 2.8, it follows that there exists some prolongation $w \in \mathcal{W}$ of v to M such that denoting by $\Lambda := M_w^h$ the Henselization of M with respect to w , one has: $A_\Lambda(a, b) \neq 0$ in $\text{Br}(\Lambda)$. By abuse of language, we will denote by w the Henselian prolongation of w to Λ , etc.

For w as above, let $L := K_w^h \subseteq \Lambda$ denote the (unique) Henselization of K with respect to (the restriction of) w which is contained in Λ . Then the compositum $LM \subseteq \Lambda$ is Henselian with respect to w , hence we must have $LM = \Lambda$. Note that $L' = K'L$ by Lemma 2.3, 3), and $K'|M$ is finite, because $\text{im}(s')$ is finite, and $M = (K')^{\text{im}(s')}$. We conclude that $L' = LK'$ is finite over $\Lambda = LM$; and $A_\Lambda(a, b) \neq 0$ in $\text{Br}(\Lambda)$ implies $A_L(a, b) \neq 0$ in $\text{Br}(L)$, as $L \subset \Lambda$.

Lemma 3.1. *The valuation w is a p -adic valuation of L .*

Proof. As in the discussion above, let w_1 and v_1 be the canonical coarsenings of w , respectively v , i.e., the valuations with valuation rings $\mathcal{O}_w[1/p]$, respectively $\mathcal{O}_v[1/p]$. We denote the corresponding residue fields by $k_0 := kv_1$, and $L_0 := Lw_1$, and $\Lambda_0 := \Lambda w_1$; and recall that $v_0 := v/v_1$ on k_0 and $w_0 := w/w_1$ on L_0 and Λ_0 are rank one valuations (as $w \in \mathcal{W}$). Recall/note that the following hold:

- a) w_1 prolongs v_1 to L and Λ , and w_0 prolongs v_0 to L_0 and Λ_0 , as w prolongs v to L .
- b) w_1 and v_1 , and w_0 and v_0 are Henselian, as w and v were so.
- c) $L'w_1|Lw_1$ is the maximal \mathbb{Z}/p elementary abelian extension of $L_0 = Lw_1$, by Lemma 2.3, 2), hence $L'w_1$ equals the maximal \mathbb{Z}/p elementary abelian extension L'_0 of L_0 .
- d) Further, since $L'|\Lambda$ is finite by the discussion above, it follows that $L'w_1|\Lambda w_1$ is finite, by the fundamental inequality. Since $L'w_1 = L'_0$ and $\Lambda w_1 = \Lambda_0$, we get: $L'_0|\Lambda_0$ is finite.

Recall the v -unramified extension $k_1 := k[\sqrt[p]{a}]$ with $\text{Gal}(k_1|k) =: G$ defined above. We set $\Lambda_1 := \Lambda k_1$, and remark that $\Lambda_1|\Lambda$ is an w -unramified cyclic extension with Galois group $\cong G$ canonically. Moreover, since $k_1|k$ is v -unramified, $k_1|k$ is also v_1 -unramified, as v_1 is a coarsening of v . Correspondingly, $L_1|L$ is w_1 -unramified. We denote the corresponding residue fields by $k_{01} := k_1 v_1$, and $\Lambda_{01} := \Lambda_1 w_1$. And remark that $k_{01}|k_0$ is a v_0 -unramified cyclic extension with Galois group $\cong G$ canonically. Correspondingly, $\Lambda_{01}|\Lambda_0$ is a w_0 -unramified cyclic extensions with Galois group $\cong G$ canonically.

We next consider the resulting commutative diagram of Brauer/cohomology groups deduced from the extension of valued fields $(\Lambda, w_1)|(k, v_1)$, and the corresponding residue fields, as discussed in Section 1, F):

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Br}(k_{01}|k_0) & \rightarrow & \text{Br}(k_1|k) & \rightarrow & \text{Hom}(G, (\mathbb{Q} \otimes v_1 k)/v_1 k) & \rightarrow & 0 \\ & & \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res} & & \\ 0 & \rightarrow & \text{Br}(\Lambda_{01}|\Lambda_0) & \rightarrow & \text{Br}(\Lambda_1|\Lambda) & \rightarrow & \text{Hom}(G, (\mathbb{Q} \otimes w_1 \Lambda)/w_1 \Lambda) & \rightarrow & 0 \end{array}$$

We recall that $v_1 k$ is divisible, hence $\mathbb{Q} \otimes v_1 k = v_1 k$, and therefore, $(\mathbb{Q} \otimes v_1 k)/v_1 k = (0)$. Hence we deduce that $\text{Br}(k_{01}|k_0) \rightarrow \text{Br}(\Lambda_{01}|\Lambda_0) \subseteq \text{Br}(\Lambda_0)$ is non-trivial.

Now let us set $L_1 := Lk_1$ and denote $L_{01} := L_1 w_1$. Then reasoning as above we get: $L_1|L$ is w -unramified, hence w_1 -unramified. And further, $L_{01}|L_0$ is a w_0 -unramified extension with Galois group $\cong G$ canonically. And it is obvious that $\text{Br}(k_{01}|k_0) \rightarrow \text{Br}(\Lambda_0)$ factors through $\text{Br}(L_{01}|L_0)$. Therefore we have: $\text{Br}(L_{01}|L_0) \rightarrow \text{Br}(\Lambda_0)$ is non-trivial.

Hence by Lemma 2.5 applied to L_0 endowed with the Henselian rank one valuation w_0 , and the w_0 -unramified extension $L_{01}|L_0$, and the extension $\Lambda_0|L_0$ such that $L'_0|\Lambda_0$ is finite, we get: w_0 is discrete and has finite residue field (of characteristic p , as w_0 prolongs v_0). Equivalently, w is a (Henselian) p -adic valuation of L , as claimed. \square

Lemma 3.2. *The p -adic valuation w from Lemma 3.1 has p -adic rank equal to the p -adic rank of v and satisfies: $\text{im}(s') \subseteq Z_w$.*

Proof. The proof is a refinement of the arguments in the proof of the previous Lemma. As remarked there, the canonical restriction map

$$\text{res} : \text{Br}(k_{01}|k_0) \rightarrow \text{Br}(L_{01}|L_0) \rightarrow \text{Br}(\Lambda_0)$$

is non-trivial. Since completion does not change the inertial cohomology, without loss of generality, we can replace $k_0 \subseteq L_0 \subseteq \Lambda_0$ by the corresponding sequence of completions $\hat{k}_0 \subseteq \hat{L}_0 \subseteq \hat{\Lambda}_0$ —all of which are finite extensions of \mathbb{Q}_p , and deduce that

$$\text{res} : \text{Br}(\hat{k}_{01}|\hat{k}_0) \rightarrow \text{Br}(\hat{L}_{01}|\hat{L}_0) \rightarrow \text{Br}(\hat{\Lambda}_0)$$

is non-trivial. But then by Lemma 2.6, it follows that $[\hat{\Lambda}_0 : \hat{k}_0]$ is prime to p ; and therefore, $[\Lambda_0 : k_0] = [\hat{\Lambda}_0 : \hat{k}_0]$ is prime to p . Hence from $[\Lambda_0 : k_0] = [\Lambda_0 : L_0] \cdot [L_0 : k_0]$ it follows that both $[L_0 : k_0]$ and $[\Lambda_0 : L_0]$ are prime to p . On the other hand, $\Lambda_0|L_0$ is a sub-extension of the \mathbb{Z}/p elementary abelian extension $L'_0|L_0$. Thus finally $\Lambda_0 = L_0$.

Now recall that $M = (K')^{\text{im}(s')}$ is the fixed field of $\text{im}(s') = s'(\text{Gal}(k'|k))$ in K' ; further, $L' = LK'$, and $\Lambda = ML$ inside L' , by the discussion at above at the beginning of the proof. From this we deduce the following sequence of inequalities:

$$(*) \quad [k' : k] = |\text{Gal}(k'|k)| = [K' : M] \geq [LK' : LM] = [L' : \Lambda].$$

Further, since k is p -adically closed, hence $pr_k : \text{Gal}(k'|k) \rightarrow \text{Gal}(k'_0|k_0)$ is an isomorphism, one has $[k' : k] = [k'_0 : k_0]$; and by the fundamental inequality we have $[L' : \Lambda] \geq [L'w_1 : \Lambda w_1]$. On the other hand, we have $L'w_1 = L'_0$, and $\Lambda w_1 := \Lambda_0$; and $\Lambda_0 = L_0$ by the remarks above. Thus the above sequences of inequalities can be extended as follows:

$$(**) \quad [k'_0 : k_0] = [k' : k] = [K' : M] \geq [LK' : LM] = [L' : \Lambda] \geq [L'w_1 : \Lambda w_1] = [L'_0 : L_0].$$

On the other hand, by Lemma 2.6, 2), we have: $[k'_0 : k_0] = p^{e_{k_0}}$, where $e_{k_0} := [\hat{k}_0 : \mathbb{Q}_p]$ and $[L'_0 : L_0] = p^{e_{L_0}}$, with $e_{L_0} := [\hat{L}_0 : \mathbb{Q}_p]$. Hence the inequality (**) above implies $e_{k_0} \geq e_{L_0}$. On the other hand, $k_0 \subseteq L_0$, implies $e_{k_0} \leq e_{L_0}$. Hence finally $e_{k_0} = e_{L_0}$, and $\hat{k}_0 = \hat{L}_0$. Equivalently, w is a p -adic valuation having p -adic rank equal to

$$d_w = [\hat{L}_0 : \mathbb{Q}_p] = [\hat{k}_0 : \mathbb{Q}_p] = d_v$$

hence equal to the p -adic rank of v . Moreover, because of this, all the inequalities in the formulas (*) and (**) above are actually equalities. Hence $[K' : M] = [LK' : LM]$, and the restriction map $\text{Gal}(L'|L) = \text{Gal}(L'|L) \rightarrow Z_w \subset \text{Gal}(K'|K)$, which maps $\text{Gal}(L'|L)$ isomorphically onto Z_w by the fact that $L' = K'L$, defines an isomorphism

$$\text{Gal}(L'|\Lambda) \rightarrow \text{Gal}(K'|M) = s'(\text{Gal}(k'|k)).$$

Equivalently, $\text{im}(s') \subseteq Z_w$, as claimed. \square

Coming back to the proof of Theorem B, we have the following: Let $M \subseteq K'$ be the fixed field of $\text{im}(s')$ in K' . Then there exists a p -adic valuation w of M such that w prolongs v to M and has p -adic rank d_w equal to the p -adic rank d_v of v ; and moreover, $\text{im}(s')$ is contained in the decomposition group Z_w of w in $\text{Gal}(K'|K)$.

Remark 3.3. The precise structure of Z_w can be deduced as follows: First, let w_1 is the canonical coarsening of w , and $T_{w_1} \subset Z_{w_1}$ be the inertia/decomposition groups above w_1 in $\text{Gal}(K'|K)$. Then $Z_w = Z_{w_1}$, and $pr'_K : \text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k)$ gives rise to an exact sequence:

$$1 \rightarrow T_{w_1} \rightarrow Z_{w_1} \xrightarrow{pr'_K} \text{Gal}(k'|k) \rightarrow 1,$$

such that $s'(\text{Gal}(k'|k)) \subseteq Z_{w_1} = Z_w$ is a complement of T_{w_1} . And if T_{w_1} is non-trivial, then $T_{w_1} \cong \mu_p$ as a $\text{Gal}(k'|k)$ -module, thus $T_{w_1} \cong \mathbb{Z}/p$ non-canonically as a $\text{Gal}(k'|k)$ -module.

Lemma 3.4. *The p -adic valuation w from Lemma 3.2 satisfying $\text{im}(s') \subseteq Z_w$ is unique.*

Proof. Indeed, consider p -adic valuations w^1 and w^2 such that $\text{im}(s') \subset Z_{w^i}$, $i = 1, 2$. We claim that $w^1 = w^2$. Indeed, let w be the maximal common coarsening of w^1, w^2 . By contradiction, suppose that $w < w^1, w^2$. Then the valuations w^1/w and w^2/w are independent p -adic valuations on Kw , both of which prolonging the p -adic valuation of the p -adically closed field kw . Further, by Lemma 2.3, 2, it follows that $K'w$ is the maximal \mathbb{Z}/p elementary abelian extension of Kw ; and moreover, since $\text{im}(s') \subset Z_{w^i}$, $i = 1, 2$, it follows by general decomposition theory for valuations that $s'_w(\text{Gal}(k'|k)) \subset Z_{w^i/w}$, $i = 1, 2$. On the other hand, by the construction of w , it follows that w^1/w and w^2/w are independent valuations of Kw . On the other hand, since w^1/w and w^2/w are independent, it follows by Lemma 2.3, 2, that $Z_{w^1/w} \cap Z_{w^2/w}$ is trivial. Contradiction, as $\text{im}(s'_w) \subset Z_{w^i/w}$, $i = 1, 2$. \square

The proof of Theorem B is complete.

4. PROOF OF THEOREM A

The following stronger assertion holds (from which Theorem A immediately follows):

Theorem 4.1. *Let $k|\mathbb{Q}_p$ be a finite extension containing the p^{th} roots of unity, and let $k_0 \subseteq k$ a subfield which is relatively algebraically closed in k . Let X_0 be a complete smooth curve over k_0 , and $K_0 = k_0(X)$ the function field of X_0 .*

1) *Every k -rational point $x \in X_0$ gives rise to a bouquet of conjugacy classes of liftable sections $s'_x : \overline{G}'_{k_0} \rightarrow \overline{G}'_{K_0}$ above x .*

2) *Let $s' : \overline{G}'_{k_0} \rightarrow \overline{G}'_{K_0}$ be a liftable section. Then there exists a unique k -rational point $x \in X_0$ such that s' equals one of the sections s'_x mentioned above.*

Proof. To 1): Let v be the valuation of k . We notice that by Section 2, H), b), there exists a bijection from the p -adic valuations w of $\kappa(X_0)$ with $d_w = d_v$ and the k -rational points x of X_0 by sending each w to the center x of the canonical coarsening w_1 on $X = X_0 \times_{k_0} k$. Conclude by applying Theorem B, 1).

To 2): Since $k_0 \subseteq k$ is relatively algebraically closed, it follows that k_0 is p -adically closed. Let v be the valuation of k and of all subfields of k . Since k_0 is p -adically closed, we can apply Theorem B and get: For every section $s' : \overline{G}'_{k_0} \rightarrow \overline{G}'_{K_0}$, there exists a unique p -adic valuation w of K_0 which prolongs v to K_0 and has p -adic rank equal to the p -adic rank of v , such that s' is a section above w . Let w_1 be the canonical coarsening of v . Then we have:

Case 1. The valuation w_1 is trivial.

Then w is a discrete valuation of K prolonging v to K , and having the same residue field and the same value group as v . Equivalently, the completions \hat{k}_0 and \hat{K}_0 are equal, hence equal to k . Therefore, w is uniquely determined by the embedding $\iota_w : (K_0, w) \hookrightarrow (k, v)$. In geometric terms, ι_w defines a k -rational point x of X_0 , etc.

Case 2. The valuation w_1 is not trivial.

Then w_1 is a k_0 -rational place of K_0 , hence defines a k_0 -rational point x_0 of X_0 ; hence a k -rational point x of X_0 , etc. \square

5. PROOF OF THEOREM B⁰

First, the proof of assertion 1) is identical with the proof of assertion 1) of Theorem B, thus we omit it. Concerning assertion 2), let $s'_L : \text{Gal}(k'|l) \rightarrow \text{Gal}(K'|L)$ be a liftable section of the canonical projection $pr'_L : \text{Gal}(K'|L) \rightarrow \text{Gal}(k'|l)$. Then the restriction of s'_L to $\text{Gal}(k'|k) \subseteq \text{Gal}(k'|l)$ gives rise to a liftable section $s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K)$ of $pr'_K : \text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k)$. Hence by Theorem B, there exists a unique p -adic valuation w^1 of K which prolongs the p -adic valuation v_k of k to K and has $d_{w^1} = d_{v_k}$, and $s' = s_{w^1}$ in the usual way. Let $w = w^1|_L$ be the restriction of w^1 to L . Then w prolongs the valuation v of l to L . We claim that w^1 is the unique prolongation of w to K . Indeed, let $w^2 := w^1 \circ \sigma_0$ with $\sigma_0 \in \text{Gal}(k|l)$, be a further prolongation of w to K . Then if $(w^i)'$ is a prolongation of w^i to K' , $i = 1, 2$, and $\sigma \in \text{im}(s'_L)$ is a preimage of σ_0 , then $(w^2)' := (w^1)' \circ \sigma$ is a prolongation of w^2 to K' . Therefore, if $Z_{w^1} \subset \text{Gal}(K'|K)$ is the decomposition group above w^1 , then $Z_{w^2} := \sigma Z_{w^1} \sigma^{-1}$ is the decomposition group above w^2 . On the other hand, $\text{im}(s') \subseteq Z_{w^1}$ by Theorem B (precisely, by Lemma 3.2 in the proof of Theorem B). Since $\sigma \in \text{im}(s'_L)$, and $\text{Gal}(k'|k)$ is a normal subgroup of $\text{Gal}(k'|l)$, thus $\text{im}(s')$ is normal in $\text{im}(s'_L)$, it follows that $\sigma(\text{im}(s'))\sigma^{-1} = \text{im}(s')$. Hence $\text{im}(s') \subseteq Z_{w^1} \cap Z_{w^2}$. But then by Theorem B (precisely, by Lemma 3.4 in the proof of Theorem B), we must have $w^1 = w^2$. Equivalently, $\text{im}(s'_L)$ is contained in $Z_w \subset \text{Gal}(K'|L)$. Finally conclude that $d_w = d_v$, as claimed.

6. PROOF OF THEOREM A⁰

The following stronger assertion holds (from which Theorem A⁰ follows immediately):

Theorem 6.1. *Let $l|\mathbb{Q}_p$ be a finite extension. Let $l_0 \subset l$ a relatively algebraically closed subfield, and $k_0|l_0$ a finite Galois extension with $\mu_p \subset k_0$. Let Y_0 be a complete smooth geometrically integral curve over l_0 . Let $L_0 = \kappa(Y_0)$ the function field of Y_0 , and $K_0 = L_0 k_0$.*

1) *Every l -rational point $y \in Y_0$ gives rise to a bouquet conjugacy classes of liftable sections $s'_y : \text{Gal}(k'_0|l_0) \rightarrow \text{Gal}(K'_0|L_0)$ above y .*

2) *Let $s' : \text{Gal}(k'_0|l_0) \rightarrow \text{Gal}(K'_0|L_0)$ be a liftable section. Then there exists a unique l -rational point $y \in Y_0(l)$ such that s' equals one of the sections s'_y mentioned above.*

Proof. The proof is identical with the proof of Theorem 4.1 above, with the only difference that one uses Theorem B⁰, in stead of Theorem B. □

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