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The birational anabelian conjecture — r e v i s i t e d —

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Introduction

Let ℓ be a fixed rational prime number. For every field K of positive characteristic $p \neq \ell$, let $K^{\rm cyclc}$ be a maximal cyclotomic extension of K. Thus $K^{\rm cyclc} = K\,\overline{\mathbb{F}}_p$, where $\overline{\mathbb{F}}_p$ is an algebraic closure of the prime field $\mathbb{F}_p$ of K. Further let K^ℓ be a maximal Galois pro-ℓ extension of $K^{\rm cycl}$. By general Galois theory, it is clear that $K^\ell|K$ is a Galois extension. We denote by $G^\ell_K = {\rm Gal}(K^\ell|K)$ its Galois group.

The aim of this paper is to prove the following *cyclotomic by pro-ℓ version* of Grothendieck's birational anabelian conjecture in positive characteristic:[1]

Theorem. *Let $\mathcal{F}$ be the category of all finitely generated fields of positive characteristic having absolute transcendence degree* ${\rm td}\,(K) > 1$. *There exists a group theoretic recipe by which we can recover every field $K \in \mathcal{F}$ from G^ℓ_K, up to a pure inseparable extensions. This recipe is invariant under profinite group isomorphisms. In particular, if K and L are in $\mathcal{F}$, there exists a canonical bijection*

$$\mathrm{Isom}^{\rm i}\,(L^{\rm i}, K^{\rm i}) \longrightarrow \mathrm{Out}\,(G^\ell_K, G^\ell_L)\,,$$

where $(\ )^{\rm i}$ *denotes pure inseparable closure, and* ${\rm Isom}^{\rm i}$ *means up to Frobenius twists, and* Out *denotes outer isomorphisms of profinite groups.*

Equivalently, if $\Phi : G^\ell_K \to G^\ell_L$ is an isomorphism of profinite groups, then up to a Frobenius twist, there exists a unique field isomorphism $\phi : L^{{\rm i}\ell} \to K^{{\rm i}\ell}$ such that $\Phi(g) = \phi^{-1}\, g\, \phi$ for all $g \in G^\ell_K$. In particular, $\phi(L^{\rm i}) = K^{\rm i}$.

Remarks.

1) The Theorem above implies corresponding "full Galois assertions", i.e., the corresponding assertions for the full Galois group G_K in stead of the quotient G^ℓ_K. In particular the Galois characterization of finitely generated fields

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[1] A similar result is true as well if we include function fields of characteristic zero. This will be shown in a subsequent paper.

of Kronecker dimension > 1 (in positive characteristic), see [P2], [P3], follows from the Main Theorem above. Thus the above result *generalizes* in a non-trivial way the celebrated result by NEUKIRCH, IKEDA, IWASAWA, UCHIDA concerning the Galois characterization global fields.

2) The proof goes by induction on $d = \mathrm{td}(K) > 1$, but information from the pro-ℓ Galois theory of global function fields will be used in an essential way. We remark the a corresponding cyclic by pro-ℓ result for global function fields would bring a major simplification of the proof of the general case. Unfortunately, in spite of several efforts in the last years, the case of global function fields is still open. Second, we strongly believe that a stronger assertion holds, namely that the above recipe for detecting the field structure from G^ℓ_K is actually invariant under *open homomorphisms* of profinite groups. Thus in fact a "cyclic by pro-ℓ Hom-form" of the birational anabelian conjecture should hold. See also the comments at point 3) below. Finally, we remark that in stead of working with the full cyclic by pro-ℓ Galois group, one could work as well with truncations of the pro-ℓ part. This is doable, but the resulting assertions are quite technical, and at the moment maybe too complicated in order to be interesting...

3) In characteristic zero, one should mention here the very strong results by MOCHIZUKI [M1], [M2], concerning anabelian schemes over sub-p-adic fields k. Among other things, he shows the following: Given regular function fields $K|k$ and $L|k$, every *open* G_k*-homomorphism* $G_K \to G_L$ arises canonically from a k-embedding of fields $L \to K$. This shows that one could expect better functorial properties of the "recipe" for detecting K from G^ℓ_K. Nevertheless, there are some essential differences between results proved earlier and ours in the present manuscript: First, in MOCHIZUKI's approach, one deals with a *relative full Galois situation,* i.e., G_k-morphisms of the corresponding full Galois groups G_K and G_L. Second, comparing with previous results in positive characteristic, e.g., by TAMAGAWA and/or by the author, in the present situation we are missing a very essential previous ingredient, namely the *full Frobenius:* It is not possible to detect it at a pro-ℓ level.

Rough idea of Proof

Let $\mathfrak{K}|\mathfrak{k}$ be an extension of fields of characteristic $\neq \ell$. We consider Galois extensions $\mathfrak{K}'|\mathfrak{K}$ with the following properties:

i) $\mathfrak{K}'$ contains a separable closure $\mathfrak{k}'$ of $\mathfrak{k}$. Set $\mathfrak{K}_0 = \mathfrak{K}\mathfrak{k}'$ in $\mathfrak{K}'$.

ii) $\mathfrak{K}'|\mathfrak{K}_0$ is a pro-ℓ extension.

iii) There exists a Galois sub-extension $\mathfrak{K}_1$ of $\mathfrak{K}'|\mathfrak{K}$ containing $\mathfrak{k}'$ such that $\mathfrak{K}'|\mathfrak{K}_1$ is a maximal pro-ℓ Abelian extension of $\mathfrak{K}_1$.

We remark that the extension $K^\ell|K$ as introduced before the Main Theorem satisfies the conditions i), ii), iii). Thus working with $K'|K$ is a little bit more general than working with $K^\ell|K$, and offers more flexibility.

To fix notations, let $G'_{\mathfrak{K}} = \mathrm{Gal}(\mathfrak{K}'|\mathfrak{K})$, and $G'_{\mathfrak{k}} = \mathrm{Aut}(\mathfrak{k}'|\mathfrak{k})$, and let denote $pr : G'_{\mathfrak{K}} \to G'_{\mathfrak{k}}$ the canonical projection.

Let us fix an isomorphism $\imath : \mathbb{T}_\ell \to \mathbb{Z}_\ell(1)$. By Kummer Theory there is a functorial isomorphism

$$(*) \qquad \widehat{\delta}_{\mathfrak{K}} : \widehat{\mathfrak{K}} \longrightarrow \mathrm{H}^1\big(G'_{\mathfrak{K}}, \mathbb{T}_\ell\big) \xrightarrow{\imath} \mathrm{H}^1\big(\mathfrak{K}, \mathbb{Z}_\ell(1)\big),$$

where $\widehat{\mathfrak{K}}$ denotes the ℓ-adic completion of the multiplicative group $\mathfrak{K}^\times$ of $\mathfrak{K}$. Suppose that $\mathfrak{K}|\mathfrak{k}$ is a regular field extension, thus in particular, the canonical projection $pr : G'_{\mathfrak{K}} \to G'_{\mathfrak{k}}$ is surjective. By the functoriality of $\widehat{\delta}$, we get a commutative diagram of the form

$$\begin{array}{ccc} \widehat{\mathfrak{k}} & \longrightarrow & \widehat{\mathfrak{K}} \\ \big\downarrow{\scriptstyle \hat{\delta}_{\mathfrak{k}}} & & \big\downarrow{\scriptstyle \hat{\delta}_{\mathfrak{K}}} \\ \mathrm{H}^1\big(\mathfrak{k}, \mathbb{Z}_\ell(1)\big) & \longrightarrow & \mathrm{H}^1\big(\mathfrak{K}, \mathbb{Z}_\ell(1)\big) \end{array}$$

Setting $\widehat{\mathcal{H}}_{\mathfrak{K}/\mathfrak{k}} = \widehat{\mathfrak{K}}/\widehat{\mathfrak{k}}$, and $\mathrm{H}^1\big(\mathfrak{K}/\mathfrak{k}, \mathbb{Z}_\ell(1)\big) = \mathrm{H}^1\big(\mathfrak{K}, \mathbb{Z}_\ell(1)\big)/\mathrm{H}^1\big(\mathfrak{k}, \mathbb{Z}_\ell(1)\big)$ we have a canonical isomorphism $\hat{\delta}_{\mathfrak{K}/\mathfrak{k}} : \widehat{\mathcal{H}}_{\mathfrak{K}/\mathfrak{k}} \to \mathrm{H}^1\big(\mathfrak{K}/\mathfrak{k}, \mathbb{Z}_\ell(1)\big)$.

Now suppose that $\mathfrak{K}|\mathfrak{k}$ is a regular function field. Then $\mathfrak{K}^\times/\mathfrak{k}^\times$ is a free Abelian group, and the ℓ-adic completion homomorphism is an embedding:

$$\jmath_{\mathfrak{K}/\mathfrak{k}} : \mathfrak{K}^\times/\mathfrak{k}^\times \hookrightarrow \widehat{\mathcal{H}}_{\mathfrak{K}/\mathfrak{k}} \xrightarrow{\hat{\delta}_{\mathfrak{K}/\mathfrak{k}}} \mathrm{H}^1\big(\mathfrak{K}, \mathbb{Z}_\ell(1)\big)/\mathrm{H}^1\big(\mathfrak{k}, \mathbb{Z}_\ell(1)\big).$$

Now let us denote $\mathcal{P}(\mathfrak{K}) = \mathfrak{K}^\times/\mathfrak{k}^\times$, and view $\mathcal{P}(\mathfrak{K})$ as the projectivization of the (infinite) dimensional $\mathfrak{k}$-vector space $(\mathfrak{K}, +)$. **Suppose** that we have a Galois theoretic recipe in order to detect: First the *image of* $\mathcal{P}(\mathfrak{K})$ inside the Galois theoretically "known" $\widehat{\mathcal{H}}_{\mathfrak{K}/\mathfrak{k}}$. Second, the *projective lines in* $\mathcal{P}(\mathfrak{K})$. Remark that the multiplication l_x by any non-zero element $x \in \mathfrak{K}$ defines an "automorphisms" of $\mathcal{P}(\mathfrak{K})$ which respects "co-lineations". This automorphism is noting but the translation $l_{\jmath_{\mathfrak{K}/\mathfrak{k}}(x)}$ in $\mathcal{P}(\mathfrak{K})$, this time viewed again as the multiplicative group $\mathcal{P}(\mathfrak{K}) = \mathfrak{K}^\times/\mathfrak{k}^\times$. Now **using** the *Fundamental Theorem of projective geometry*, see e.g. Artin [A], it follows that the additive structure of $\mathfrak{K}$ can be deduced from the knowledge of all the projective lines in $\mathcal{P}(\mathfrak{K})$. And finally, since the "multiplications" $l_{\jmath_{\mathfrak{K}/\mathfrak{k}}(x)}$ do respect this structure, we finally deduce from this the field structure of $\mathfrak{K}$, as well the field extension $\mathfrak{K}|\mathfrak{k}$.

Now coming back to the case of a finitely generated field of positive characteristic K, let k is its field of constants, i.e., the field of absolute algebraic elements in K. Then k is a finite field, and K is a regular function field over

k. In order to use the idea of the proof above, we will give *Galois theoretic recipes* in order to detect:

I) *The projection* $pr : G_K^\ell \to G_k$ *and the* ℓ*-Frobenius of* k.

Here, by the ℓ-Frobenius of k we mean the Frobenius of the Galois group of the ℓ^∞-cyclotomic extension of k. In particular, from this we can deduce the ℓ-adic completions of $K^\times$ and of $k^\times$. Thus finally $\widehat{\mathcal{H}}_{K/k}$.

II) $K_{(\ell)} := K^\times \otimes \mathbb{Z}_{(\ell)}$ *inside* $\widehat{K}$.

This almost answers the question about detecting $\mathcal{P}(K)$: We know namely its $\mathbb{Z}_{(\ell)}$-version, but not $\mathcal{P}(K)$ itself.

III) $\mathcal{P}(K)$ *and its projective lines.*

Organization of the paper

Part I: Here we put together the necessary tools for the proof as follows:

a) First, a "cyclotomic by pro-ℓ" form of the *Local theory* from [P1]. It relies on results by WARE [W], KOENIGSMANN [Ko], see also BOGOMOLOV–TSCHINKEL [B–T], EFRAT [Ef], ENGELER–KOENIGSMANN [E–K]. Alternatively, one could develop the ideas from [P1] in this new (simpler) context. The aim of this local theory is to recover in a functorial way from G_K^ℓ the set $\mathfrak{D}_K$ of all the Zariski prime divisors of K, where K is any finitely generated field of positive characteristic with $\mathrm{td}\,(K) > 0$. In particular, if $\mathrm{td}\,(K) = 1$, the set $\mathfrak{D}_K$ is in a natural bijection with the set of closed points of the (unique) complete normal model X of the global function field K. Nevertheless, if $\mathrm{td}\,(K) > 1$, then $\mathfrak{D}_K$ is "huge", and much bigger than the set of Zariski prime divisors $\mathfrak{D}_X$ defined by the Weil prime divisors of any model X of K.

b) As a consequence of a) above, we show that given a finitely generated field K of positive characteristic with $\mathrm{td}\,(K) > 0$, say with constant field k, the following invariants of $K|k$ are group theoretically encoded in G_K^ℓ: First, the projection $G_K^\ell \to G_k$. Second, the cardinality q of k, and the ℓ-Frobenius of k. Thus finally, the ℓ-cyclotomic character $\chi : G_K^\ell \to \mathbb{Z}_\ell(1)$ is group theoretically encoded in G_K^ℓ. In particular, if $\mathrm{td}\,(K) = 1$, the degree d_x of every closed point $x \in X$ of the complete normal model X of K is encoded in G_K^ℓ.

c) Third, a more technical result by which we recover the "geometric sets of prime divisors" of any finitely generated field K of positive characteristic. By definition, a geometric set of prime divisors of K is the set of Zariski prime divisors $\mathfrak{D}_X \subset \mathfrak{D}_K$ defined by the Weil prime divisors of some quasi-projective normal model X of K. This result itself relies on DE JONG's *theory of alterations* [J].

Part II: Following a suggestion by Deligne [D2], we first develop a kind of "abstract non-sense" which we call abstract *pre-Galois formations*, reminding one in some sense of the abstract class field theory. The aim of this theory is to lay an axiomatic strategy for the proof of the main result.

The main example of pre-Galois formations are the *concrete* ones, which arise from geometry and arithmetic.

Elaborating on this, we next define abstract *Galois formations,* which come closer to approximating the concrete Galois formations. A main results here is Theorem 4.12 of Section 4), which shows that for a concrete Galois formation based on a finitely generated field $\mathfrak{K}$ with $\mathrm{char}(\mathfrak{K}) > 0$, the concrete field formation $\mathfrak{K}_{(\ell)}$ defining it is actually Galois theoretically encoded in $G^{\ell}_{\mathfrak{K}}$.

We finally mention Theorem 5.12, which asserts that under certain hypotheses, a *Galois formation* $\mathcal{G}$ can carry at most one structure of a concrete Galois formation. This result is the last key point in proving the main Theorem above.

PART I

1. Local theory (revisited)

In this section we will recall the main facts concerning the Local theory from [P1], but in a pro-ℓ setting. Our aim is to give to give a Galois theoretic recipe for finding information about the *space of Zariski prime divisors* in a functorial way.

Let $\mathfrak{K}|\mathfrak{k}$ be a function field over some base field $\mathfrak{k}$, and $\mathrm{td}(\mathfrak{K}|\mathfrak{k}) = d > 0$. We consider the family of all models $X_i \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$ such that the structure sheaf of X_i is a sheaf of sub-rings of $\mathfrak{K}$ with inclusions as structure morphisms. On the family of all the X_i's there exists a naturally defined domination relation as follows: $X_j \geq X_i$ if there exists a surjective $\mathfrak{k}$-morphism $\varphi_{ji} : X_j \to X_i$ which at the structure sheaf level is defined by inclusions. Let $\mathfrak{Proj}_{\mathfrak{K}}$ be the subfamily of all projective, normal models of the function field $\mathfrak{K}$. The following is well known, see e.g. Zariski–Samuel [Z–S], Ch.VI, especially §17:

• Every complete model is dominated by some $X_i \in \mathfrak{Proj}_{\mathfrak{K}}$ (Chow Lemma).

• The set $\mathfrak{Proj}_{\mathfrak{K}}$ is increasingly filtered with respect to $\geq$, hence it is a surjective projective system.

• Denote $\mathfrak{R}_{\mathfrak{K}} = \varprojlim_i X_i$ as topological spaces. We will call $\mathfrak{R}_{\mathfrak{K}}$ the *Riemann space* of $\mathfrak{K}$. Then the points of $\mathfrak{R}_{\mathfrak{K}}$ are in bijection with the space of all valuation $\mathfrak{k}$-rings of $\mathfrak{K}$. For $v = (x_i)_i$ in $\mathfrak{R}_{\mathfrak{K}}$ one has: x_i is the center of v on X_i in the usual sense.

Fact/Definition 1.1. Using e.g. [BOU], Ch.IV, §3, one shows that for a point $v = (x_i)_i$ in $\mathfrak{R}_{\mathfrak{K}}$ the following conditions are equivalent:

i) For i sufficiently large, x_i has co-dimension 1, or equivalently, x_i is the generic point of a prime Weil divisor of X_i. Hence v is the discrete $\mathfrak{k}$-valuation of $\mathfrak{K}$ with valuation ring $\mathcal{O}_{X_i,x_i}$.

ii) $\mathrm{td}(\mathfrak{K}v \,|\, \mathfrak{k}) = \mathrm{td}(\mathfrak{K}|\mathfrak{k}) - 1$.

We will say that a point $v = (x_i)_i$ in $\mathfrak{R}_{\mathfrak{K}}$ satisfying the above equivalent conditions is a *Zariski prime divisor* of $\mathfrak{K}|\mathfrak{k}$. We denote the space of all Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$ by $\mathfrak{D}_{\mathfrak{K}} \subset \mathfrak{R}_{\mathfrak{K}}$. One has: The space $\mathfrak{D}_{\mathfrak{K}} \subset \mathfrak{R}_{\mathfrak{K}}$ of all Zariski prime divisors of $\mathfrak{K}$ is the union of the spaces $\mathcal{P}_i$ of prime Weil divisors of all the models X_i of $\mathfrak{K}|\mathfrak{k}$ in $\mathfrak{Proj}_{\mathfrak{K}}$ (if we identify every prime Weil divisor with the discrete valuation on $\mathfrak{K}$ it defines).

More generally, let $\mathfrak{K}'|\mathfrak{K}$ be an arbitrary algebraic extension, and v' a valuation on $\mathfrak{K}'$. Then we will say that v' is a Zariski prime divisor (of $\mathfrak{K}$ or of $\mathfrak{K}'$), if its restriction v to $\mathfrak{K}$ is a Zariski prime divisor of $\mathfrak{K}|\mathfrak{k}$.

A) *On the decomposition group*

Consider the following context: $\mathfrak{K}'|\mathfrak{K}$ is some Galois field extension, and v is a valuation on $\mathfrak{K}'$. We set $p = \mathrm{char}(\mathfrak{K}'v)$ the residue characteristic. Further let Z_v, T_v, and V_v be respectively the decomposition group, the inertia group, and the ramification group of v in $\mathrm{Gal}(\mathfrak{K}'|\mathfrak{K})$. We denote by $\mathfrak{K}^V$, $\mathfrak{K}^T$, and $\mathfrak{K}^Z$ the corresponding fixed fields.

Fact 1.2. The following are well known facts from Hilbert decomposition, and/or ramification theory for general valuations:

1) $\mathfrak{K}'v \,|\, \mathfrak{K}v$ is a normal field extension, and $V_v \subset T_v$ are normal subgroups of Z_v. Moreover, one has a canonical exact sequence

$$1 \to T_v \to Z_v \to G_v \to 1\,.$$

One has $v(\mathfrak{K}^T) = v(\mathfrak{K}^Z) = v\mathfrak{K}$, and $\mathfrak{K}v = \mathfrak{K}^Z v$. Further, $\mathfrak{K}^T v \,|\, \mathfrak{K}v$ is the separable part of the normal extension $\mathfrak{K}'v \,|\, \mathfrak{K}v$, thus it is the maximal Galois sub-extension of $\mathfrak{K}'v \,|\, \mathfrak{K}v$.

2) There exists a pairing $\Psi_{\mathfrak{K}'} : T_v \times v\mathfrak{K}' \to \mu_{\mathfrak{K}'v}$, $(g,\, vx) \mapsto (gx/x)v$, and the following hold: The left kernel of $\Psi_{\mathfrak{K}'}$ is exactly V_v, and the right kernel of $\Psi_{\mathfrak{K}'}$ is the p-divisible hull of $v\mathfrak{K}$ in $v\mathfrak{K}'$. In particular, T_v/V_v is Abel Ian, and V_v is the unique Sylow p-group of T_v. Further, Ψ is compatible with the action of $\mathrm{Aut}(\mathfrak{K}'v \,|\, \mathfrak{K}v)$.

Fact 1.3. In the above context, let $\mathfrak{K}|\mathfrak{k}$ be as in the Introduction, and $\mathfrak{K}'|\mathfrak{K}$ satisfying the conditions i), ii), iii), from there. Suppose that v is trivial on $\mathfrak{k}$. Then $V_v = \{1\}$, $v\mathfrak{K}'$ is the ℓ-divisible hull of $v\mathfrak{K}$, and the residue field extension $\mathfrak{K}'v \,|\, \mathfrak{K}v$ also satisfies the properties i), ii), iii). In particular, we will denote $G'_{\mathfrak{K}v} = \mathrm{Gal}\,(\mathfrak{K}'v \,|\, \mathfrak{K}v)$

Further, for every $n = \ell^e$, there exists a unique sub-extension $\mathfrak{K}_n|\mathfrak{K}^T$ of $N|\mathfrak{K}^T$ such that $v\mathfrak{K}_n = \frac{1}{n}\,v\mathfrak{K}^T$ (thus $= \frac{1}{n}\,v\mathfrak{K}$). Clearly, $\mathfrak{K}_n|\mathfrak{K}^Z$ is a Galois sub-extension of $\mathfrak{K}'|\mathfrak{K}^Z$. On the other hand, the multiplication by n gives rise to a canonical isomorphism $\frac{1}{n}\,v\mathfrak{K} \,/\, v\mathfrak{K} \cong v\mathfrak{K}/n$. Therefore, the pairing $\Psi_{\mathfrak{K}_n}$ gives rise to a non-degenerate pairing

$$\Psi_n : T_v/n \times v\mathfrak{K}/n \to \mu_n \xrightarrow{\imath} \mathbb{Z}/n(1),$$

hence there result canonical isomorphisms $\theta^{v,n} : v\mathfrak{K}/n \to \mathrm{Hom}(T_v, \mu_n)$, and $\theta_{v,n} : T_v/n \to \mathrm{Hom}(v\mathfrak{K}, \mu_n)$. In particular, taking limits over all $n = \ell^e$, one obtains canonical isomorphisms

1^ℓ) $\quad \theta^v : \widehat{v\mathfrak{K}} \to \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell(1)\big)$ and $\theta_v : T_v \to \mathrm{Hom}\big(v\mathfrak{K}, \mathbb{Z}_\ell(1)\big)$.

Definition/Remark 1.4. In the above context, an element $g \in T_v$ is called a v-inertia element. A v-inertia element is called arithmetical inertia element, w.r.t. identification $\imath : \mathbb{T}_\ell \to \mathbb{Z}_\ell(1)$, if it satisfies the following equivalent conditions:

(i) $\theta_v(g)\big(v\mathfrak{K}\big) \subseteq \mathbb{Z}(1) \subset \mathbb{Z}_\ell(1)$.

(ii) $\theta^v(v\mathfrak{K})(g) \subseteq \mathbb{Z}(1) \subset \mathbb{Z}_\ell(1)$.

The arithmetical v-inertia elements build a subgroup $\mathfrak{T}_v \subset T_v$ which is dense in every closed subgroup of T_v. Moreover, if $v\mathfrak{K}$ is a finitely generated group, then $\mathfrak{T}_v$ is canonically isomorphic to the $\mathbb{Z}$-dual group of $v\mathfrak{K}$. This is so for instance if the valuation v is a discrete valuation on $\mathfrak{K}$.

2^ℓ) Recalling the notations and remarks from Introduction, one gets a commutative diagram of the form:

$$\begin{array}{ccc} \mathfrak{K}^\times & \xrightarrow{v} & v\mathfrak{K} \\ \big\downarrow{\scriptstyle \jmath_{\mathfrak{K}}} & & \big\downarrow{\scriptstyle \theta^v} \\ \mathrm{H}^1\big(G^\ell_{\mathfrak{K}}, \mathbb{Z}_\ell(1)\big) & \xrightarrow{\jmath_v} & \mathrm{H}^1\big(T_v, \mathbb{Z}_\ell(1)\big) \end{array}$$

Next let $\mathcal{B} = (vx_i)_i$ be an $\mathbb{F}_\ell$-basis of the vector space $v\mathfrak{K} \,/\, \ell$. For every x_i, choose a system of roots $(\alpha_{i,n})_n$ in N such that $\alpha_{i,n}^\ell = \alpha_{i,n-1}$ (all $n > 0$), where $\alpha_{i,0} = x_i$. Then setting $\mathfrak{K}^0 = \mathfrak{K}[(\alpha_{i,n})_{i,n}] \subset N$, it follows that v is totally ramified in $\mathfrak{K}^0|\mathfrak{K}$, and $v\mathfrak{K}^0$ is ℓ-divisible. Therefore, $\mathfrak{K}^0 v = \mathfrak{K}v$, and we finally obtain:

3^ℓ) In the above context, T_v has complements in Z_v, thus $Z_v \cong T_v \rtimes G_v$. Equivalently, we have $Z_v \cong T_v \rtimes G'_{\mathfrak{K}v} \cong \mathbb{Z}_\ell^{\mathcal{B}}(1) \rtimes G'_{\mathfrak{K}v}$ as profinite groups. In particular, $\mathrm{vcd}(Z_v) = \mathrm{vcd}(G'_{\mathfrak{K}v}) + |\mathcal{B}|$, where vcd denotes the virtual cohomological dimension.

4^ℓ) Let $U_v \subset \mathfrak{K}^\times$ denote the v-units, and $U_v^1 = 1 + \mathfrak{m}_v \subset U_v$ the principal v-units. Then U_v^1 becomes ℓ-divisible in $\mathfrak{K}^Z$, and $\mathfrak{K}v^\times = U_v/U_v^1$. We have a commutative diagram of the form:

$$\begin{array}{ccc} U_v & \longrightarrow & \mathfrak{K}v \\ \Big\downarrow{\scriptstyle \jmath_{\mathfrak{K}}} & & \Big\downarrow{\scriptstyle \jmath_{\mathfrak{K}v}} \\ \mathrm{H}^1\big(G^\ell_{\mathfrak{K}}, \mathbb{Z}_\ell(1)\big) & \xrightarrow{\jmath_v} & \mathrm{H}^1\big(G^\ell_{\mathfrak{K}v}, \mathbb{Z}_\ell(1)\big) \end{array}$$

B) *Recovering* $\mathfrak{D}_{\mathfrak{K}}$ (compare [P1])

Definition/Remark 1.5. Let $\mathfrak{K}|\mathfrak{k}$ be a function field, and suppose that $\mathrm{td}(\mathfrak{K}|\mathfrak{k}) = d > 0$.

1) The decomposition group $Z_v \subset G^\ell_{\mathfrak{K}}$ of some divisorial valuation $v \in \mathfrak{D}_{\mathfrak{K}^\ell}$ is called a *divisorial subgroup.*

2) A subgroup $Z \subset G^\ell_{\mathfrak{K}}$ which is isomorphic to a divisorial subgroup (of some function field) and has $\mathrm{vcd}(Z) = \mathrm{vcd}(\mathfrak{K})$ is called a *divisorial like subgroup.*

Proposition 1.6. *Let K be a finitely generated field of positive characteristic with $d = \mathrm{td}(K) > 0$. Let $Z_v \subset G^\ell_K$ be a divisorial subgroup, say defined by a divisorial valuation v on K^ℓ. Let T_v be the inertia group of v. Then the following hold:*

(1) *Z_v is self-normalising in G^ℓ_K. Further, if $Z_{v'} \neq Z_v$ is another divisorial subgroup, then one has: $Z_{v'} \cap Z_v = 1$.*

(2) *$T_v \cong \mathbb{Z}_\ell(1)$ as a G^ℓ_{Kv}-module. Further T_v is the unique maximal pro-ℓ abelian normal subgroup of Z_v. And $Z_v \cong T_v \rtimes G^\ell_{Kv} \cong \mathbb{Z}_\ell(1) \rtimes G^\ell_{Kv}$.*

Proof. To (1): Both assertions follow using a result of F. K. Schmidt, see e.g., Pop [P1], Proposition 1.3; see also the proof of Proposition 1.14 from loc.cit.. Concerning (2), the only non-obvious part is the fact that in the case $\mathrm{td}(K) > 1$, the Galois group G^ℓ_{Kv} has no non-trivial abelian normal subgroups. This follows from the Hilbertianity of Kv. □

A first content of the local theory is that "morally" the converse of the above Proposition is also true, i.e., if $Z \subset G^\ell_K$ is a divisorial like subgroup, then it comes from a Zariski prime divisor.

Proposition 1.7. *Let K be a finitely generated field of positive characteristic. Then one has:*

(1) *For every divisorial like subgroup Z of G^ℓ_K there exists a unique divisorial valuation v of K^ℓ such that $Z \subset Z_v$ and $\mathrm{char}(Kv) \neq \ell$.*

(2) *Moreover, if $T \subset Z$ is the unique maximal pro-cyclic normal subgroup of Z, then $T = Z \cap T_v$, where T_v is the inertia subgroup of Z_v.*

Therefore, the space $\mathfrak{D}_{K^\ell}$ of all Zariski prime divisors of K^ℓ is in bijection with the divisorial subgroups of G^ℓ_K. This bijection is given by $v \mapsto Z_v$.

Proof. The main step in the proof is the following ℓ-Lemma below, which replaces the q-Lemma from [P1], Local theory. After having the ℓ-Lemma, the remaining steps in the proof are similar to (but easier than) the ones from the proof of Theorem 1.16 from [P1], Local theory. We will skip the remaining details. □

THE ℓ-LEMMA (revisited). *In the context of Proposition 1.7, let $Z \subset G^\ell_K$ be a solvable closed subgroup with* $\mathrm{vcd}(Z) = \mathrm{td}(K) + 1$. *Then there exists a valuation v of K^ℓ satisfying: $Z \subset Z_v$ is an open subgroup and Kv a finite field. In particular, $Z \cong \mathbb{Z}^d_\ell(1) \rtimes G_v$, with G_v some open subgroup of G_k.*

There are several ways to prove the ℓ-Lemma above: First, one could develop the corresponding model theoretic machinery, and proceed as in [P1], Local theory. Second, one can apply the results from WARE [W], and KOENIGSMANN [Ko]; see also EFRAT [Ef], BOGOMOLOV–TSCHINKEL [B–T], ENGELER–KOENIGSMANN [E–K].

2. The constant field and the ℓ-Frobenius

Let K be a finitely generated field of positive characteristic with $\mathrm{td}(K) > 0$. Let k be the field of constants of K. We remark that for all prime numbers $\ell' \neq \ell$ we have $\mathrm{cd}_{\ell'}(G^\ell_K) = 1$, and that further, $\mathrm{cd}_\ell(G^\ell_K) = \mathrm{td}(K) + 1$. Thus in particular, both ℓ and $\mathrm{td}(K)$ are encoded in G^ℓ_K. Therefore, if $v \in \mathfrak{D}_{K^\ell}$ is a Zariski prime divisor, and Z_v is its decomposition group, then $\mathrm{td}(Kv)$ is encoded in Z_v via cd_ℓ of $G^\ell_{Kv} = Z_v/T_v$.

A) *Detecting the canonical projection $pr : G^\ell_K \to G_k$*

Let $T^\ell_K \subset G^\ell_K$ be the closed subgroup generated by all the inertia subgroups T_v (all $v \in \mathfrak{D}_{K^\ell}$. Then T^ℓ_K is a normal subgroup, and $\pi^\ell_{1,K} = G^\ell_K/T^\ell_K$ is the Galois group of the maximal sub-extension of $K^\ell|K$ in which no Zariski prime divisor of K is ramified. Thus is X is any normal model of K, and $\pi^\ell_1(X)$ is its cyclotomic by pro-ℓ étale fundamental group, then $\pi^\ell_{1,K}$ is a quotient of it. (N.B., if X is a smooth complete model, then the two groups coincide.) Looking at the abelianization $\pi^{\ell,\mathrm{ab}}_{1,K}$, and using the result of KATZ–LANG [K–L], it follows that $\pi^{\ell,\mathrm{ab}}_{1,K}$ fits into an exact sequence of the form:

$$1 \to (\text{finite } \ell\text{-group}) \to \pi^{\ell,\mathrm{ab}}_{1,K} \to G_k \to 1\,,$$

where k is the field of constants of k. Since $G_k \cong \widehat{\mathbb{Z}}$, it finally follows that in the above exact sequence, "(finite ℓ-group)" is exactly the torsion subgroup of $\pi_{1,K}^{\ell,\mathrm{ab}}$. Thus we have:

FACT 2.1. The canonical projection $pr : G_K^\ell \to G_k$ is encoded in G_K^ℓ as follows: First consider $G_K^\ell \to G_K^\ell / T_K^\ell =: \pi_{1,K}^\ell$. Then G_k is the quotient of $\pi_{1,K}^{\ell,\mathrm{ab}}$ by its torsion subgroup (which turns out to be a finite ℓ-group).

We conclude this subsection by the following result for later use:

FACT 2.2. In the above context, $\ker(pr) = G_{K\overline{k}}^\ell$ is encoded group theoretically in G_K^ℓ. In particular, we also "know" the geometric variant of $\pi_{1,K}^\ell$, which by definition is the following (here remark that $T_K^\ell = T_{K\overline{k}}^\ell$):

$$\overline{\pi}_{1,K}^\ell := G_{K\overline{k}}^\ell \,/\, T_K^\ell$$

1) We remark that in the case $\mathrm{td}(K) = 1$, the structure of $\overline{\pi}_{1,K}^\ell$ is well known by Grothendieck's specialisation Theorem: It is the pro-ℓ group on $2g_X$ generators and one relation (product of commutators), where g_X is the geometric genus of the complete normal model of K.

2) In general, $\overline{\pi}_{1,K}^\ell$ is a quotient of the pro-ℓ geometric fundamental group of any normal model X of $K\overline{k}$. Further, if $K\overline{k}$ has a smooth projective model $\overline{X}$, then $\overline{\pi}_{1,K}^\ell$ is the pro-ℓ quotient of the geometric fundamental group of $\overline{X}$.

B) *Detecting the constant field k of K*

We first consider the case $\mathrm{cd}\, G_K^\ell = 2$, or equivalently, $\mathrm{td}(K) = 1$. Then K is a global function field, and $\mathfrak{D}_K$ is in bijection with the closed points of the complete normal model X of K. Let $v \in \mathfrak{D}_{K^\ell}$ be a Zariski prime divisor, say corresponding to a closed point $x \in X$. Let $d_v = [\kappa(x) : k]$ be the degree of x. Then d_v is encoded in $Z_v \subset G_K^\ell$ in an obvious way: If namely $G_v = pr(Z_v)$ is the image of Z_v via the canonical projection, then $d_v = \big(G_k : G_v\big)$. In particular, the number N_d of points of degree d of X is encoded in G_K^ℓ as follows: This is exactly the number of conjugacy classes of divisorial subgroups Z_v of G_K^ℓ such that $d_v = d$.

On the other hand, by the Riemann Hypothesis for curves, it follows that $N_d = q^d + O(q^{d/2})$, where $q = |k|$ is the cardinality of the constant field k of K, and $O(\cdot)$ is a constant of absolute value bounded by $2g_X$ the genus of X. We concude detecting $q = |k|$ by the following obvious fact:

LEMMA 2.3. *The cardinality $q = |k|$ of the constant field k of K is the unique natural number q such that for constants $O(\cdot)$ as above for "sufficiently many" exponents d one has*

$$N_d = q^d + O(q^{d/2}).$$

Now in the case $\mathrm{cd}_\ell G^\ell_K > 2$, we proceed by induction. Taking any Zariski prime divisor v of K^ℓ, let Kv be its residue field, and let k_v be the constant field of Kv. Then $G^\ell_{Kv} \cong Z_v/T_v$ is "known", and $\mathrm{cd}_\ell G^\ell_{Kv} < \mathrm{cd}_\ell G^\ell_K$. Thus by induction on cd_ℓ, the cardinlity $q_v = |k_v|$ of the constant field k_v of Kv is encoded in G^ℓ_{Kv}. Finally, using the Weil bounds, we deduce the following:

Fact 2.4. The cardinality $q = |k|$ of the field of constants of K is exactly the g.c.d. of the set of all the local cardinalities $q_v = |k_v|$ (all $v \in \mathfrak{D}_{K^\ell}$).

C) *Detecting the ℓ-Frobenius of k*

In order to detect the k-Frobenius of k, we recall the by Propostion 1.6 we have: $Z_v \cong T_v \rtimes G^\ell_{Kv}$ (all divisorial Z_v). Further, G^ℓ_{Kv} acts on T_v via its quotient $G_v := pr(G^\ell_{Kv})$, which is the open subgroup of G_k of index d_v. (Here d_v is defined by $|k_v| = |k|^{d_v}$.) Moreover, the action of some $\sigma_v \in G_v$ on T_v is given by the conjugation with any preimage $\sigma'_v \in Z_v$ of σ_v. In other words, there exists a unique ℓ-adic unit $\epsilon_{\sigma_v} \in \mathbb{Z}_\ell^\times$ such that $\sigma_v(\tau) = \epsilon_{\sigma_v}\tau$ for all $\tau \in T_v$. Finally, the mapping

$$\chi_v : G_v \to \mathbb{Z}_\ell^\times, \quad \sigma_v \mapsto \epsilon_{\sigma_v}$$

is exactly the ℓ-cyclotomic character of G_v. We now have the following description of the the ℓ-Frobenius of the constant field k_v: Let $G_{1,v} = \ker(\chi_v)$. Then the fixed field $k_{0,v}$ of $G_{1,v}$ in k^ℓ is exactly the ℓ^∞-cyclotomic extension of k_v. Thus setting $G_{0,v} = \mathrm{Gal}(k_{0,v}|k_v)$, we have the following:

Lemma 2.5. *The ℓ-Frobenius of k_v is the unique element* $\mathrm{Frob}_{0,v}$ *of $G_{0,v}$ which acts on T_v as the multiplication by q^{d_v}. Equivalently, we have $\chi_v(\mathrm{Frob}_{0,v}) = q^{d_v}$ viewed as an ℓ-adic unit.*

Finally, from this we easily deduce the ℓ-cyclotomic character and the ℓ-Frobenius of k as follows. First, as at the Fact 2.4 above, it follows that the subgroup generated by all the kernels $G_{1,v}$ (all v) is exactly the kernel $G_1 \subset G_k$ of the ℓ-cyclotomic character $\chi : G_k \to \mathbb{Z}_\ell^\times$. Thus we have:

Fact 2.6. Let $G_0 = G_k/G_1$. Then G_0 is canonically the Galois group of the ℓ^∞ cyclotomic extension of k. Then the ℓ-Frobenius of k is the unique element $\mathrm{Frob}_0 \in G_0$ such that for all $v \in \mathfrak{D}_{K^\ell}$ we have: $\mathrm{Frob}_0^{d_v}$ is the image of $\mathrm{Frob}_{0,v}$ via the canonical projection $G_{0,v} \to G_0$.

In particular, Frob_0 also defines the ℓ-cyclotomic character of k in a canonical way.

3. Geometric families of Zariski prime divisors

We will work in the following context: $\mathfrak{K}|\mathfrak{k}$ is a function field with $\mathfrak{k}$, say a perfect field (maybe finite), and $\mathrm{td}(\mathfrak{K}|\mathfrak{k}) > 1$. Let $X \to \mathfrak{k}$ be a normal model of $\mathfrak{K}|\mathfrak{k}$. For such a model $X \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$, we identify each Weil prime divisor with its generic point x_1, as well as with its local ring $\mathcal{O}_{X,x_1}$ (which is a discrete valuation ring of $\mathfrak{K}|\mathfrak{k}$), thus with the Zariski prime divisor v_{x_1} of $\mathfrak{K}|\mathfrak{k}$ defined by it.

Fact/Definition 3.1. In the above context:

1) We denote by $\mathfrak{D}_X \subset \mathfrak{D}_{\mathfrak{K}|\mathfrak{k}}$ the set of all Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$ which are defined by Weil prime divisors of X.

2) We will say that a subset $\mathfrak{D} \subset \mathfrak{D}_{\mathfrak{K}|\mathfrak{k}}$ is a *geometric set of Zariski prime divisors,* if it satisfies the following equivalent conditions:

(i) There exists a normal projective model $X \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$ such that $\mathfrak{D}$ and $\mathfrak{D}_X$ are almost equal.[2]

(ii) There exists a quasi-projective normal model $X' \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$ such that $\mathfrak{D} = \mathfrak{D}_{X'}$.

Our aim in this section is to recall a criterion for describing the geometric sets $\mathfrak{D}$ of divisorial valuations of $\mathfrak{K}|\mathfrak{k}$. The source is Pop, [P2] and/or [P3], see loc.cit. for more details.

A) *Alterations and lines on varieties*

A *line* on a $\mathfrak{k}$-variety is by definition an integral $\mathfrak{k}$-subvariety $\mathfrak{l} \subseteq X$, which is a curve of geometric genus equal to 0. We denote by X^{line} the union of all the lines on X.

Definition 3.2. We will say that a variety $X \to \mathfrak{k}$ is very unruly if the set X^{line} is not dense in X. In particular, a curve X is very unruly if and only if its geometric genus g_X is positive.

Remarks 3.3. The following facts are more or less obvious:

1) Being *very unruly* is a birational notion.

2) If X is very unruly, and $f : Y \to X$ is a dominant generically finite morphism of $\mathfrak{k}$-varieties, then Y is very unruly. (This is a consequence of the Lüroth Theorem.)

3) Let $\ell \neq \mathrm{char}(\mathfrak{k})$ be a prime number, say $\ell \neq \mathrm{char}(\mathfrak{k})$. Let $X \to \mathfrak{k}$ be a geometrically integral variety of dimension $d = \dim(X)$. Then X has "many" finite ℓ-elementary covers $Z \to X$ of degree $\leq \ell^d$ such that Z is geometrically integral, and Z^{line} is empty. In particular, Z is very unruly.

Indeed, since the question is of birational nature, we can suppose that X is a finite cover of the affine d-dimensional space $\mathbb{A}^d$, where $d = \dim(X)$. To

[2] We say that two sets are almost equal, if their symmetric difference is finite.

preceed, for every $1 \leq j \leq d$, let $X^{(j)} \to \mathbb{A}^1$ be a cyclic cover of degree ℓ, such that $X^{(j)} \to \mathfrak{k}$ is geometrically integral, and has geometric genus ≥ 1. We set $Y = \times_j X^{(j)}$, and consider the resulting finite cover $Y \to \mathbb{A}^d$. Then we have:

a) Y is geometrically integral, and Y^{line} is empty.

b) $\mathrm{Aut}_{\mathbb{A}^d}(Y) \cong (\mathbb{Z}/\ell)^d$, thus ℓ-elementary of degree ℓ^d.

Finally we let $Z' = X \times_{\mathbb{A}^d} Y$ be the fibre product of X and Y over $\mathbb{A}^d$, and let $p' : Z' \to X$ and $q' : Z' \to Y$ be the structural projections. Since q' is a finite morphism, and there are no lines on Y, it follows (by Lüroth Theorem) that there are no lines on Z'. To conclude, we choose a connected component $Z \subset Z'$ such that the resulting $p : Z \hookrightarrow Z' \xrightarrow{p'} X$ is dominant. Then there are no lines on Z, and we have $\mathrm{Aut}_X(Z) \subset \mathrm{Aut}_{\mathbb{A}^d}(Y)$, thus a finite ℓ-elementary group of order $\leq \ell^d$.

4) Let $X \to \mathfrak{k}$ be a normal very unruly $\mathfrak{k}$-variety. Then almost all prime Weil divisors X_1 of X are very unruly (when viewed as $\mathfrak{k}$-varieties).

The birational interpretation of the facts above is as follows. Let $\mathfrak{K}|\mathfrak{k}$ be a regular function field with $\mathrm{td}(\mathfrak{K}|\mathfrak{k} = d > 0$. We will say that $\mathfrak{K}|\mathfrak{k}$ is *very unruly,* if $\mathfrak{K}|\mathfrak{k}$ has models $X \to \mathfrak{k}$ which are very unruly.

By the Remark 3) above, every function field $\mathfrak{K}|\mathfrak{k}$ has "many" finite ℓ-elementary extensions $N|\mathfrak{K}$ such that $N|\mathfrak{k}$ is very unruly.

Suppose that $d > 1$. We call a Zariski prime divisor of $\mathfrak{K}|\mathfrak{k}$ *very unruly,* if $\mathfrak{K}v \,|\, \mathfrak{k}$ is very unruly. A Zariski prime divisor v of $\mathfrak{K}|\kappa$ is very unruly if and only if there exists a normal model $X \to \kappa$ of $\mathfrak{K}|\mathfrak{k}$, and a very unruly prime Weil divisor X_1 of X such that $v = v_{X_1}$.

One has the following birational variant/sharpening of the remark 4) above. This is a very essential step in the Global theory. Its proof relies on DE JONG's theory of alterations.

THEOREM 3.4. *Let $\mathfrak{K}|\mathfrak{k}$ be a regular function field with $\mathrm{td}(\mathfrak{K}|\mathfrak{k}) > 1$, and $X \to \mathfrak{k}$ a normal model for $\mathfrak{K}|\mathfrak{k}$. For every Zariski prime divisor v of $\mathfrak{K}|\mathfrak{k}$, let x_v the centre of v on X, and X_v the Zariski closure of x_v in X. Then X_v is a prime Weil divisor of X for* almost all *very unruly Zariski prime divisors v of $K|\kappa$.*

In other words, the set of all very unruly Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$ is almost equal to the set of all very unruly Weil prime divisors of the model $X \to \mathfrak{k}$.

For a proof, see loc.cit.. As an application we have:

COROLLARY 1. *Let $\mathfrak{K}|\mathfrak{k}$ be a very unruly function field with $\mathrm{td}(\mathfrak{K}|\mathfrak{k}) > 1$, and $X \to \mathfrak{k}$ a normal model for $\mathfrak{K}|\mathfrak{k}$. Then the set of Zariski prime divisors*

$\mathfrak{D}_X$ defined by X is almost equal to the set of the very unruly Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$.

B) *Geometric sets of prime divisors*

As in the previous subsection, let $\mathfrak{K}|\mathfrak{k}$ be a regular function field. Let $\mathfrak{D}$ be a set of Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$. For every finite extension $\mathfrak{K}_i|\mathfrak{K}$, let $\mathfrak{D}_i$ be the prolongation of $\mathfrak{D}$ to $\mathfrak{K}_i$. Thus in particular, $\mathfrak{D}_i$ is a set of Zariski prime divisors of $\mathfrak{K}_i|\mathfrak{k}$.

We have now the following first step in giving a recipe for detecting the geometric sets of Zariski prime divisors:

PROPOSITION 3.5. *In the context above, set $d = \mathrm{td}(\mathfrak{K}|\mathfrak{k})$. A set $\mathfrak{D}$ of Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$ is geometric if and only if the following conditions are satisfied:*

(i) *There exists a finite ℓ-elementary extension $\mathfrak{K}_0|\mathfrak{K}$ of degree $\leq \ell^d$ such that $\mathfrak{D}_0$ is almost equal to the set of all very unruly prime divisors of $\mathfrak{K}_0$.*

(ii) *If $\mathfrak{K}_2|\mathfrak{K}$ is any ℓ-elementary extension of degree $\leq \ell^d$, and $\mathfrak{K}_1 = \mathfrak{K}_2\mathfrak{K}_0$ is the compositum, then $\mathfrak{D}_1$ is almost equal to the set of all very unruly prime divisors of $\mathfrak{K}_1$.*

Proof. Let $X \to \mathfrak{k}$ be a quasi-projective normal model of $\mathfrak{K}|\mathfrak{k}$. Let $\mathfrak{K}_2|\mathfrak{K}$ be a finite field extension such that $\mathfrak{K}_2|\mathfrak{k}$ is very unruly, and let $Y \to X$ be the normalisation of X in $\mathfrak{K}_2|\mathfrak{K}$. Thus $Y \to \mathfrak{k}$ is a normal model of $\mathfrak{K}_2|\mathfrak{k}$. Further, the set of Zariski prime divisors $\mathfrak{D}_Y$ of $\mathfrak{K}_2|\mathfrak{k}$ is exactly the prolongation of $\mathfrak{D}_X$ to $\mathfrak{K}_2$. On the other hand, by the Corollary above, it follows that $\mathfrak{D}_Y$ is almost equal to the set of all very unruly Zariski prime divisors of $\mathfrak{K}_2|\mathfrak{k}$. Therefore we have: If $\mathfrak{K}_2|\mathfrak{k}$ is very unruly, then the prolongation of $\mathfrak{D}_X$ to $\mathfrak{K}_2$ is almost equal to the set of all very unruly Zariski prime divisors of $\mathfrak{K}_2|\mathfrak{k}$.

Now coming back to the proof of the Proposition, we proceed as follows: Suppose first that $\mathfrak{D}$ is geometric. Thus $\mathfrak{D}$ and $\mathfrak{D}_X$ are almost equal, and the same is true for their prolongations to any finite field extension $\mathfrak{K}_i|\mathfrak{K}$ of $\mathfrak{K}$. Next consider a finite irreducible cover $Z \to X$ as in Remark 3.3, 3), and let $\mathfrak{K}_0|\mathfrak{K}$ be the corresponding ℓ-elemetary extension of degree $\leq \ell^d$. Since Z contains no lines, $\mathfrak{K}_0$ is very unruly; and the same is true for all finite extensions $\mathfrak{K}_i$ of $\mathfrak{K}_0$, in particular, for the ones considered in the Theorem. Counting everything togehter we get: The prolongations of $\mathfrak{D}$, and that of $\mathfrak{D}_X$, to $\mathfrak{K}_i$ are both almost equal to the set of all very unruly Zariski prime divisors of $\mathfrak{K}_i|\mathfrak{k}$.

Conversally, given a set $\mathfrak{D}$ with the properties i) and ii), from the Theorem. To the given finite extension $\mathfrak{K}_0|\mathfrak{K}$, we choose $\mathfrak{K}_2|\mathfrak{K}$ to be the finite ℓ-elementary extension of degree $\leq \ell^d$ defined by some irreducible finite cover $Z \to X$ as in the Remark 3.3, 3). Then $\mathfrak{K}_2|\mathfrak{k}$ is very unruly, and the same is true for the finite extension $\mathfrak{K}_1 = \mathfrak{K}_0\mathfrak{K}_2$ of $\mathfrak{K}_2$. As above, it follows that the set of all very unruly Zariski prime divisors of $\mathfrak{K}_1$ is geometric. Therefore, if $Y \to X$ is the normalisation of X in the finite field extension $\mathfrak{K}_1|\mathfrak{K}$, it follows that $\mathfrak{D}_Y$ is almost equal to the set of all very unruly Zariski prime divisors of $\mathfrak{K}_1|\mathfrak{k}$. On the other hand, by the hypotheses on i) and ii), it follows that $\mathfrak{D}_1$ is almost equal to the set of all very unruly Zariski prime divisors of $\mathfrak{K}_1|\mathfrak{k}$. Thus finally, $\mathfrak{D}_1$ is almost equal to $\mathfrak{D}_Y$. But then the same is true restrictions of $\mathfrak{D}_1$ and $\mathfrak{D}_Y$ to $\mathfrak{K}$. These sets are on the other hand exactly $\mathfrak{D}$ and $\mathfrak{D}_X$. □

Criterion 3.6.

Finally, we reamrk that the above Proposition gives an inductive way to reduce the description of geometric set of Zariski prime divisors of $\mathfrak{D}$ to a criterion for its Zariski prime divisors v to be very unruly. This can be done inductively on $d = \mathrm{td}(\mathfrak{K}\,|\,\mathfrak{k})$ as follows:

1) Case $d = 1$:

$\mathcal{P}_{\mathrm{geom}}(1)$: A set of Zariski prime divisors $\mathfrak{D}$ of $\mathfrak{K}\,|\,\mathfrak{k}$ is geometric if and only if it is almost equal to the set of all Zariski prime divisors $\mathfrak{D}_{\mathfrak{K}\,|\,\mathfrak{k}}$.

$\mathcal{P}_{\mathrm{v.u.}}(1)$: $\mathfrak{K}\,|\,\mathfrak{k}$ is very unruly if and only if the geometric genus of the unique complete normal model of $\mathfrak{K}\,|\,\mathfrak{k}$ is > 0.

2) Case $d > 1$: Then by induction on d, we already have criteria $\mathcal{P}_{\mathrm{geom}}(k)$ and $\mathcal{P}_{\mathrm{v.u.}}(k)$ which assure that sets of Zariski prime divisors are geometric, respectively that function fields are very unruly (all $1 \le k < d$). We now make the induction step $d = \mathrm{td}(\mathfrak{K}\,|\,\mathfrak{k})$ as follows:

$\mathcal{P}_{\mathrm{geom}}(d)$: A set of Zariski prime divisors $\mathfrak{D}$ of $\mathfrak{K}\,|\,\mathfrak{k}$ is geometric if and only if the criterion given by Lemma 3.5 is satisfied. Here we remark that the assertion from loc.cit. "v is a very unruly Zariski prime divisor of $\mathfrak{K}\,|\,\mathfrak{k}$" is actually equivalent to "$\mathfrak{K}v\,|\,\mathfrak{k}$ satisfies $\mathcal{P}_{\mathrm{v.u.}}(d-1)$".

$\mathcal{P}_{\mathrm{v.u.}}(d)$: $\mathfrak{K}\,|\,\mathfrak{k}$ is very unruly if and only if for some set $\mathfrak{D} \subset \mathfrak{D}_{\mathfrak{K}\,|\,\mathfrak{k}}$ satisfying $\mathcal{P}_{\mathrm{geom}}(d)$ one has: $\mathcal{P}_{\mathrm{v.u.}}(d-1)$ holds for almost all $\mathfrak{K}v\,|\,\mathfrak{k}$.

C) *The case of a finite base field*

Here we show that in the case the base field k is finite, and $K|k$ is a (regular) function field with $d = \mathrm{td}(K|k) > 0$, the above Criterion 3.6 for checking whether a set of Zariski prime divisors can be interpreted in G^ℓ_K.

Indeed, by Proposition 1.7, the Zariski prime divisors of K^ℓ are in bijection with the divisorial subgroups of G^ℓ_K via $v \mapsto Z_v$. Thus a bijection $\mathfrak{D}^\ell \mapsto \mathcal{Z}_{\mathfrak{D}^\ell}$ from sets of Zariski prime divisors $\mathfrak{D}^\ell \subset \mathfrak{D}_{K^\ell}$ to sets of divisorial subgroups $\mathcal{Z}$ of G^ℓ_K. Moreover, a set $\mathfrak{D}^\ell$ is the prolongation to K^ℓ of a set of Zariski prime divisors $\mathfrak{D} \subset \mathfrak{D}_K$ if and only if $\mathcal{Z}_{\mathfrak{D}^\ell}$ is invarinat under conjugation in G^ℓ_K. If this is the case, then every conjugacy class represents an element of $\mathfrak{D}$.

Let $\mathcal{Z}$ be a set of divisorial subgroups of G^ℓ_K, which is closed under conjugation in G^ℓ_K. Thus the corresponding set of Zariski prime divisors $\mathfrak{D}^\ell_{\mathcal{Z}}$ is the prolongation to K^ℓ of a set of Zariski prime divisors $\mathfrak{D}_{\mathcal{Z}} \subset \mathfrak{D}_K$. By induction on $d = \mathrm{td}\,(K) = \mathrm{cd}\,(G^\ell_K) - 1$, we have the following Galois translation of Criterion 3.6:

1) Case $d = 1$:

$\mathrm{Gal}\,\mathcal{P}_{\mathrm{geom}}(1)$ is equivalent to: $\mathcal{Z}$ contains almost all conjugacy classes of divisorial subgroups of G^ℓ_K.

$\mathrm{Gal}\,\mathcal{P}_{\mathrm{v.u.}}(1)$ is equivalent to: $\overline{\pi}^\ell_{1,K} := G^\ell_{K\overline{k}}\,/\,T^\ell_K$ as defined at Fact 2.2 is non-trivial.

2) Case $d > 1$:

Then by induction on d, we already have the Galois translation $\mathrm{Gal}\,\mathcal{P}_{\mathrm{geom}}(\delta)$ and $\mathrm{Gal}\,\mathcal{P}_{\mathrm{v.u.}}(\delta)$ of the criteria $\mathcal{P}_{\mathrm{geom}}(\delta)$ and $\mathcal{P}_{\mathrm{v.u.}}(\delta)$ (all $1 \leq \delta < d$). We now make the induction step $d = \mathrm{td}\,(K|k)$ as follows:

$\mathrm{Gal}\,\mathcal{P}_{\mathrm{geom}}(d)$: $\mathcal{Z}$ defines a geometric set of Zariski prime divisors $\mathfrak{D}_{\mathcal{Z}}$ of $K|k$ if and only if (the Galois translation of) the criterion given by Lemma 3.5 is satisfied. Here we remark that the assertion from loc.cit. "v is a very unruly Zariski prime divisor of $K|k$" is actually equivalent to "G^ℓ_{Kv} satisfies $\mathrm{Gal}\,\mathcal{P}_{\mathrm{v.u.}}(d-1)$".

$\mathrm{Gal}\,\mathcal{P}_{\mathrm{v.u.}}(d)$: $K|k$ is very unruly if and only if for some set $\mathcal{Z}$ as above satisfying $\mathrm{Gal}\,\mathcal{P}_{\mathrm{geom}}(d)$ one has: For almost all conjugacy classes of Z_v, the condition $\mathrm{Gal}\,\mathcal{P}_{\mathrm{v.u.}}(d-1)$ holds for G^ℓ_{Kv}.

Part II

4. Abstract Γ by pro-ℓ Galois theory

In this section we develop a "Γ by pro-ℓ Galois theory", in other words a theory of "Γ by pro-ℓ Galois groups", which in some sense has a flavor similar to one of the abstract class field theory. The final aim of this theory is to provide a machinery for recovering fields from their pro-ℓ Galois theory in an axiomatic way.

A) *The axioms*

The context is the following: Let ℓ be a fixed prime number, and $\mathcal{Z}_\ell$ a quotient of $\mathbb{Z}_\ell$. Let further Γ_0 be a profinite group endowed with an "$\mathcal{Z}_\ell$-cyclotomic character" $\chi_{\Gamma_0} : \Gamma_0 \to \mathcal{Z}^\times_\ell$.

Definition. An *abstract* (Γ by pro-ℓ) *level* $\delta \geq 0$ *pre-Galois formation* $\mathcal{G} = \big(G, pr, (Z_v)_v\big)$ is defined by induction on δ to be a profinite group G endowed with extra structure as follows:

Axiom I) The level $\delta = 0$:

A level $\delta = 0$ pre-Galois formation is a pro-finite group G, which is an extension of a closed subgroup $\Gamma \subseteq \Gamma_0$ by a pro-ℓ-group:

$$1 \to \overline{G} \longrightarrow G \xrightarrow{pr} \Gamma \to 1$$

We will say that $\mathcal{G}$ *has level* $\delta > 0$, if the following Axioms II, III, are inductively satisfied.

Axiom II) Decomposition structure:

G is endowed with a set of of closed subgroups $(Z_v)_v$ closed under conjugation. We call Z_v the decomposition group at v, and suppose that:

i) *Every decomposition group Z_v is endowed with a non-trivial pro-cyclic normal subgroup $T_v \subseteq \ker(pr)$, such that the set $\{T_v\}_v$ is closed under conjugation.*

ii) *$T_v \cong \mathcal{Z}_\ell$ as abstract pro-ℓ groups (all v), and $T_v \cap T_w = \{1\}$ if $v \neq w$, and the trivial group is the only accumulation point of the set $\{T_v\}_v$.*

We call T_v the inertia subgroup at v, and let $\Gamma_v = pr(G_v)$ be the image of G_v in Γ. We set $G_v := Z_v/T_v$, and call G_v the residue group at v. We remark that since $T_v \subseteq \ker(pr)$, the restriction of pr to Z_v factorizes through G_v. Thus G_v is an extension of Γ_v by a pro-ℓ group of the form

$$(*) \qquad 1 \to \overline{G}_v \longrightarrow G_v \xrightarrow{pr_v} \Gamma_v \to 1$$

We also remark that G, as well as all the G_v act on $\mathcal{Z}_\ell$ via their "cyclotomic characters" $\chi := \chi_{\Gamma_0} \circ pr$, respectively $\chi_v := \chi \circ pr_v$.

iii) *The canonical action of G_v on T_v resulting from the exact sequence $1 \to T_v \to Z_v \to G_v \to 1$ is via the cyclotomic character χ_v of G_v.*

Axiom III) Induction:

Every G_v caries itself the structure of an abstract pre-Galois formation of level $(\delta - 1)$ via the exact sequence $(*)$ above.

Convention. Let $\mathcal{G} = \big(G, pr, (Z_v)_v\big)$ be a level $\delta \geq 0$ pre-Galois formation. In order to have a uniform notation, we enlarge the index set v (which in the case $\delta = 0$ is empty) by a new symbol, which we denote v_*, by setting $Z_{v_*} = G$ and $T_{v_*} = \{1\}$. We will say that v_* is the "trivial valuation". Thus G is the decomposition group of the trivial valuation, and its "inertia group" is the trivial group. In particular, the residue Galois group at v_* is $G_{v_*} = G$.

Definition/Remark 4.1. Let $\mathcal{G} = \big(G, pr, (Z_v)_v\big)$ be a pre-Galois formation of some level $\delta \geq 0$. Consider any α such that $0 \leq \alpha \leq \delta$.

1) By induction on δ it is easy to see that one can view $\mathcal{G} = \big(G, pr, (Z_v)_v\big)$ canonically as a pre-Galois formation of level α.

2) We define inductively the system $\mathcal{G}^{(\alpha)} = (\mathcal{G}_{\alpha,i})_i$ of the *α-residual abstract pre-Galois formations* of $\mathcal{G}$ as follows: First, if $\alpha = 0$, then this system consists of $\mathcal{G}$ only. In general, if $0 < \alpha \leq \delta$, we recall that every residue group G_v carries the structure of an abstract pre-Galois formation of level $(\delta - 1)$; thus of level $(\alpha - 1)$ by remark 1) above. Let $\mathcal{G}_v$ be this abstract pre-Galois formation. Then by induction, the system of the $(\alpha-1)$-residual abstract pre-Galois formations $\mathcal{G}_v^{(\alpha-1)}$ of each $\mathcal{G}_v$ are defined. We then set $\mathcal{G}^{(\alpha)} = \big(\mathcal{G}_v^{(\alpha-1)}\big)_v$.

We remark that the "correct" notation for the system of the α-residual abstract pre-Galois formations $(\mathcal{G}_{\alpha,i})_i$ is to index it by multi-indices of length α of the form $\mathbf{v} = (v_{i_1} \dots v_{i_\alpha})$, where v_{i_1} is a valuation of G, v_{i_2} is a valuation of G_{v_1}, etc..

3) For every multi-index $\mathbf{v}$ as above, let $G_\mathbf{v}$ be the profinite group on which the $\mathbf{v}$-residual abstract pre-Galois formation $\mathcal{G}_\mathbf{v}$ is based. Then we will call $G_\mathbf{v}$ a *$\mathbf{v}$-residual group* of $\mathcal{G}$. One can further elaborate here as follows: Given a multi-index $\mathbf{v} = (v_{i_1} \dots v_{i_\alpha})$, we can define inductively the following:

a) The decomposition group $Z_\mathbf{v}$ of G, as being —inductively on α— the pre-image of $Z_{(v_{i_2} \cdots v_{i_\alpha})} \subseteq G_{v_{i_1}}$ in $Z_{v_{i_1}}$ via $Z_{v_{i_1}} \to G_{v_{i_1}}$.

b) The inertia group $T_\mathbf{v}$ of $Z_\mathbf{v}$, as being the kernel of $Z_\mathbf{v} \to G_\mathbf{v}$. From the definition it follows that $T_\mathbf{v} \cong \mathcal{Z}_\ell^\alpha$.

c) A canonical projection $pr_\mathbf{v} : G_\mathbf{v} \to \Gamma_\mathbf{v} \subset \Gamma$ induced by pr.

4) We can/will associate to $\mathcal{G}$ its "geometric part" $\overline{\mathcal{G}} = \big(\overline{G}, \underline{pr}, (\overline{Z}_v)_v\big)$, which is defined as follows: $\overline{G} := \ker(pr)$, pr is trivial morphism, and $\overline{Z}_v = Z_v \cap \overline{G}$, etc.. Clearly, if $\mathcal{G}$ has level δ, then $\overline{\mathcal{G}}$ has level δ too. Further, one can define the geometric part $\overline{\mathcal{G}_\mathbf{v}}$ of each $\mathbf{v}$-residual pre-Galois formation $\mathcal{G}_\mathbf{v}$. Remark that the process of "going/taking $\mathbf{v}$-residual" is compatible with "going/taking geometric parts", i.e., $\overline{\mathcal{G}_\mathbf{v}} = \overline{\mathcal{G}}_\mathbf{v}$.

B) *From abstract pre-Galois formations to abstract Galois ones*

Definition/Remark 4.2. Let $\mathcal{G} = \big(G, pr, (Z_v)_v\big)$ be an abstract pre-Galois formation of some level $\delta' \geq 0$. Because of the action of G on $\mathcal{Z}_\ell$ via its "cyclotomic character" χ, we can speak about the corresponding (abstract) G-twists $\mathcal{Z}_\ell(r)$, and the resulting ℓ-adic cohomology. We do correspondingly the same for all closed subgroups of G, e.g., for Z_v and T_v (all v).

1) We denote $\widehat{\mathcal{L}}_\mathcal{G} = \mathrm{H}^1\big(G, \mathcal{Z}(1)\big)$ and call it the ℓ-adic completion of the *abstract pre-field formation defining* $\mathcal{G}$. Set further, $\widehat{\kappa}_\mathcal{G} := \mathrm{H}^1\big(\Gamma, \mathcal{Z}(1)\big)$ and

$\widehat{\mathcal{L}}_{\overline{\mathcal{G}}} = \mathrm{H}^1\big(\overline{G}, \mathcal{Z}(1)\big)$. Via the exact sequence $1 \to \overline{G} \hookrightarrow G \xrightarrow{pr} \Gamma \to 1$, we get canonical embeddings

$$\widehat{\kappa}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \hookrightarrow (\widehat{\mathcal{L}}_{\overline{\mathcal{G}}})^{\Gamma} \hookrightarrow \widehat{\mathcal{L}}_{\overline{\mathcal{G}}}$$

We denote $\widehat{\mathcal{H}}_{\mathcal{G}} = \widehat{\mathcal{L}}_{\mathcal{G}}/\widehat{\kappa}_{\mathcal{G}}$, and call it the geometric part of $\widehat{\mathcal{L}}_{\mathcal{G}}$.

Further, consider α satisfying $0 \leq \alpha \leq \delta'$, and $\mathbf{v}$ some multi-index of length α. Then we do the same correspondingly for the δ-residual abstract pre-Galois formation $\mathcal{G}_{\mathbf{v}}$.

From now on suppose that $\delta' > 0$.

2) Let $T \subseteq G$ be the closed subgroup generated by all the inertia groups T_v (all v). We denote $\pi_{1,\mathcal{G}} := G/T$ and call it the abstract fundamental group of $\mathcal{G}$. Thus there exists a canonical exact sequence $1 \to T \to G \to \pi_{1,\mathcal{G}} \to 1$. Taking ℓ-adic cohomology, and taking into account that T acts trivially on $\mathcal{Z}_\ell$, it follows that we have an exact sequence of the form

$$0 \to \mathrm{H}^1\big(\pi_{1,\mathcal{G}}, \mathcal{Z}_\ell(1)\big) \xrightarrow{\mathrm{can}} \mathrm{H}^1\big(G, \mathcal{Z}_\ell(1)\big) = \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\mathcal{G}}} \big(\mathrm{Hom}\big(T, \mathcal{Z}_\ell\big)\big)^G$$

We set $\widehat{U}_{\mathcal{G}} = \mathrm{H}^1\big(\pi_{1,\mathcal{G}}, \mathcal{Z}_\ell(1)\big)$ and call it the *unramified part* of $\widehat{\mathcal{L}}_{\mathcal{G}}$. In particular, if no confusion is possible, we will identify $\widehat{U}_{\mathcal{G}}$ with its image in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

3) We now have a closer look at the structure of $\widehat{\mathcal{L}}_{\mathcal{G}}$. For an arbitrary v we have inclusions $T_v \hookrightarrow Z_v \hookrightarrow G$. Thus we can/will consider/denote restriction maps as follows:

$$\jmath^v : \widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{H}^1\big(G, \mathcal{Z}_\ell(1)\big) \xrightarrow{\mathrm{res}_{Z_v}} \mathrm{H}^1\big(Z_v, \mathcal{Z}_\ell(1)\big) \xrightarrow{\mathrm{res}_v} \mathrm{Hom}\big(T_v, \mathcal{Z}_\ell\big)$$

Let T_v and $T_{v'}$ be conjugated, say by σ. It is well known that σ acts trivially on $\widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{H}^1\big(G, \mathcal{Z}_\ell(1)\big)$, and that $\mathrm{H}^1(\sigma) \circ \jmath^v = \jmath^{v'}$. Therefore we have: Let $\mathfrak{v}$ be a system of representatives for the G-conjugacy classes of the inertia (equivalently: decomposition) groups T_v. Then by the Axiom II), ii), the family $(\jmath^{\mathfrak{v}})_{\mathfrak{v}}$ gives rise canonically to a continuous homomorphism $\widehat{\oplus_{\mathfrak{v}}}\, \jmath^{\mathfrak{v}}$ of ℓ-adically complete $\mathbb{Z}_\ell$-modules

$$\widehat{\oplus_{\mathfrak{v}}}\, \jmath^{\mathfrak{v}} : \widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{H}^1\big(G, \mathcal{Z}_\ell(1)\big) \to \widehat{\oplus_{\mathfrak{v}}}\, \mathrm{Hom}\big(T_{\mathfrak{v}}, \mathcal{Z}_\ell\big) .$$

It obvious that $\widehat{\oplus_{\mathfrak{v}}}\, \jmath^{\mathfrak{v}}$ factors trough $\mathrm{Hom}\big(T, \mathcal{Z}_\ell\big)^G$. And one checks without difficulty that the resulting canonical homomorphism is an embedding (of ℓ-adically complete $\mathcal{Z}_\ell$-modules):

$$\mathrm{Hom}\big(T, \mathcal{Z}_\ell\big)^G \hookrightarrow \widehat{\oplus_{\mathfrak{v}}}\, \mathrm{Hom}\big(T_{\mathfrak{v}}, \mathcal{Z}_\ell\big) .$$

We will identify $\mathrm{Hom}\big(T, \mathcal{Z}_\ell\big)^G$ with its image inside $\widehat{\oplus_{\mathfrak{v}}}\, \mathrm{Hom}\big(T_{\mathfrak{v}}, \mathcal{Z}_\ell\big)$. Thus $\jmath^{\mathcal{G}} = \widehat{\oplus_{\mathfrak{v}}}\, \jmath^{\mathfrak{v}}$ on $\widehat{\mathcal{L}}_{\mathcal{G}}$.

We will denote $\widehat{\mathrm{Div}}_{\mathcal{G}} := \widehat{\oplus}_{\mathfrak{v}} \operatorname{Hom}\big(T_{\mathfrak{v}}, \mathcal{Z}_\ell\big)$ and call it the *abstract divisor group* of $\mathcal{G}$. We further say that the image of $\widehat{\mathcal{L}}_{\mathcal{G}}$ in $\widehat{\mathrm{Div}}_{\mathcal{G}}$ is the *divisorial quotient* (or the divisorial part) of $\widehat{\mathcal{L}}_{\mathcal{G}}$.

We further set $\widehat{\mathfrak{Cl}}_{\mathcal{G}} = \operatorname{coker}(\jmath^{\mathcal{G}})$, and call it the *abstract divisor class group* of $\mathcal{G}$. Therefore, we finally have a canonical exact sequence

$$0 \to \widehat{U}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\mathcal{G}}} \widehat{\mathrm{Div}}_{\mathcal{G}} \xrightarrow{\text{can}} \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0\,.$$

4) Recall that $\overline{G} = \ker(\,pr\,)$, thus $T \subset \overline{G}$. Let $T \to T^{\mathrm{ab}}$, and $\overline{G} \to \overline{G}^{\mathrm{ab}}$ be the abelianisations of T and $\overline{G}$. Thus there is a canonical homomorphism $T^{\mathrm{ab}} \to \overline{G}^{\mathrm{ab}}$, which need not be injective. Let $\overline{v}$ denote a representative for the $\overline{G}$-conjugacy classes of T_v (all v).

We say that $\mathcal{G}$ is *complete curve like* if the following holds:

i) The canonical map $T^{\mathrm{ab}} \to \overline{G}^{\mathrm{ab}}$ is injective, and the abelianisation homomorphism $T \to T^{\mathrm{ab}}$ is injective on each T_v, i.e., $T_v \cong T^{\mathrm{ab}}_{\overline{v}}$.

ii) There exist generators $\tau^{\mathrm{ab}}_{\overline{v}}$ of $T^{\mathrm{ab}}_{\overline{v}}$ such that $\sum_{\overline{v}} \tau^{\mathrm{ab}}_{\overline{v}} = 0$, and this is the only pro-relation satisfied by the system of elements $\mathfrak{T} = (\tau^{\mathrm{ab}}_{\overline{v}})_{\overline{v}}$.

Now consider $0 \leq \delta < \delta'$. We say that $\mathcal{G}$ is *δ-residually complete curve like* if all the δ-residual pre-Galois formations $\mathcal{G}_{\mathbf{v}}$ are residually complete curve like. In particular, "0-residually complete curve like" is the same as "residually complete curve like".

5) For every v consider the exact sequence $1 \to T_v \to Z_v \to G_v \to 1$ given by Axiom II, iii). Recall that $T_v \cong \mathcal{Z}_\ell(1)$ as a Z_v-module, and as a G_v-module. Using the above exact sequence, we get an inflation homomorphism $\mathrm{inf}_v : \mathrm{H}^1\big(G_v, \mathcal{Z}_\ell(1)\big) \to \mathrm{H}^1\big(Z_v, \mathcal{Z}_\ell(1)\big)$.

We set $\widehat{U}_v = \ker(\jmath^v)$ and call it the *abstract v-units* in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Thus the unramified part of $\widehat{\mathcal{L}}_{\mathcal{G}}$ is exactly $\widehat{U}_{\mathcal{G}} = \cap_v \ker(\jmath^v)$. Since $T_v = \ker(Z_v \to G_v)$, it follows that $\mathrm{res}_{Z_v}(\widehat{U}_v)$ is the image of the inflation map infl_v. Therefore there exists a canonical continuous homomorphism, which we call the *v-reduction homomorphism:*

$$\jmath_v : \widehat{U}_v \to \mathrm{H}^1\big(G_v, \mathcal{Z}_\ell(1)\big) =: \widehat{\mathcal{L}}_{\mathcal{G}_v}$$

where $\widehat{\mathcal{L}}_{\mathcal{G}_v}$ is the ℓ-adic completion of the abstract v-residual field, i.e., the one attached to the v-residual pre-Galois formation $\mathcal{G}_v$.

Denote $\mathcal{L}_{\mathcal{G},\mathrm{fin}} = \{\, x \in \widehat{\mathcal{L}}_{\mathcal{G}} \mid \jmath^v(x) = 0 \text{ for almost all } v\,\}$. Remark/recall that $\widehat{\kappa}_{\mathcal{G}} \subseteq \widehat{U}_{\mathcal{G}} \subseteq \mathcal{L}_{\mathcal{G},\mathrm{fin}}$. A closed submodule $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ is said to have *finite co-rank,* if $\Delta \subset \mathcal{L}_{\mathcal{G},\mathrm{fin}}$, and $\Delta / \widehat{U}_{\mathcal{G}}$ is a finite $\mathcal{Z}_\ell$-module (or equivalently, Δ is contained in $\ker(\jmath^v)$ for almost all v). Clearly, the sum of two finite co-rank submodules of $\widehat{\mathcal{L}}_{\mathcal{G}}$ is again of finite co-rank. Thus the set of such submodules is inductive. And one has:

$$\mathcal{L}_{\mathcal{G},\mathrm{fin}} = \cup_\Delta \Delta \text{ (all finite co-rank } \Delta)$$

6) Next let v be arbitrary, and $\mathcal{G}_v$ the corresponding residual pre-Galois formation. To $\mathcal{G}_v$ we have the corresponding exact sequence as defined for $\mathcal{G}$ at point 3) above:

$$0 \to \widehat{U}_{\mathcal{G}_v} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}_v} \xrightarrow{\jmath^{\mathcal{G}_v}} \widehat{\mathrm{Div}}_{\mathcal{G}_v}$$

We will say that $\mathcal{G}$ is *ample,* if for every finite co-rank submodule $\Delta \subseteq \widehat{\mathcal{L}}_{\mathcal{G}}$ there exists v such that:

i) $\Delta \subseteq \widehat{U}_v$ and $\jmath_v(\Delta) \subseteq \mathcal{L}_{\mathcal{G}_v,\mathrm{fin}}$.

ii) $\ker\big(\,\Delta \xrightarrow{\jmath_v} \widehat{\mathcal{L}}_{\mathcal{G}_v} \xrightarrow{\jmath^{\mathcal{G}_v}} \widehat{\mathrm{Div}}_{\mathcal{G}_v}\,\big) \subseteq \widehat{U}_{\mathcal{G}}$.

Now consider $0 \leq \delta \leq \delta'$. We say that $\mathcal{G}$ is *ample up to level* δ*,* if either $\delta = 0$, or by induction on δ the following hold: First, if $\delta = 1$, then $\mathcal{G}$ is ample in the sense defined above. Second, if $\delta > 1$, then $\mathcal{G}_v$ is ample up to level $(\delta-1)$ for all v.

C) *From abstract Galois formations to abstract field formations*

Convention/Definition 4.3.

1) In order avoid too technical formulations, we will suppose from now —if not explicitly otherwise stated— that $\mathcal{Z}_\ell = \mathbb{Z}_\ell$. In particular, $\mathbb{Z} \subset \mathbb{Z}_{(\ell)}$ are subgroups/subrings of $\mathcal{Z}_\ell$.

2) Let M be an arbitrary $\mathcal{Z}_\ell$-module. We will say subsets M_1, M_2 of M are *ℓ-adically equivalent,* if there exists an ℓ-adic unit $\epsilon \in \mathcal{Z}_\ell$ such that $M_2 = \epsilon \cdot M_1$ inside M. Correspondingly, given systems $S_1 = (x_i)_i$ and $S_2 = (y_i)_i$ of elements of M, we will say that S_1 and S_2 are ℓ-adically equivalent, if there exists an ℓ-adic unit $\epsilon \in \mathcal{Z}_\ell$ such that $x_i = \epsilon\, y_i$ (all i).

3) Let M be an arbitrary ℓ-adically complete module. We will say that a $\mathbb{Z}_{(\ell)}$-submodule $\mathcal{M}_{(\ell)} \subseteq M$ of M is a $\mathbb{Z}_{(\ell)}$*-lattice* in M, (for short, a lattice) if $\mathcal{M}_{(\ell)}$ is a free $\mathbb{Z}_{(\ell)}$-module, and it is ℓ-adically dense in M, and satisfies the following equivalent conditions:

a) $M/\ell \cong \mathcal{M}_{(\ell)}/\ell$

b) $\mathcal{M}_{(\ell)}$ has a $\mathbb{Z}_{(\ell)}$-basis $\mathfrak{B}$ which is ℓ-adically independent in M.

c) The condition b) above is satisfied for every $\mathbb{Z}_{(\ell)}$-basis of $\mathcal{M}_{(\ell)}$.

More general, let $N \subseteq \mathcal{M}_{(\ell)} \subseteq M$ be $\mathbb{Z}_{(\ell)}$-submodules of M such that M/N is again ℓ-adically complete. We will say that $\mathcal{M}_{(\ell)}$ is an N*-lattice* in M, if $\mathcal{M}_{(\ell)}/N$ is a lattice in M/N.

4) Finally, in the context from 3) above, a *true lattice* in M is a free abelian subgroup $\mathcal{M}$ of M such that $\mathcal{M}_{(\ell)} := \mathcal{M} \otimes \mathbb{Z}_{(\ell)}$ is a lattice in M in the sense of 3) above. And we will say that $\mathbb{Z}$-submodule $\mathcal{M} \subseteq M$ containing N is a *true N-lattice* in M, if $\mathcal{M}/N$ is a true lattice in M/N.

CONSTRUCTION 4.4. Let $\mathcal{G} = \big(G, pr, (Z_v)_v\big)$ be a pre-Galois formation which is both *ample up to level* δ and δ*-residually complete curve like.* Recall the last exact sequence from point 3) of subsection B):

$$0 \to \widehat{U}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\mathcal{G}}} \widehat{\mathrm{Div}}_{\mathcal{G}} \xrightarrow{\mathrm{can}} \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0\,.$$

The aim of this subsection is to describe the ℓ-adic equivalence class of a $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$, which —in the case it exists— will called an *abstract field formation defining* $\mathcal{G}$.

The case $\delta = 0$, i.e., $\mathcal{G}$ complete curve like.

In the notations from Definition/Remark 4.2, 4) above, let $\mathfrak{T} = (\tau^{\mathrm{ab}}_{\overline{v}})_{\overline{v}}$ be the system of generators of the groups $T^{\mathrm{ab}}_{\overline{v}}$ as there. Let as call such a system a *distinguished system of inertia generators.* We remark that any two distinguished system of inertia generators are *strictly* ℓ*-adically equivalent.* Indeed, if $\tau'_{\overline{v}} \in T^{\mathrm{ab}}_{\overline{v}}$ is another generator of $T^{\mathrm{ab}}_{\overline{v}}$, then $\epsilon_{\overline{v}}\tau'_{\overline{v}} = \tau_{\overline{v}}$ for some ℓ-adic units $\epsilon_{\overline{v}} \in \mathcal{Z}_\ell$. Thus if $\mathfrak{T}' = (\tau'_{\overline{v}})_{\overline{v}}$ does also satisfy condition ii) from Definition/Remark 4.2, 4), then we have also $\sum_{\overline{v}} \epsilon^{-1}_{\overline{v}}\, \tau^{\mathrm{ab}}_{\overline{v}} = 0$. By the uniqueness of the relation $\sum_{\overline{v}} \tau^{\mathrm{ab}}_{\overline{v}} = 0$, it follows that $\epsilon_{\overline{v}} = \epsilon$ for some fixed ℓ-adic unit $\epsilon \in \mathcal{Z}_\ell$, as claimed.

Now let $\mathfrak{T} = (\tau^{\mathrm{ab}}_{\overline{v}})_{\overline{v}}$ be a distinguished system generators of T. Further let $\mathcal{F}_{\mathfrak{T}}$ be the Abelian pro-ℓ free group on the system $\mathfrak{T}$. Then one has a canonical exact sequence of pro-ℓ groups

$$0 \to \mathcal{Z}_\ell\, \tau \to \mathcal{F}_{\mathfrak{T}} \to T^{\mathrm{ab}} \to 0\,,$$

where $\tau = \sum_{\overline{v}} \tau^{\mathrm{ab}}_{\overline{v}}$ in $\mathcal{F}_{\mathfrak{T}}$ is the pro-ℓ sum of the generators $\tau^{\mathrm{ab}}_{\overline{v}}$ (all $\overline{v}$). Taking ℓ-adically continuous Homs we get an exact sequence

$$0 \to \mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell(1)\big) \to \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}_\ell(1)\big) \to \mathrm{Hom}\big(\mathcal{Z}_\ell\,\tau, \mathcal{Z}_\ell(1)\big) \to 0$$

and remark that last homomorphism is simply the summation:

$$\varphi \mapsto \big(\tau \mapsto \textstyle\sum_{\overline{v}} \varphi(\tau^{\mathrm{ab}}_{\overline{v}}\big)$$

Thus $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell(1)\big)$ consists of all the homomorphisms $\varphi \in \mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}_\ell\big)$ with trivial "trace". Next consider the system $\mathfrak{B} = (\varphi_{\overline{v}})_{\overline{v}}$ of all the functionals $\varphi_{\overline{v}} \in \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}_\ell\big)$ defined by $\varphi_{\overline{v}}(\tau^{\mathrm{ab}}_{\overline{v}'}) = 1$ if $\overline{v} = \overline{v}'$, and 0, otherwise (all $\overline{v}, \overline{v}'$). We denote by $\mathcal{D}_{\mathfrak{T},(\ell)}$ the $\mathbb{Z}_{(\ell)}$-submodule of $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}_\ell\big)$ generated by $\mathfrak{B}$, and set

$$\mathcal{D}^0_{\mathfrak{T},\ell} := \{\, \textstyle\sum_{\overline{v}} a_{\overline{v}}\, \varphi_{\overline{v}} \in \mathcal{D}_{\mathfrak{T},(\ell)} \mid \sum_{\overline{v}} a_{\overline{v}} = 0 \,\} = \mathcal{D}_{\mathfrak{T},(\ell)} \cap \mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell\big)$$

We have: $\mathfrak{B}$ *is an* ℓ*-adic basis* of $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}_\ell\big)$, i.e., $\mathcal{D}_{\mathfrak{T},(\ell)}$ is ℓ-adically dense in $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}_\ell\big)$, and there are no non-trivial ℓ-adic relations between the elements of $\mathfrak{B}$. We will say that $\mathfrak{B} = (\varphi_{\overline{v}})_{\overline{v}}$ is the "dual basis" to $\mathfrak{T}$. In particular, $\mathcal{D}_{\mathfrak{T},(\ell)}$

is a lattice in $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell\big)$, and $\mathcal{D}^0_{\mathfrak{T},(\ell)}$ is a lattice in $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell\big)$. Precisely, if $\varphi_{\overline{v}_0}$ is fixed, then the system $e_{\overline{v}} = \varphi_{\overline{v}} - \varphi_{\overline{v}_0}$ (all $\overline{v} \neq \overline{v}_0$) is an ℓ-adically independent $\mathbb{Z}_{(\ell)}$-basis of $\mathcal{D}^0_{\mathfrak{T},(\ell)}$.

We now explain the dependence of $\mathcal{D}_{\mathfrak{T},(\ell)}$ on the distinguished system of generators $\mathfrak{T} = (\tau^{\mathrm{ab}}_{\overline{v}})_{\overline{v}}$ of of T. Let namely $\mathfrak{T}' = (\tau'_{\overline{v}})_{\overline{v}}$ be another such system. Let $\epsilon \in \mathcal{Z}_\ell$ be the unique ℓ-adic unit such that $\epsilon \cdot \mathfrak{T}' = \mathfrak{T}$. If $\mathfrak{B} = (\varphi'_{\overline{v}})_{\overline{v}}$ is the dual basis to $\mathfrak{T}'$, then $\varphi'_{\overline{v}} = \epsilon\, \varphi_{\overline{v}}$ inside $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell\big)$. Thus $\mathfrak{B}' = \epsilon \cdot \mathfrak{B}$. In other words, $\mathfrak{B}$ and $\mathfrak{B}'$ are strictly ℓ-adically equivalent. Hence we have: $\epsilon \cdot \mathcal{D}_{\mathfrak{T},(\ell)} = \mathcal{D}_{\mathfrak{T}',(\ell)}$ and $\epsilon \cdot \mathcal{D}^0_{\mathfrak{T},(\ell)} = \mathcal{D}^0_{\mathfrak{T}',(\ell)}$. Moreover, ϵ is unique modulo rational ℓ-adic units.

1) Therefore, all the subgroups of $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell\big)$ the form $\mathcal{D}_{\mathfrak{T},(\ell)}$, respectively of the form $\mathcal{D}^0_{\mathfrak{T},(\ell)}$, are ℓ-adically rationally strictly equivalent (all distinguished $\mathfrak{T}$). Hence the ℓ-adic equivalence classes of $\mathcal{D}_{\mathfrak{T},(\ell)}$ and $\mathcal{D}^0_{\mathfrak{T},(\ell)}$ do not depend on $\mathfrak{T}$, but only on $\mathcal{G}$.

2) We now consider G-invariants: First we remark that G acts on both T^{ab} and the distinguished systems of inertia generators $\mathfrak{T}$ by conjugation, and on $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}_\ell(1)\big)$ functorially. Thus this action is takes place actually via the quotient $pr : G \to \Gamma$. Thus we have:

a) If $\mathfrak{T} = (\,\tau^{\mathrm{ab}}_{\overline{v}})_{\overline{v}}$ is a distinguished system of inertia generators, then $\mathfrak{T}^{\,\gamma}$ is also such a system. Taking into account Axiom II), iii), it follows that $\mathfrak{T}^{\,\gamma} = (\tau^{\mathrm{ab}}_{\overline{v}})^{\gamma} = \chi(\gamma)\, \tau^{\mathrm{ab}}_{\overline{v}_\gamma}$, where χ is the cyclotomic character of Γ, and $\overline{v}_\gamma$ is the corresponding Γ-conjugate of $\overline{v}$. Thus up to permutation of the coordinates we have $\mathfrak{T}^\gamma = \chi(\gamma) \cdot \mathfrak{T}$. Since for every v the image $pr(Z_v) = \Gamma_v \subseteq \Gamma$ is open, it follows that the orbit of each $T^{\mathrm{ab}}_{\overline{v}}$ under Γ is finite, say of cardinality $d_{\overline{v}}$.

b) From this one easily deduces the action of γ on the generators $\varphi_{\overline{v}}$. It is given by $(\varphi_{\overline{v}})^\gamma = \varphi_{\overline{v}^\gamma}$. Hence from this it follows that the Γ-invariant part of $\mathcal{D}_{\mathfrak{T},(\ell)}$ and $\mathcal{D}^0_{\mathfrak{T},(\ell)}$ can be described as follows:

- $\big(\mathcal{D}_{\mathfrak{T},(\ell)}\big)^G = \{\, \sum_{\overline{v}} a_{\overline{v}}\, \varphi_{\overline{v}} \in \mathcal{D}_{\mathfrak{T},(\ell)} \mid a_{\overline{v}} = a_{\overline{v}^\gamma}$ for all $\overline{v}$ and all $\gamma \in \Gamma\,\}$.
- $\big(\mathcal{D}^0_{\mathfrak{T},(\ell)}\big)^G = \{\, \varphi \in \big(\mathcal{D}_{\mathfrak{T},(\ell)}\big)^G \mid \sum_{\overline{v}} a_{\overline{v}} = 0\,\}$.

For every $\mathfrak{T}$, we denote $\mathrm{Div}_{\mathfrak{T},(\ell)} = (\mathcal{D}_{\mathfrak{T},(\ell)})^G = \mathcal{D}_{\mathfrak{T},(\ell)} \cap \widehat{\mathrm{Div}}_{\mathcal{G},(\ell)}$. We have the following:

Fact 4.5. In the above context, let $\mathcal{L}_{\mathfrak{T},(\ell)}$ be the pre-image of $\mathrm{Div}_{\mathfrak{T},(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Further consider all the finite co-rank submodules Δ of $\widehat{\mathcal{L}}_{\mathcal{G}}$ containing $\widehat{U}_{\mathcal{G}}$. Then the following assertions are equivalent:

(i) $\mathcal{L}_{\mathfrak{T},(\ell)}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

(ii) $\Delta \cap \mathcal{L}_{\mathfrak{T},(\ell)}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in Δ (all Δ as above).

Proof. Clear. □

The case: $\delta > 0$.

For every v we denote by $\mathcal{G}_v$ the corresponding v-residual abstract pre-Galois formation, etc.. In particular, $\mathcal{G}_v$ is both $(\delta-1)$-residually complete curve like, and ample up to level $(\delta-1)$. Now let Δ be a finite co-rank $\mathcal{Z}_\ell$-submodule of $\widehat{\mathcal{L}}_{\mathcal{G}}$ such that $\Delta \cap \widehat{U}_{\mathcal{G}} = 1$. For every v such that $\Delta \subset \widehat{U}_v$, we set $\Delta_v = \jmath_v(\Delta)$. Since $\mathcal{G}$ is *ample up to level* δ, there exists some v such that:

$(*)$ $\Delta \subset \widehat{U}_v$, and Δ_v has finite co-rank in $\widehat{\mathcal{L}}_{\mathcal{G}_v}$, and $\Delta_v \cap \widehat{U}_{\mathcal{G}_v} = 1$. In particular, Δ and Δ_v have the same $\mathcal{Z}_\ell$-rank.

FACT 4.6. For every v and the corresponding $\mathcal{G}_v$, let a $\widehat{U}_{\mathcal{G}_v}$-lattice $\mathcal{L}_{v,(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}_v}$ be given. Then up to ℓ-adic equivalence, there exits at most one $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ such that for every finite co-rank $\mathcal{Z}_\ell$-module $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ with $\Delta \cap \widehat{U}_{\mathcal{G}} = 1$ and v as at $(*)$ above the following hold:

i) $\mathcal{L}_\Delta := \Delta \cap \mathcal{L}_{(\ell)}$, $\mathcal{L}_{\Delta_v} := \Delta_v \cap \mathcal{L}_{v,(\ell)}$ are lattices in Δ, respectively Δ_v.

ii) The lattices $\jmath_v(\mathcal{L}_\Delta)$ and $\mathcal{L}_{\Delta_v}$ are ℓ-adically equivalent.

Proof. Let $\mathcal{L}_{(\ell)}, \mathcal{L}'_{(\ell)}$ be $\widehat{U}_{\mathcal{G}}$-latices in $\widehat{\mathcal{L}}_{\mathcal{G}}$ satisfying i), ii), above. For Δ as in Fact above, set $\mathcal{L}'_\Delta = \Delta \cap \mathcal{L}'_{(\ell)}$. Then by hypothesis ii) it follows that both latices $\jmath_v(\mathcal{L}_\Delta)$ and $\jmath_v(\mathcal{L}'_\Delta)$ are ℓ-adically equivalent to the lattice $\mathcal{L}_{\Delta_v}$ inside Δ_v. Thus they are equivalent. After replacing $\mathcal{L}'_{(\ell)}$ by some multiple, we can suppose that $\mathcal{L}_\Delta = \mathcal{L}'_\Delta$. As Δ was arbitrary, it finally follows that $\mathcal{L}_{(\ell)} = \mathcal{L}'_{(\ell)}$. $\square$

DEFINITION/REMARK 4.7. Let $\mathcal{G}$ be an abstract pre-Galois formation which is both δ-residual complete curve like and ample up to level δ. An *abstract field formation* $\mathcal{L}_{(\ell)}$ defining $\mathcal{G}$ —if it exists— is defined inductively on δ as follows:

First, if $\delta = 0$, suppose that $\mathcal{G}$ satisfies the equivalent conditions (i), (ii), from Fact 4.5. Then define $\mathcal{L}_{(\ell)}$ to be one of the $\mathcal{L}_{\mathfrak{T},(\ell)}$.

Second, if $\delta > 0$, *suppose that for all* v, corresponding v-residual abstract field formations $\mathcal{L}_{v,(\ell)}$ defining $\mathcal{G}_v$ exist. We define $\mathcal{L}_{(\ell)}$ to be any $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$ which satisfies the conditions i), ii) from Fact 4.6.

We will say that $\mathcal{G}$ as above is a *level* δ *abstract Galois formation,* if there exist abstract field formations $\mathcal{L}_{(\ell)}$ defining $\mathcal{G}$ as explained above.

We remark the following:

Let $\mathcal{G}$ be a level δ abstract Galois formation, and $\mathcal{L}_{(\ell)}$ an abstract field formation defining $\mathcal{G}$. Then $\jmath^v(\mathcal{L}_{(\ell)})$ is a lattice in $\mathrm{Hom}\big(T_v, \mathcal{Z}_\ell(1)\big) \cong \mathcal{Z}_\ell$ (all v). Recalling that $\mathfrak{v}$ are representatives for the conjugacy classes of the valuations v, we will set $\mathrm{Div}_{\mathcal{L}_{(\ell)}} = \oplus_{\mathfrak{v}} \mathbb{Z}_{(\ell)}\mathfrak{v}$, and $\mathfrak{Cl}_{\mathcal{L}_{(\ell)}} = \mathrm{Div}_{\mathcal{L}_{(\ell)}}/\jmath^{\mathcal{G}}(\mathcal{L}_{(\ell)})$, and call them the abstract divisor group of $\mathcal{L}_{(\ell)}$, respectively the abstract ideal class group of $\mathcal{L}_{(\ell)}$. Thus one has a commutative diagram of the form

$$(*)\qquad \begin{array}{ccccccccc} 0 \to \widehat{U}_{\mathcal{G}} & \hookrightarrow & \mathcal{L}_{(\ell)} & \xrightarrow{\jmath^{\mathcal{G}}} & \mathrm{Div}_{\mathcal{L}_{(\ell)}} & \xrightarrow{\text{can}} & \mathfrak{Cl}_{\mathcal{L}_{(\ell)}} \to 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \jmath_{\mathfrak{Cl}}} \\ 0 \to \widehat{U}_{\mathcal{G}} & \hookrightarrow & \widehat{\mathcal{L}}_{\mathcal{G}} & \xrightarrow{\jmath^{\mathcal{G}}} & \widehat{\mathrm{Div}}_{\mathcal{G}} & \xrightarrow{\text{can}} & \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0 \end{array}$$

where the first three vertical morphisms are the canonical inclusions, whereas the last one is the ℓ-adic completion homomorphism.Thus in particular, $\mathfrak{Cl}_{\mathcal{L}_{(\ell)}}$ does not necessarily embed in $\widehat{\mathfrak{Cl}}_{\mathcal{G}}$.

D) *Example: Concrete Galois formations / concrete field formations*

Let $\mathfrak{k}$ be a perfect base field, e.g., a finite field or a number field. Let further $\mathfrak{K}|\mathfrak{k}$ be a field extension such that the relative (separable) algebraic closure of $\mathfrak{k}$ in $\mathfrak{K}$ is finite over $\mathfrak{k}$. Consider any field extension $\mathfrak{K}'|\mathfrak{K}$ satisfying the conditions i), ii), iii), from Introduction. Suppose that $\mathfrak{K}'$ contains a separable closure $\mathfrak{k}^{\mathrm{sep}} = \overline{\mathfrak{k}}$ of $\mathfrak{k}$. We set as usually $G = \mathrm{Gal}(\mathfrak{K}'|\mathfrak{K})$, and $\Gamma_0 = G_{\mathfrak{k}}$, and let $pr : G \to \Gamma := pr(G) \subseteq \Gamma_0$ be the canonical projection. We further set $\mathcal{Z}_\ell = \mathbb{Z}_\ell$, and endow Γ_0 with its $\mathcal{Z}_\ell$-cyclotomic character $\chi_{\Gamma_0} : \Gamma_0 \to \mathcal{Z}_\ell^\times$. Then we get:

FACT 4.8. $\mathcal{G} = \big(G, pr, \emptyset\big)$ is a level $\delta = 0$ pre-Galois formation.

Nevertheless, in the case $\mathfrak{K}|\mathfrak{k}$ is a *function field* with $d = \mathrm{td}(\mathfrak{K}|\mathfrak{k}) > 0$, we can refine the pre-Galois formation above by making it into a pre-Galois formation of level δ for every $0 \le \delta \le d$ as follows: First, let us endow every function field $\mathfrak{K}|\mathfrak{k}$ as above with a geometric set $\mathfrak{D}_X$ of Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$. Here $X \to \mathfrak{k}$ is a quasi-projective normal model of $\mathfrak{K}|\mathfrak{k}$, and $\mathfrak{D}_X$ is its set of Weil prime divisors. Without loss of generality, we will suppose that X *geometrically integral,* or equivalently, that the function field $\mathfrak{K}|\mathfrak{k}$ is regular. Second, for every field extension $\mathfrak{K}'|\mathfrak{K}$ as above, let $\mathfrak{D}'_X$ be the prolongation of $\mathfrak{D}_X$ to $\mathfrak{K}'$. For every $v \in \mathfrak{D}'_X$ we have: The residue field $\mathfrak{K}v$ is again a function field over $\mathfrak{k}$ with $\mathrm{td}(\mathfrak{K}v \,|\, \mathfrak{k}) = (d-1)$. Furthermore, using the facts presented in Section 2, it is an easy exercise to show that one has the following:

j) Let $T_v \subset Z_v$ be the inertia, respectively the decomposition group of v in G. Then $T_v \cong \mathbb{Z}_\ell$ is pro-ℓ cyclic. Further, if $v \neq w$, then $T_v \cap T_w = \{1\}$.

jj) For every v, the residue field extension $\mathfrak{K}'v \,|\, \mathfrak{K}v$ satisfies the conditions i), ii), iii) from the Introduction.

jjj) Let $G_v = \mathrm{Gal}(\mathfrak{K}'v \,|\, \mathfrak{K}v)$ be the residual Galois group at v. Then denoting $pr_v : G_v \to \Gamma$ the canonical projection, it follows that $\Gamma_v := pr_v(G_v)$ is an open subgroup of Γ.

Thus the field extension $\mathfrak{K}'|\mathfrak{K}$ defines in a canonical way "local" field extensions $\mathfrak{K}'v|\mathfrak{K}v$ which also satisfy the conditions j), jj), jjj), above. Thus if $d>1$, each such function field $\mathfrak{K}v|\mathfrak{k}$ comes equipped with a geometric set of Zariski prime divisors $\mathfrak{D}_{X_v}$ (with X_v a quasi-projective normal model of $\mathfrak{K}v\mid\mathfrak{k}$).

Now consider any δ with $0\le\delta\le d$. Then by induction on the transcendence degree over $\mathfrak{k}$, we can suppose that each G_v endowed with the projection $pr_v: G_v\to\Gamma$ and the set of decomposition groups defined by $\mathfrak{D}'_{X_v}$ is an pre-Galois formation of level $(\delta-1)$. Thus we have the following:

Proposition/Definition 4.9. *Let* $\mathcal{G}_X=\big(G,pr,(Z_v)_v\big)$ *be as constructed above. Then* $\mathcal{G}_X$ *is in a canonical way a level* δ *abstract pre-Galois formation, for every* δ *satisfying* $0\le\delta\le\mathrm{td}(\mathfrak{K}|\mathfrak{k})$.

An abstract pre-Galois formation of the form $\mathcal{G}_X$ *will be called a concrete pre-Galois formation of level* δ.

Remarks 4.10. Let $\mathcal{G}_X=\big(G,pr,(Z_v)_v\big)$ be a concrete pre-Galois formation as constructed/defined above. Then $\widehat{\mathfrak{K}}=\mathrm{H}^1\big(G'_{\mathfrak{K}},\mathcal{Z}_\ell(1)\big)=\widehat{\mathcal{L}}_{\mathcal{G}_X}$. In order to compute $\widehat{U}_{\mathcal{G}_X}$, $\widehat{\mathrm{Div}}_{\mathcal{G}_X}$, and $\widehat{\mathfrak{Cl}}_{\mathcal{G}_X}$ we do the following: First, let $\mathcal{H}(X)$ denote the group of principal divisors on X, and consider the canonical exact sequence $0\to\mathcal{H}(X)\longrightarrow\mathrm{Div}(X)\xrightarrow{pr}\mathfrak{Cl}(X)\to 0$.

1) Passing to ℓ-adic completions, we get an exact sequence of ℓ-adic complete groups of the form: $0\to\mathcal{T}_{\ell,X}\longrightarrow\widehat{\mathcal{H}}(X)\longrightarrow\widehat{\mathrm{Div}}(X)\longrightarrow\widehat{\mathfrak{Cl}}(X)\to 0$, where $\mathcal{T}_{\ell,X}=\varprojlim\,{}_{\ell^n}\mathfrak{Cl}(X)$ (with multiplication as homomorphisms), and the last three objects the corresponding ℓ-adic completions.

2) On the other hand one has $\mathrm{Div}(X)=\oplus_{\mathfrak{v}\in\mathfrak{D}_X}\mathfrak{v}\mathfrak{K}$. For every $\mathfrak{v}$, and the fixed prolongation v of $\mathfrak{v}$ to $\mathfrak{K}'$, consider the commutative diagram from subsection A), 2^{ℓ}). Recalling that $\mathfrak{K}^\times\to\mathcal{H}(X)$, $x\mapsto\oplus\,\mathfrak{v}(x)$ is surjective, we get a commutative diagram of the following form:

$$\begin{array}{ccccccc}
\widehat{\mathfrak{K}} & \to & \widehat{\mathrm{Div}}(X)=\widehat{\oplus_v}\,v\mathfrak{K} & \to & \widehat{\mathfrak{Cl}}(X) & \to & 0\\
\big\downarrow{\scriptstyle \jmath_{\mathfrak{K}}} & & \big\downarrow{\scriptstyle \oplus\jmath_v} & & \big\downarrow{\scriptstyle \jmath_X} & & \\
\mathrm{H}^1\big(G^{\ell}_{\mathfrak{K}},\mathbb{Z}_\ell(1)\big) & \xrightarrow{\jmath^{\mathcal{G}_X}} & \widehat{\oplus_v}\,\mathrm{H}^1\big(T_v,\mathbb{Z}_\ell(1)\big) & \xrightarrow{\text{can}} & \widehat{\mathcal{P}}_X & \to & 0
\end{array}$$

where the vertical maps are isomorphisms, and $\widehat{\mathcal{P}}_{\mathfrak{K}}$ is simply the quotient of the middle group by the second. We remark/recall that $\oplus\jmath_v$ is defined as follows, see Definition/Remark 1.4: For every v, let $\gamma_v=1\cdot v$ be the unique positive generator of $v\mathfrak{K}=\mathbb{Z}v$. Then there exists a unique generator of the arithmetical inertia $\tau_v\in\mathfrak{T}_v$ such that $\jmath_v(\gamma_v)\big(\tau_v\big)=1$. Hence, the commutativity of the diagram above follows directly from the definition of the homomorphisms.

3) From this we deduce: $\widehat{\mathrm{Div}}_{\mathcal{G}_X}=\widehat{\mathrm{Div}}(X)$ and $\widehat{\mathfrak{Cl}}_{\mathcal{G}_X}=\widehat{\mathfrak{Cl}}(X)$.

4) Finally, we remark that one has a canonical exact sequence of the form $1 \to \widehat{U}(X) \to \widehat{U}_{\mathcal{G}_X} \to \mathcal{T}_{\ell,X} \to 0$, where $U(X)$ is the group of global unit sections on X. Thus this gives a complete description of $\widehat{U}_{\mathcal{G}_X}$.

We next want to discuss the issue of a concrete pre-Galois formation being δ-residually complete curve like, respectively ample up to level δ. Let us say that a quasi-projective normal variety $X \to \mathfrak{k}$ is almost complete if the group of global units is $U(X) = \mathfrak{k}^\times$. Remark that projective normal varieties are almost complete in the sense above.

Proposition 4.11. *Let $\mathfrak{k}$ be a perfect base field, and $\mathfrak{K}|\mathfrak{k}$ a function field. Let $\mathcal{G}_X$ be a concrete Galois formation as introduced/defined in Proposition above. Consider some $\delta \le \mathrm{td}(\mathfrak{K}|\mathfrak{k})$, and let us view $\mathcal{G}_X$ as an abstract pre-Galois formation of level δ. We denote by $\mathcal{G}_{X_\mathbf{v}}$ the residual pre-Galois formations of $\mathcal{G}_X$ with $\mathbf{v}$ multi-indices of length $\le \delta$. In particular, $X_\mathbf{v} \to \mathfrak{k}$ are quasi-projective normal varieties. Then one has:*

(1) $\mathcal{G}_X$ is δ-residually complete curve like if (and only if) $\delta = \mathrm{td}(\mathfrak{K}|\mathfrak{k})-1$, and all the δ-residual varieties $X_\mathbf{v}$ are complete normal curves.

(2) Suppose that either $\mathfrak{k}$ is a hilbertian field, or that $\delta < \mathrm{td}(\mathfrak{K}|\mathfrak{k})$. Further suppose that $X_\mathbf{v}$ is almost complete (all $\mathbf{v}$ as above). Then $\mathcal{G}_X$ is ample up to level δ.

(3) Let $\jmath_\mathfrak{K} : \mathfrak{K}^\times \to \widehat{\mathfrak{K}}$ be the completion homomorphism, and inside $\widehat{\mathfrak{K}}$ set $\mathfrak{K}_{(\ell)} := \jmath_\mathfrak{K}(\mathfrak{K}^\times) \otimes \mathbb{Z}_{(\ell)}$. Suppose that $\mathcal{G}_X$ is δ-residually curve like, and ample up to level δ, thus $\delta = \mathrm{td}(\mathfrak{K}|\mathfrak{k}) - 1$. Then $\mathcal{L}_{X,(\ell)} := \widehat{U}(X) \cdot \mathfrak{K}_{(\ell)}$ is an abstract field formation defining $\mathcal{G}_X$, called the concrete field formation *defining $\mathcal{G}_X$.*

Proof. To (1) and (3): Clear. It is nevertheless more/quite difficult to prove the "only if" part of (1), which we will not directly use, thus omit the proof here...

To (2): In the notations from subsection 4.2, let $\Delta \subset \widehat{\mathfrak{K}}$ be a co-finite rank submodule such that $\Delta \cap \widehat{U}_{\mathcal{G}_X} = 0$. Then by definition, the set of all $\mathfrak{v} \in \mathfrak{D}_X$ such that $\Delta \subset \widehat{U}_v$ has a finite complement in $\mathfrak{D}_X$. Thus there exists an open subset $X' \subset X$ such that for all $\mathfrak{v} \in \mathfrak{D}_{X'}$ one has: $\Delta \subset \widehat{U}_v$. In other words, after replacing X by X', it is sufficient to prove that there exist infinitely many $v \in \mathfrak{D}_X$ such that $\Delta = \ker(\jmath^{\mathcal{G}_X})$ is mapped isomorphically into $\widehat{\mathfrak{K}v}$ by the reduction map $\jmath_v : \widehat{U}_v \to \widehat{\mathfrak{K}v}$. This now immediately follows from the Lemma below. $\square$

We begin with the following little preparation. For the beginning, let $\mathfrak{K}|\mathfrak{k}$ be any field extension, and $\ell \ne \mathrm{char}\,(\mathfrak{k})$ a fixed rational prime number. We will say that a finitely generated subgroup $\Delta \subset \mathfrak{K}^\times$, say of rational rank r, is $\mathfrak{k}, \ell$*-pure in* $\mathfrak{K}$, if $\Delta \left(\mathrm{mod}\ \mathfrak{k}^\times \cdot \mathfrak{K}^{\times\ell}\right)$ has the same rank as Δ. Equivalently, setting $\mathfrak{L} = \mathfrak{K}[\Delta^{\frac{1}{\ell}}]$ one has: Δ is $\mathfrak{k}, \ell$-pure in $\mathfrak{K} \Leftrightarrow [\mathfrak{L} : \mathfrak{K}] = \ell^r$ and $\mathfrak{L}$ is linearly disjoint to the compositum $\mathfrak{K}\,\mathfrak{k}'$, where $\mathfrak{k}'$ is the Galois extension of $\mathfrak{k}$ obtained by adjoining to $\mathfrak{k}$ all the ℓ^∞ roots from all the elements of $\mathfrak{k}$.

Now consider the context of the Proposition. One has an exact sequence of the form $1 \to \mathfrak{k}^\times \to U(X) \to \mathcal{H}_{U(X)} \to 0$, where $\mathcal{H}_{U(X)} \subset \mathfrak{K}^\times/\mathfrak{k}^\times$ is a finite free $\mathbb{Z}$-module, say of rank r. Let Δ be a complement of $\mathfrak{k}^\times$ in $U(X)$. Then Δ is a finite free $\mathbb{Z}$-module of rank r. Setting $\mathfrak{L} = \mathfrak{K}[\Delta^{\frac{1}{\ell}}]$, one has: First, since $\mathfrak{K}^\times/\mathfrak{k}^\times$ is torsion free, $[\mathfrak{L} : \mathfrak{K}] = \ell^r$. Second, since $\mathfrak{K}|\mathfrak{k}$ is a regular field extension, $\mathfrak{L}$ is $\mathfrak{K}$-linearly disjoint to $\mathfrak{K}\,\mathfrak{k}^{\mathrm{s}}$. In particular, Δ is $\mathfrak{k}, \ell$-pure in $\mathfrak{K}$.

Lemma 4.12. *Under the hypothesis of the Proposition above, there exist infinitely many v such that $U(X) \subset U_v$ consists of non-principal v-units, and moreover, Δv is $\mathfrak{k}, \ell$-pure in $\mathfrak{K}v$. In particular, $\widehat{U}(X)$ is mapped isomorphically into $\widehat{\mathfrak{K}v}$ by the reduction map $\jmath_v$.*

Proof. First remark that by induction on $\mathrm{td}(\mathfrak{K}|\mathfrak{k})$, there exist "many" subfield extensions $\kappa|\mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$ such that $\mathfrak{K}|\kappa$ is a regular function field in one variable, and $\Delta \cap \kappa = \{1\}$ (WHY?). Thus $\kappa|\mathfrak{k}$ is a function field. We remark that under the hypothesis of point (2) of the Proposition, κ is hilbertian. Let $\tilde{\kappa} \subset \kappa' \subset \mathfrak{K}'$ denote the Galois extension of κ obtained by adjoining to κ all the ℓ^∞ roots from all the elements of κ. Since κ is Hilbertian, it follows by Kuyk's Theorem, see e.g. [We], Satz 9.8, that $\kappa[\mu_{\ell^\infty}]$ is hilbertian too. Therefore, by loc.cit., the same holds for $\tilde{\kappa}$, as it is an Abelian extension of $\kappa[\mu_{\ell^\infty}]$.

By the remarks above, $\mathfrak{L} = \mathfrak{K}[\Delta^{\frac{1}{\ell}}]$ is linearly disjoint to $\mathfrak{K}\tilde{\kappa}$ over $\mathfrak{K}$. To simplify notations, set $\tilde{\mathfrak{L}} = \mathfrak{L}\tilde{\kappa}$ and $\tilde{\mathfrak{K}} = \mathfrak{K}\tilde{\kappa}$. Then $[\tilde{\mathfrak{L}} : \tilde{\mathfrak{K}}] = [\mathfrak{L} : \mathfrak{K}]$. Since $\tilde{\kappa}$ is hilbertian, there exist infinitely many places $\tilde{v}$ of $\tilde{\mathfrak{L}}|\tilde{\kappa}$ which are inert in the function field extension $\tilde{\mathfrak{L}}|\tilde{\mathfrak{K}}$.[3] In particular, there exist such places $\tilde{v}$ that Δ consists of $\tilde{v}$-units.

Therefore, for such places $\tilde{v}$ we have: $\tilde{\mathfrak{L}}\tilde{v} \,|\, \tilde{\mathfrak{K}}\tilde{v}$ is a separable extension of the same degree as $\tilde{\mathfrak{L}}|\tilde{\mathfrak{K}}$. On the other hand, $\tilde{\mathfrak{L}}\tilde{v} = \tilde{\mathfrak{K}}\tilde{v}\,[\,(\Delta\tilde{v})^{\frac{1}{\ell}}]$. Therefore, $\mathfrak{L}\tilde{v}$ is linearly disjoint to $\tilde{\mathfrak{K}}\tilde{v}$ over $\mathfrak{K}\tilde{v}$. On the other hand we have: First, $\mathfrak{L}\tilde{v} = \mathfrak{K}\tilde{v}\,[\,(\Delta\tilde{v})^{\frac{1}{\ell}}]$, and second, $\tilde{\kappa} \subset \tilde{\mathfrak{K}}\tilde{v}$. From this we deduce that $\Delta\tilde{v}$ is κ, ℓ-pure in $\mathfrak{K}\tilde{v}$. In particular, it follows that Δ does not contain non-trivial principal units, and second, that $\Delta\tilde{v}$ is $\mathfrak{k}, \ell$-pure in $\mathfrak{K}\tilde{v}$. Finally, the $\tilde{v}$-reduction homomorphism $\jmath_{\tilde{v}} : U_{\tilde{v}} \to \mathfrak{K}\tilde{v}$ maps $U(X)$ injectively into $\mathfrak{K}\tilde{v}$.

In order to finish the proof of the Lemma, it is sufficient to remark that almost all places $\tilde{v}$ of $\mathfrak{K}|\kappa$ are actually defined by Weil prime divisors $\mathfrak{v}$ of X. In order to do this, we remark that the field extension $\kappa \hookrightarrow \mathfrak{K}$ id defined by some rational dominant rational map $\psi : X \dashrightarrow Y$, where Y is some model

[3] This is nothing but an equivalent way to say that $\tilde{\kappa}$ is a hilbertian field.

$Y \to \mathfrak{k}$ of the filed extension $\kappa|\mathfrak{k}$. Since $\mathfrak{K}|\kappa$ is a function field of one variable, it follows that the general fiber of $\psi : X \dashrightarrow Y$ has dimension 1. Thus making a base change of ψ to κ (via the canonical morphism $\operatorname{Spec}\kappa \hookrightarrow Y$), it follows that $X_\kappa \to \kappa$ is a geometrically integral normal model of $\mathfrak{K}|\kappa$. Thus X_κ is an open subset in its completion. Thus almost all places of $\mathfrak{K}|\kappa$ are defined by closed points of X_κ, thus by Zariski prime divisors $\mathfrak{v}$ of X. (Precisely, by Zariski prime divisors corresponding to points $x_1 \in X$ of co-dimension 1, such that $\psi(x_1)$ is not the generic point of Y.) $\square$

E) *The case of a finite base field*

Let K be a finitely generated field with $d = \operatorname{td}(K) > 0$. Let k be the constant field of K. Further let $G^\ell_K = \operatorname{Gal}(K^\ell|K)$ be as usually defined. The aim of this section is to show that the system of all concrete pre-Galois formations $\mathcal{G}_X$ are actually group theoretically encoded in G^ℓ_K. Moreover, the extra information concerning such a pre-Galois formation, e.g., δ-residually complete curve like, and/or ample up to level δ, is also encoded in the Galois theory.

1) First, by Section 2, A) and C), the projection $pr : G^\ell_K \to G_k$ and the cyclotomic character $\chi : G^\ell_K \to \mathbb{Z}_\ell^\times$ are group theoretically encoded in G^ℓ_K.

2) Second, by Section 1, b), the set of all Zariski prime divisors $\mathfrak{D}_{K^\ell}$ of K^ℓ is in bijection with the divisorial subgroups Z_v of G^ℓ_K. And $T_v \subset Z_v$ is the unique pro-ℓ cyclic normal subgroup of Z_v. Thus in particular, $G^\ell_{Kv} = Z_v/T_v$ is encoded in Z_v.

3) Moreover, by subsection C) of Section 3, the geometric sets of Zariski prime divisors are encoded in G^ℓ_K.

From 1)—3) above we deduce that the pre-Galois formations $\mathcal{G}_X$ based on K are encoded in G^ℓ_K.

4) Further, using Proposition 4.13, the fact that $\mathcal{G}_X$ is complete curve like is equivalent to the follows two facts: First, $\operatorname{cd} G^\ell_K = 2$, thus $\operatorname{td}(K) = 1$. And second, $(Z_v)_v$ is the set of <u>all</u> divisorial subgroups of G^ℓ_K.

In the same way, if $d = \operatorname{td}(K) > 1$, then combining Proposition 4.13 with the discussion above, we see that the fact that some $\mathcal{G}_X$ is *δ-residually complete curve like* is encoded in the Galois information carried by $\mathcal{G}_X$: One must have $\delta = \operatorname{td}(K) - 1$, and all the $(\delta - 1)$-residual varieties $X_{\mathbf{v}}$ must be complete curves.

5) Finally, the questions whether some given $\mathcal{G}_X$ is *ample up to level* δ follows immediately using Proposition 4.13: Recall first the discussion preceding Proposition 4.13 above. By point 4) of loc.cit., $\widehat{U}_{\mathcal{G}_X} \cong \widehat{U}(X)$. Thus X is almost complete if and only if $\widehat{U}_{\mathcal{G}_X}$ equals $\widehat{k}$, or equivalently, it is finite. The same is true correspondingly for the residual varieties $X_{\mathbf{v}}$. Thus applying Proposition

4.13, 2), we have: Suppose that all residual varieties $X_{\mathbf{v}}$ are almost complete (all multi-indices $\mathbf{v}$ of length $\leq \delta$). Then $\mathcal{G}_X$ is ample up to level δ if and only if $\delta < \mathrm{td}(K)$.

6) Nevertheless, the questions about the existence of *canonically defined abstract fields formations* $\mathcal{L}_{(\ell)}$ defining $\mathcal{G}_X$ is more subtle... Recalling the notation $K_{(\ell)} = \jmath_K(K^\times) \otimes \mathbb{Z}_{(\ell)}$ inside $\widehat{K}$, we show the following:

THEOREM 4.13. *In the above context, let $\mathcal{G}_X$ be δ-residually complete curve like, and let all residual varieties $X_{\mathbf{v}}$ be almost complete. Thus in particular, $\delta = \mathrm{td}(K) - 1$, and $\mathcal{G}_X$ is ample up to level δ. Then the "concrete" field formation $\mathcal{L}_{K,(\ell)} := K_{(\ell)} \otimes \mathbb{Z}_{(\ell)}$ defining $\mathcal{G}_X$ is —up to ℓ-adic equivalence— the unique abstract field formation defining $\mathcal{G}_X$.*

Proof. Indeed, let us consider the exact sequence defined at 2) of loc.cit. above. Since $\mathcal{T}_{\ell,X} = 0$, we get

$$(*) \qquad 0 \to \widehat{\mathcal{H}}(X) \longrightarrow \widehat{\mathrm{Div}}(X) \longrightarrow \widehat{\mathfrak{Cl}}\,(X) \to 0\,.$$

Case $\delta = 1$, i.e., X is a complete curve, and $\mathcal{G}_X$ is complete curve like. Then the above exact sequence has a quite well understood shape, and everything is completely encoded in G^ℓ_K. Indeed, the concrete pre-Galois formation $\mathcal{G}_X$ is encoded in G^ℓ_K. Further, let us consider the "geometric part" $\overline{\mathcal{G}}_X$ of $\mathcal{G}_X$. Then $\overline{G} = G^\ell_{K\overline{k}}$. Choosing canonical generators of inertia $\tau^{\mathrm{ab}}_{\overline{v}}$ (all $\overline{v}$) as indicated at Definition/Remark 1.4, it is well known that the resulting $\mathfrak{T} = (\tau^{\mathrm{ab}}_{\overline{v}})_{\overline{v}}$ is a distinguished system of inertia generators. Consider the commutative diagram mentioned in Remark 4.10, which in our context looks like the following:

$$\begin{array}{ccccccccc}
1 \to & \widehat{k} & \to & \widehat{K} & \to & \widehat{\mathrm{Div}}(X) & \to & \widehat{\mathfrak{Cl}}\,(X) & \to 0 \\
 & \big\downarrow{\scriptstyle \jmath_k} & & \big\downarrow{\scriptstyle \jmath_K} & & \big\downarrow{\scriptstyle \oplus \jmath_v} & & \big\downarrow{\scriptstyle \mathrm{can}} & \\
1 \to & \mathrm{H}^1\big(G_k, \mathbb{Z}_\ell(1)\big) & \to & \mathrm{H}^1\big(G^\ell_K, \mathbb{Z}_\ell(1)\big) & \xrightarrow{\jmath^{\mathcal{G}_X}} & \widehat{\oplus_v}\, \mathrm{H}^1\big(T_v, \mathbb{Z}_\ell(1)\big) & \to & \widehat{\mathcal{P}}_X & \to 0
\end{array}$$

It then immediately follows that the canonical lattice $\mathcal{L}_{\mathfrak{T},(\ell)}$ as considered in Fact 4.5 is exactly the (pre-image under $\jmath_K$ of the) pre-image of $\mathcal{H}(X) \otimes \mathbb{Z}_{(\ell)}$ in $\widehat{K}$, where $\mathcal{H}(X)$ is the group of principal divisors on X. In other words,

$$\mathcal{L}_{\mathfrak{T},(\ell)} = K_{(\ell)} := \jmath_K(K^\times) \otimes \mathbb{Z}_{(\ell)} \quad \text{inside} \quad \widehat{K} = \widehat{K}_{\mathcal{G}_X}\,.$$

General case $\mathrm{td}(K) > 0$: By induction on td, suppose that we know that $K_{v,(\ell)}$ is —up to ℓ-adic equivalence— the unique abstract field formation defining $\mathcal{G}_{X_v}$ (all v). By Proposition 4.6, we have only to show that the $\widehat{k}$-lattice $K_{(\ell)}$ satisfies the conditions i), ii), from loc.cit.. Thus consider any finite co-rank submodule Δ of $\widehat{K}$, say containing $\widehat{k}$. Then $\Delta/\widehat{k}$ is a finite free $\mathcal{Z}_\ell$-module, say of rank r. From the exact sequence $(*)$ above, it follows that

$\Delta \cap K_{(\ell)}$ is a $\widehat{k}$-lattice in Δ. Further, by Proposition 4.13 it follows that there exists "many" v such that $\Delta \subset \widehat{U}_v$, and the reduction map $\jmath_v : \widehat{U}_v \to \widehat{Kv}$ is injective on Δ.

For any such v we have: $\Delta \cap \jmath(K^\times)$ is mapped by $\jmath_v$ injectively into $Kv^\times$. Further, $\jmath_v(\Delta)$ is the ℓ-adic closure of $\jmath_v\big(\Delta \cap \jmath(K^\times)\big)$ in $\widehat{Kv}$. Therefore, $\jmath_v(\Delta)$ has finite co-rank in $\widehat{Kv}$, and further,

$$\jmath_v(\Delta \cap K_{(\ell)}) \subseteq \jmath_v(\Delta) \cap K_{v,(\ell)}\,.$$

Both sides of the above inclusion are finite $\mathbb{Z}_{(\ell)}$-modules, and they have equal ℓ-adic completions, namely equal to $\jmath_v(\Delta)$. Therefore the above inclusion is actually an equality. Thus the conditions i), ii), from Fact 4.6 are verified. $\square$

5. From abstract field formations to concrete ones

A) *Morphisms of pre-Galois formations*

Let $\Gamma_{0,\mathcal{G}}$ and $\Gamma_{0,\mathcal{H}}$ be given profinite groups endowed with "pro-ℓ cyclotomic characters" $\chi_{\Gamma_{0,\mathcal{G}}} : \Gamma_{0,\mathcal{G}} \to \mathbb{Z}_\ell^\times$ and $\chi_{\Gamma_{0,\mathcal{H}}} : \Gamma_{0,\mathcal{H}} \to \mathbb{Z}_\ell^\times$. Let further $\Phi_0 : \Gamma_{0,\mathcal{G}} \to \Gamma_{0,\mathcal{H}}$ be an open homomorphism which is compatible with the cyclotomic characters.

Definition. Let $\mathcal{G} = \big(G, pr_{\mathcal{G}}, (Z_v)_v\big)$ and $\mathcal{H} = \big(H, pr_{\mathcal{H}}, (Z_w)_w\big)$ be given abstract pre-Galois formations of some levels $\delta_{\mathcal{G}}$ and $\delta_{\mathcal{H}}$. Let δ be an integer satisfying $0 \le \delta \le \delta_{\mathcal{G}}, \delta_{\mathcal{H}}$.

1) A *level* $\delta = 0$ *morphism* $\Phi : \mathcal{G} \to \mathcal{H}$ is any open group Φ_0-morphism $\Phi : G \to H$, i.e., satisfying $\Phi_0 \circ pr_{\mathcal{G}} = pr_{\mathcal{H}} \circ \Phi$.

2) A *level* $\delta > 0$ *morphism* $\Phi : \mathcal{G} \to \mathcal{H}$ is defined to be any open group Γ-homomorphism satisfying further conditions as follows:

i) For almost all w, there exits v such that $\Phi(T_v) \neq \{1\}$ and $\Phi(T_v) \subseteq T_w$, and if $w \neq w_*$ is non-trivial, the set of conjugacy classes of v with this property is finite. Further, for almost all conjugacy classes of v with $\Phi(T_v) \neq \{1\}$, there exists w such that $\Phi(T_v) \subseteq T_w$.

In the context above, if $\Phi(T_v) \neq \{1\}$ and $\Phi(T_v) \subseteq T_w$, we will say that w *corresponds to* v. In particular, if $w \neq w_*$ corresponds to v, then $v \neq v_*$ is non-trivial. And w_* corresponds to all v such that $\Phi(T_v) = 1$. And to almost all conjugacy classes of valuations v there do correspond (conjugacy classes of) valuations w. We remark that if w corresponds to v, then Φ induces a group homomorphism

$$\Phi_v : G_v \to H_w$$

which is compatible with the projection homomorphisms $pr_v : G_v \to \Gamma_{\mathcal{G}}$ and $pr_w : H_w \to \Gamma_{\mathcal{H}}$. This means $\Phi_0 \circ pr_v = pr_w \circ \Phi_v$.

ii) If w corresponds to v, then inductively on δ and on $\delta_{\mathcal{G}}$, the residual group homomorphism $\Phi_v : G_v \to H_w$ gives rise to: A level $(\delta - 1)$ morphism of pre-Galois formations $\mathcal{G}_v \to \mathcal{H}_w$, if w is non-trivial, respectively, a level δ morphism $\mathcal{G}_v \to \mathcal{H}$, if w is trivial (all w, and v).

Remarks 5.1. In the above context, let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level δ morphism of pre-Galois formations.

1) Then Φ gives rise to a *Kummer homomorphism*

$$\hat{\varphi} : \widehat{\mathcal{L}}_{\mathcal{H}} = \mathrm{H}^1\big(H, \mathcal{Z}_\ell(1)\big) \to \mathrm{H}^1\big(G, \mathcal{Z}_\ell(1)\big) = \widehat{\mathcal{L}}_{\mathcal{G}}\,.$$

which is compatible with the embeddings $\widehat{\mathcal{L}}_{\mathcal{H}} \hookleftarrow \widehat{\kappa}_{\mathcal{H}} \xrightarrow{\text{can}} \widehat{\kappa}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}}$.

From now on *suppose that* $\delta > 0$.

2) Let $G_{\mathbf{v}}$ and $H_{\mathbf{w}}$ be the δ-residual groups of $\mathcal{G}$, respectively $\mathcal{H}$, see Remark 4.1, especially 3). Then by induction on δ, it is clear how to define the fact that $\mathbf{w}$ *corresponds to* $\mathbf{v}$. Using the elaboration from loc.cit., one has the following: $\mathbf{w}$ *corresponds to* $\mathbf{v}$ *if and only if* $\Phi(T_{\mathbf{v}}) \subseteq T_{\mathbf{w}}$.

3) In particular, let $\mathbf{w}$ correspond to $\mathbf{v}$, and suppose that $\mathbf{w}$ is a multi-index of length $\delta_{\mathbf{w}}$. Then $\Phi : G_{\mathbf{v}} \to H_{\mathbf{w}}$ gives rise to a residual level $(\delta - \delta_{\mathbf{w}})$ morphism of abstract pre-Galois formations $\Phi_{\mathbf{v}} : \mathcal{G}_{\mathbf{v}} \to \mathcal{H}_{\mathbf{w}}$. Thus $\Phi_{\mathbf{v}}$ also gives rise to a Kummer $\mathbf{v}$-homomorphism

$$\hat{\varphi}_{\mathbf{v}} : \widehat{\mathcal{L}}_{\mathbf{w}} = \mathrm{H}^1\big(H_{\mathbf{w}}, \mathcal{Z}_\ell(1)\big) \to \mathrm{H}^1\big(G_{\mathbf{v}}, \mathcal{Z}_\ell(1)\big) = \widehat{\mathcal{L}}_{\mathbf{v}}$$

4) For every finite co-rank module $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{H}}$, and let $\hat{\varphi}\,(\Delta) \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ be its image by the Kummer homomorphism. Then by condition i) above, it follows that $\hat{\varphi}\,(\Delta)$ has finite co-rank. Thus in the notations from Definition/Remark 4.2, 5), it follows that $\hat{\varphi}$ maps $\mathcal{L}_{\mathcal{H},\mathrm{fin}}$ into $\mathcal{L}_{\mathcal{G},\mathrm{fin}}$.

Definition/Remark 5.2. Let $\Phi : \mathcal{G} \to \mathcal{H}$ be a level δ morphism of pre-Galois formations. We will say that:

1) Φ is *proper,* if first, each w corresponds to some v, and vice-versa, to each v there is some w corresponding to it (which might be trivial). Second, if residually $\mathbf{w}$ corresponds to $\mathbf{v}$, then the residual morphism $\Phi_{\mathbf{v}} : \mathcal{G}_{\mathbf{v}} \to \mathcal{G}_{\mathbf{w}}$ is —inductively— a proper one.

2) Φ defines $\mathcal{H}$ as a *level* δ *quotient* of $\mathcal{G}$, or that $\mathcal{H}$ is a *level* δ *quotient* of $\mathcal{G}$ via Φ, if Φ is proper, and both $\Phi(G) = H$ and $\Phi(T_{\mathcal{G}}) = T_{\mathcal{H}}$.

In particular, if $\mathcal{H}$ is a quotient of $\mathcal{G}$, then we can identify $\Gamma_{\mathcal{H}}$ with the corresponding quotient of $\Gamma_{\mathcal{G}}$.

We remark the following: Let $\Phi : \mathcal{G} \to \mathcal{H}$ be a quotient of $\mathcal{G}$. Then in the notations from Remark/Definition 4.2, 2), Φ gives rise to a commutative diagram as follows:

$$\begin{array}{ccccccccc} 1 \to & T_{\mathcal{G}} & \to & G & \to & \pi_{1,\mathcal{G}} & \to 0 \\ & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow & \\ 1 \to & T_{\mathcal{H}} & \to & H & \to & \pi_{1,\mathcal{H}} & \to 0 \end{array}$$

where the vertical homomorphisms are all induced by Φ and in particular, surjective. Thus taking ℓ-adic cohomology, and using the remarks from loc.cit., we obtain a commutative diagram of the form:

$$(*) \qquad \begin{array}{ccccccccc} 0 \to & \widehat{U}_{\mathcal{H}} & \to & \widehat{\mathcal{L}}_{\mathcal{H}} & \to & \widehat{\mathrm{Div}}_{\mathcal{H}} & \to & \widehat{\mathfrak{Cl}}_{\mathcal{H}} & \to 0 \\ & \Big\downarrow{\scriptstyle \hat{\varphi}} & & \Big\downarrow{\scriptstyle \hat{\varphi}} & & \Big\downarrow & & \Big\downarrow & \\ 0 \to & \widehat{U}_{\mathcal{G}} & \to & \widehat{\mathcal{L}}_{\mathcal{G}} & \to & \widehat{\mathrm{Div}}_{\mathcal{G}} & \to & \widehat{\mathfrak{Cl}}_{\mathcal{G}} & \to 0 \end{array}$$

Here the vertical homomorphisms are induced by the Kummer homomorphism. The first three ones are inclusions, and the last one is the canonically resulting one.

3) For later use, we also remark the following: Suppose that $\Phi(T_{\mathcal{G}}) = T_{\mathcal{H}}$. Then $\mathrm{Hom}\big(T_{\mathcal{H}}, \mathcal{Z}_\ell(1)\big)$ is pure in $\mathrm{Hom}\big(T_{\mathcal{G}}, \mathcal{Z}_\ell(1)\big)$, i.e., the torsion of quotient of the latter group by the former one is trivial. Taking Γ-invariants, from this we finally deduce that $\widehat{U}_{\mathcal{G}} \cdot \hat{\varphi}\,(\widehat{\mathcal{L}}_{\mathcal{H}})$ is pure in $\widehat{\mathcal{L}}_{\mathcal{G}}$. This the case e.g., if $\mathcal{H}$ is a quotient of $\mathcal{G}$ via Φ.

Defnition 5.3. Let $\mathcal{G}$ and $\mathcal{H}$ be level δ abstract Galois formations. Let further $\Phi : \mathcal{G} \to \mathcal{H}$ be a level δ morphism of abstract pre-Galois formations, and $\hat{\varphi} : \widehat{\mathcal{L}}_{\mathcal{H}} \to \widehat{\mathcal{L}}_{\mathcal{G}}$ be its Kummer homomorphism. We say Φ is a *level δ morphism of abstract Galois formations,* if for a proper choice of abstract field formations $\mathcal{L}_{\mathcal{G},(\ell)}$ and $\mathcal{L}_{\mathcal{H},(\ell)}$ defining $\mathcal{G}$, respectively $\mathcal{H}$, we have $\hat{\varphi}\,(\mathcal{L}_{\mathcal{H},(\ell)}) \subseteq \mathcal{L}_{\mathcal{G},(\ell)} \cdot \hat{\varphi}\,(\widehat{U}_{\mathcal{H}})$.

B) *From abstract field formations to field formations*

We will give a procedure by which —under certain hypothesis— we can identify in a canonical way "true $\widehat{U}_{\mathcal{G}}$-lattices" inside any abstract field formation defining a given Galois formation $\mathcal{G}$. In the special case of finitely generated fields $\mathfrak{K}$ of positive characteristic, such true lattices will be ℓ-adically equivalent to the image $\jmath_{\mathfrak{K}}\,(\mathfrak{K}^\times)$ of the completion homomorphism $\jmath_{\mathfrak{K}} : \mathfrak{K}^\times \to \widehat{\mathfrak{K}}$.

To begin with, let $\mathcal{G}$ be a level one Galois formation, thus in particular, a complete curve like pre-Galois formation. Recall the notations from Construction 4.4, Case $\delta = 0$. In the notations from there, let $\mathfrak{T}$ be a distinguished system of inertia generators of $T^{\mathrm{ab}}_{\mathcal{G}}$, and $\mathfrak{B}$ the dual basis to $\mathfrak{T}$. Then we defined $\mathcal{D}_{\mathfrak{T}}$ to be the $\mathbb{Z}_{(\ell)}$-submodule of $\widehat{\mathrm{Div}}_{\mathcal{G}}$ generated by $\mathfrak{B}$. And further, $\mathcal{L}_{\mathfrak{T},(\ell)}$ to be the pre-image of $\mathcal{D}_{\mathfrak{T},(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ (and called it the abstract $\mathfrak{T}$-field defining $\mathcal{G}$).

Now let $\mathrm{Div}_{\mathfrak{T}}$ be the $\mathbb{Z}$-submodule of $\widehat{\mathrm{Div}}_{\mathcal{G}}$ generated by the ℓ-adic basis $\mathfrak{B}$ dual to $\mathfrak{T}$. And further, set $\mathcal{L}_{\mathfrak{T}}$ for its pre-image in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Then $\mathcal{L}_{\mathfrak{T}}$ is a *true* $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$. And correspondingly, we can define the abstract divisor class group $\mathfrak{Cl}_{\mathcal{L}_{\mathfrak{T}}}$ of $\mathcal{L}_{\mathfrak{T}}$. Thus we have a canonical exact sequence of the from:

$$0 \to \widehat{U}_{\mathcal{G}} \hookrightarrow \mathcal{L}_{\mathfrak{T}} \xrightarrow{\jmath^{\mathcal{G}}} \mathrm{Div}_{\mathfrak{T}} \xrightarrow{\mathrm{can}} \mathfrak{Cl}_{\mathcal{L}_{\mathfrak{T}}} \to 0\,.$$

Definition 5.4. In the notations from above we define:

1) A level one Galois formation $\mathcal{G}_i$ is called *projective line like,* if the following hold: First, $\widehat{U}_{\mathcal{G}_i}$ consists of the "constants" $\widehat{\kappa}_{\mathcal{G}_i}$ of $\widehat{\mathcal{L}}_{\mathcal{G}_i}$ only. Second, there is a distinguished system of inertia generators $\mathfrak{T}_i$ such that $\mathfrak{Cl}_{\mathcal{L}_{\mathfrak{T}_i}} \cong \mathbb{Z}$. We will call $\mathcal{L}_{\mathfrak{T}_i}$ the *$\mathfrak{T}_i$-field formation defining $\mathcal{G}_i$.*

2) Let $\mathcal{G}$ be a level $\delta > 0$ Galois formation. A 1-*dimensional projection* of $\mathcal{G}$ is any level 1 morphism $\Phi_i : \mathcal{G} \to \mathcal{G}_i$ into a level 1 Galois formation $\mathcal{G}_i$ satisfying $\Gamma_{\mathcal{G}} = \Gamma_{\mathcal{G}_i}$, thus also $\widehat{\kappa}_{\mathcal{G}} = \widehat{\kappa}_{\mathcal{G}_i}$, and further conditions as follows:

i) Suppose $\delta = 1$, i.e., $\mathcal{G}$ is complete curve like. Then Φ is proper, and there exist distinguished systems of inertia generators $\mathfrak{T} = (\tau^{\mathrm{ab}}_{\overline{v}})_{\overline{v}}$ and $\mathfrak{T}_i = (\tau^{\mathrm{ab}}_{\overline{v}_i})_{\overline{v}_i}$ of $\mathcal{G}$, respectively $\mathcal{G}_i$, such that if $\overline{v}_i$ corresponds to $\overline{v}$, then $\tau^{\mathrm{ab}}_{\overline{v}_i}$ is a integer multiple of $\Phi(\tau^{\mathrm{ab}}_{\overline{v}_i})$ (all $\overline{v}_i$).

ii) Suppose that $\delta > 1$. Then there exists v such that $\Phi_i(T_v) = \{1\}$. And for all such v, the resulting residual morphism $\Phi_{i,v} : \mathcal{G}_v \to \mathcal{G}_i$ is —inductively— a 1-dimensional projection.

3) A 1-dimensional projection $\Phi : \mathcal{G} \to \mathcal{G}_i$ is called a *rational projection* if $\mathcal{G}_i$ is projective line like.

4) One defines correspondingly 1-*dimensional quotients,* respectively *rational quotients.*

Remarks 5.5. In notations as above we have:

1) Let $\mathcal{G}_i$ be a projective line like Galois formation, $\mathfrak{T}_i$ one of its distinguished systems of inertia generators. Let $\widehat{\kappa}_i$ be "constants" of $\mathcal{G}_i$, and $\mathcal{L}_i = \mathcal{L}_{\mathfrak{T}_i}$ the $\mathfrak{T}_i$-field formation defining $\mathcal{G}_i$. Let $\mathfrak{T}' = \epsilon^{-1} \cdot \mathfrak{T}_i$ with $\epsilon \in \mathcal{Z}_\ell^\times$ be a further distinguished system of inertia generators of T_i. Then in corresponding notations we have a commutative diagram of the from

$$\begin{array}{ccccccccc}
0 \to & \widehat{U}_{\mathcal{G}_i} & \to & \mathcal{L}_i & \to & \mathrm{Div}_{\mathcal{L}_i} & \to & \mathfrak{Cl}_{\mathcal{L}_i} & \to 0 \\
 & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow & \\
0 \to & \widehat{U}_{\mathcal{G}_i} & \to & \mathcal{L}'_i & \to & \mathrm{Div}_{\mathcal{L}'_i} & \to & \mathfrak{Cl}_{\mathcal{L}'_i} & \to 0
\end{array}$$

where the vertical maps are the multiplication by ϵ. Since $\mathfrak{Cl}_{\mathcal{L}_i} \cong \mathbb{Z}$, it follows that all the maps are actually isomorphisms. Thus the fact that a Galois formation $\mathcal{G}_i$ is projective line like does not depend on the specific distinguished system of inertia generators $\mathfrak{T}_i$.

a) In particular, every abstract field formation $\mathcal{L}_{i,(\ell)}$ defining $\mathcal{G}_i$ contains a field formation $\mathcal{L}_i$ defining $\mathcal{G}_i$. And all these field formations are ℓ-adic equivalent $\widehat{\kappa}_i$-latices in $\widehat{\mathcal{L}}_{\mathcal{G}_i}$.

b) Finally, let $\mathcal{L}_i = \mathcal{L}_{\mathfrak{T}_i}$ be a given a field formation defining $\mathcal{G}_i$. An element $x_i \in \mathcal{L}_i$ is called a *generating* element if $\jmath^{\mathcal{G}_i}(x_i) = \varphi_{\overline{v}'_i} - \varphi_{\overline{v}_i}$ for some valuations $\overline{v}'_i \neq \overline{v}_i$ of $\overline{\mathcal{G}}$. We denote $(x_i)_\infty = \overline{v}$ and call it the pole of x_i. If x_i is a generating element as above, we will denote

$$\mathcal{P}_{x_i} = \{\, x'_i \in \mathcal{L}_i \mid x'_i \text{ generating, and } (x'_i)_\infty = (x_i)_\infty \,\}$$

We further set $\mathcal{P}_i = \cup_{x_i} \mathcal{P}_{x_i}$ the set of all generating elements from $\mathcal{L}_i$.

2) Let $\mathcal{G}$ be a level $\delta \geq 1$ Galois formation, and $\mathcal{L}_{(\ell)}$ an abstract field formation defining $\mathcal{G}$. Let $\Phi_i : \mathcal{G} \to \mathcal{G}_i$ be a rational projection of $\mathcal{G}$, and $\hat{\varphi}_i : \widehat{\mathcal{L}}_i \to \widehat{\mathcal{L}}_{\mathcal{G}}$ the corresponding Kummer homomorphism. Then by induction on δ, one shows that there exists a unique (properly chosen) abstract field formation $\mathcal{L}_{i,(\ell)}$ defining $\mathcal{G}_i$ such that $\hat{\varphi}_i(\mathcal{L}_{i,(\ell)}) \subseteq \mathcal{L}_{(\ell)}$. Further one has:

a) Let $\mathcal{L}_{i,(\ell)}$ be the *unique abstract field formation* defining $\mathcal{G}_i$ such that $\hat{\varphi}(\mathcal{L}_{i,(\ell)}) \subset \mathcal{L}_{(\ell)}$. Then $\hat{\varphi}(\mathcal{L}_i) \subset \mathcal{L}_{(\ell)}$ for any field formation $\mathcal{L}_i$ which is contained in $\mathcal{L}_{i,(\ell)}$.

b) Further, by Remark 5.2, 1), and 2), $\hat{\varphi}_i(\widehat{\mathcal{L}}_i) \cap \widehat{U}_{\mathcal{G}} = \hat{\varphi}_i(\widehat{\kappa}_i) = \widehat{\kappa}_{\mathcal{G}}$, and $\widehat{U}_{\mathcal{G}} \cdot \hat{\varphi}_i(\widehat{\mathcal{L}}_i)$ is pure in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Thus $\hat{\varphi}_i(\mathcal{L}_{i,(\ell)}) \cap \widehat{U}_{\mathcal{G}} = \widehat{\kappa}_{\mathcal{G}}$, and further, $\widehat{U}_{\mathcal{G}} \cdot \hat{\varphi}(\mathcal{L}_{i,(\ell)})$ is pure in $\mathcal{L}_{(\ell)}$.

Definition 5.6. In notations as above, let the level δ Galois formation $\mathcal{G}$ be endowed with a family of *rational quotients* $\Phi_i : \mathcal{G} \to \mathcal{G}_i$. Let $\hat{\varphi}_i : \widehat{\mathcal{L}}_i \to \widehat{\mathcal{L}}_{\mathcal{G}}$ be the corresponding Kummer homomorphism (all i), and let $\mathcal{L}_{i,(\ell)}$ be the unique abstract field formation defining $\mathcal{G}_i$ such that $\hat{\varphi}_i(\mathcal{L}_{i,(\ell)}) \subseteq \mathcal{L}_{(\ell)}$.

1) For every v, let I_v be the set of all i_v such that $\Phi_{i_v}(T_v) = \{1\}$, if $\delta = 2$; respectively such that the resulting rational projection $\Phi_{i_v} : \mathcal{G}_v \to \mathcal{G}_{i_v}$ is a rational quotient, if $\delta > 2$. We will say that $(\Phi_i)_i$ is an *ample* family of rational quotients of $\mathcal{G}$ if the following hold:

i) $\widehat{U}_{\mathcal{G}}$ together with the $\hat{\varphi}_i(\mathcal{L}_{i,(\ell)})$ (all i) generate $\mathcal{L}_{(\ell)}$.

ii) If $\delta > 2$, then —by induction on the level— Φ_{i_v} (all $i_v \in I_v$) is an ample family of rational quotients of $\mathcal{G}_v$.

2) In the above context, suppose that $(\Phi_i)_i$ is an ample family of rational quotients of $\mathcal{G}$. Let a *field formation* $\mathcal{L}_i \subset \mathcal{L}_{i,(\ell)}$ defining $\mathcal{G}_i$ be given (all i). Let $\mathcal{L}$ be the $\mathbb{Z}$-submodule of $\mathcal{L}_{(\ell)}$ generated by the $\hat{\varphi}_i(\mathcal{L}_i)$ (all i). We will say that $\mathcal{G}$ endowed with Φ_i (all i) is a *Galois formation,* respectively that $\mathcal{L}$ endowed with the $\mathcal{L}_i$ (all i) is a *field formation* defining $\mathcal{G}$ if one has:

i) $\mathcal{L} \cap \widehat{U}_{\mathcal{G}}$ is trivial, and $\hat{\varphi}_i(\mathcal{L}_i)$ is pure in $\mathcal{L}$ (all i).

ii) $\mathcal{L} \cdot \widehat{U}_{\mathcal{G}}$ is a *true* $\widehat{U}_{\mathcal{G}}$*-lattice* in $\mathcal{L}_{(\ell)}$.

Definition/Remark 5.7. Let $\{\mathcal{G},(\Phi)_i\}$ and $\{\mathcal{H},(\Phi_j)_j\}$ be *Galois formations.* Further let $\Phi:\mathcal{G}\to\mathcal{H}$ be a level δ morphism of abstract Galois formations, and $\hat{\varphi}:\widehat{\mathcal{L}}_{\mathcal{H}}\to\widehat{\mathcal{L}}_{\mathcal{G}}$ the Kummer morphism. Then by definition, there exit abstract field formations $\mathcal{L}_{\mathcal{G},(\ell)}\subset\widehat{\mathcal{L}}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{H},(\ell)}\subset\widehat{\mathcal{L}}_{\mathcal{H}}$ such that $\hat{\varphi}\big(\mathcal{L}_{\mathcal{H},(\ell)}\big)\subset\mathcal{L}_{\mathcal{G},(\ell)}$.

Then Φ is called a *morphism of Galois formations* if for every j there exists an i and an isomorphism $\Phi_{ij}:\mathcal{G}_i\to\mathcal{G}_j$ such that $\Phi_j\circ\Phi=\Phi_{ij}\circ\Phi_i$.

In the above notations, $\hat{\varphi}_i$, and $\hat{\varphi}_j$ denote the Kummer morphisms defined by Φ_i, and Φ_j, respectively. Let $\mathcal{L}_{\mathcal{G}}$ endowed with $(\mathcal{L}_i)_i$, and $\mathcal{L}_{\mathcal{H}}$ endowed with $(\mathcal{L}_j)_j$ be field formations defining $\mathcal{G}$, respectively $\mathcal{H}$. Let i and j be such that $\Phi_j\circ\Phi=\Phi_{ij}\circ\Phi_i$. Then one has:

1) Let $\mathcal{L}_{j,(\ell)}$ be the unique abstract field formation defining $\mathcal{H}_j$ such that $\hat{\varphi}_j(\mathcal{L}_{j,(\ell)})\subseteq\mathcal{L}_{\mathcal{H},(\ell)}$. Then $\mathcal{L}_{j,(\ell)}$ is also the unique abstract field formation defining $\mathcal{H}_j$ such that $\hat{\varphi}\circ\hat{\varphi}_j\,(\mathcal{L}_{j,(\ell)})\subseteq\mathcal{L}_{\mathcal{G},(\ell)}$.

2) Moreover, after multiplication by a rational ℓ-adic unit $\epsilon\in\mathbb{Z}_{(\ell)}^{\times}$, we have $\hat{\varphi}\circ\hat{\varphi}_j(\mathcal{L}_j)=\hat{\varphi}_i(\mathcal{L}_i)$. Nevertheless, in order to have a "good" theory, *we cannot assume* that this equality holds simultaneously for all j and i as above. This is a fine point of the whole story, as we will see bellow...

C) *Example: Concrete morphisms of concrete Galois formations*

We consider notations as in subsection D) of the the previous Section. Let $\mathfrak{k}$ and $\mathfrak{l}$ be perfect base fields of characteristic $\neq\ell$. Let $\imath:\mathfrak{L}|\mathfrak{l}\hookrightarrow\mathfrak{K}|\mathfrak{k}$ be an embedding of function fields such that the relative algebraic closure of $\mathfrak{l}$ in $\mathfrak{k}$ is finite. Further let $\mathfrak{K}'|\mathfrak{K}$ and $\mathfrak{L}'|\mathfrak{L}$ be field extensions as at the beginning of subsection D) of Section 4), and consider concrete pre-Galois formations $\mathcal{G}_X$ and $\mathcal{G}_Y$ as introduced/defined in Proposition/Definition 4.13. Suppose that $\imath$ has a prolongation to an embedding of fields

$$\imath':\mathfrak{L}'\hookrightarrow\mathfrak{K}'$$

Thus in particular, $\overline{\mathfrak{l}}|\mathfrak{l}\hookrightarrow\overline{\mathfrak{k}}|\mathfrak{k}$ via $\imath'$. Therefore, $\imath$ gives rise to a canonical projection of Galois groups

$$\Phi_0:G_{\mathfrak{k}}\to G_{\mathfrak{l}},\quad \Phi:G'_{\mathfrak{K}}\to G'_{\mathfrak{L}}$$

which are compatible with the projections $G'_{\mathfrak{K}}\to G_{\mathfrak{k}}$ and $G'_{\mathfrak{L}}\to G_{\mathfrak{l}}$, and with the cyclotomic characters. Moreover, by the hypothesis on $\imath$, the homomorphism Φ is open. Thus Φ defines a *level $\delta=0$ morphism of pre-Galois formations $\mathcal{G}_X\to\mathcal{G}_Y$*. Moreover, the Kummer homomorphism

$$\hat{\varphi}:\widehat{\mathfrak{L}}=\mathrm{H}^1\big(G'_{\mathfrak{L}},\mathbb{Z}_\ell(1)\big)\to\mathrm{H}^1\big(G'_{\mathfrak{K}},\mathbb{Z}_\ell(1)\big)=\widehat{\mathfrak{K}}$$

defined by Φ is nothing but the ℓ-adic completion of $\imath$.

Consider concrete Galois formations $\mathcal{G}_X$ and $\mathcal{G}_Y$ based on $\mathfrak{K}|\mathfrak{k}$, respectively $\mathfrak{L}|\mathfrak{l}$, of levels $\mathrm{td}(\mathfrak{K}|\mathfrak{k})$, respectively $\mathrm{td}(\mathfrak{L}|\mathfrak{l})$. Let $0 \leq \delta \leq \mathrm{td}(\mathfrak{L}|\mathfrak{k})$ be given. Thus $\mathcal{G}_X$ and $\mathcal{G}_Y$ in a canonical way level δ (concrete) pre-Galois formations. We claim that Φ gives rise canonically to a level δ morphism of abstract pre-Galois formations $\Phi : \mathcal{G}_X \to \mathcal{G}_Y$.

Indeed, the $\mathfrak{k}|\mathfrak{l}$-embedding of function fields $\imath : \mathfrak{L} \to \mathfrak{K}$ is defined by some rational dominant map $f : X \dashrightarrow Y$ which maps $\mathfrak{l}$ into $\mathfrak{k}$. We have: f is defined at almost all points x_1 of co-dimension 1 of X; its value $f(x_1)$ at almost all such points is a point of co-dimension 1 in Y; and almost all points y_1 of co-dimension 1 of Y are images of such points via f. Moreover, for any such y_1 there exist only finitely many pre-images x_1 of co-dimension 1. In birational terms this means the following: For every Zariski prime divisor $\mathfrak{v}$ of $\mathfrak{K}|\mathfrak{k}$, let $\mathfrak{w} := \mathfrak{v}|_{\mathfrak{L}}$ be its restriction to $\mathfrak{L}$. Thus $\mathfrak{w}$ is either trivial, or it is a Zariski prime divisor of $\mathfrak{L}|\mathfrak{k}$. The geometric arguments explained above show the following: There are only finitely many $\mathfrak{v} \in \mathfrak{D}_X$ such that the corresponding $\mathfrak{w}$ is non-trivial <u>and</u> does not lie in $\mathfrak{D}_Y$. Second, almost all $\mathfrak{w} \in \mathfrak{D}_Y$ are restrictions of some $\mathfrak{v} \in \mathfrak{D}_X$.

Using general valuation theory, this translates into the following facts concerning $\mathcal{G}_X$ and $\mathcal{G}_Y$: Let v be a prolongation of $\mathfrak{v} \in \mathfrak{D}_X$ to $\mathfrak{K}'$. Define $T_v \subset Z_v$ correspondingly. Then one has: Let $w = v|_{\mathfrak{L}'}$ be the restriction of v to $\mathfrak{L}'$. (N.B., w might be trivial!) Then $\Phi(Z_v) \subset Z_w$, $\Phi(T_v) \subset T_w$ are open subgroups. In particular, the residual homomorphism

$$\Phi_v : G_v = G'_{\mathfrak{K}v} \to G'_{\mathfrak{L}w} = G_w$$

has an open image. Further, this homomorphism is defined by the canonical embedding of function field extensions over $\mathfrak{k}|\mathfrak{l}$ coming from valuation theory:

$$\mathfrak{L}'w \mid \mathfrak{L}w \hookrightarrow \mathfrak{K}'v \mid \mathfrak{K}v\,.$$

Hence, proceeding by induction we obtain the following:

Proposition/Definition 5.8.

(1) In the above notations, $\imath : \mathfrak{L}' \hookrightarrow \mathfrak{K}'$ gives rise in a canonical way to a level δ morphism $\Phi : \mathcal{G}_X \to \mathcal{G}_Y$ of pre-Galois formations, for all δ such that $0 \leq \delta \leq \mathrm{td}(\mathfrak{L}|\mathfrak{k})$. The Kummer homomorphism $\hat{\varphi} : \widehat{\mathfrak{L}} \to \widehat{\mathfrak{K}}$ is the completion of the embedding $\imath : \mathfrak{L} \hookrightarrow \mathfrak{K}$, thus it maps $\mathfrak{L}_{(\ell)}$ isomorphically into $\mathfrak{K}_{(\ell)}$.

If concrete pre-Galois formations $\mathcal{G}_X$ and $\mathcal{G}_Y$ as above are given, a morphism of pre-Galois formations $\Phi : \mathcal{G}_X \to \mathcal{G}_Y$ is called a concrete morphism*, if the Kummer homomorphism*

$$\hat{\varphi} : \widehat{\mathfrak{L}} = \mathrm{H}^1\big(G'_{\mathfrak{L}}, \mathbb{Z}_\ell(1)\big) \to \mathrm{H}^1\big(G'_{\mathfrak{K}}, \mathbb{Z}_\ell(1)\big) = \widehat{\mathfrak{K}}$$

defined by Φ is —up to multiplication by an ℓ-adic unit— the ℓ-adic completion of some embedding of function fields $\imath : \mathfrak{L} \mid \mathfrak{l} \hookrightarrow \mathfrak{K} \mid \mathfrak{k}$.

(2) For a given concrete morphism $\Phi : \mathcal{G}_X \to \mathcal{G}_Y$ as above, and its Kummer homomorphism $\hat{\varphi}$, the embedding of function fields $\imath : \mathfrak{L} \mid \mathfrak{l} \hookrightarrow \mathfrak{K} \mid \mathfrak{k}$ defining $\hat{\varphi}$ is unique.

Proof. The first assertion (1) follows immediately from the discussion preceding the Proposition. Let us prove (2). Let $\imath', \imath''$ be both embeddings as above defining $\hat{\varphi}$. For every $y \in \mathfrak{L}^\times$, set $y' = \imath'(y)$, and $y' = \imath'(y)$. Since both $\imath', \imath''$ define $\hat{\varphi}$, it follows that $\jmath_{\mathfrak{K}}(y') = \jmath_{\mathfrak{K}}(y'')$. Since $\ker(\jmath_{\mathfrak{K}})$ is contained in $\mathfrak{k}^\times$, it follows that there exists $a_y \in \mathfrak{k}^\times$ such that $y'' = a_y y'$. Now consider some $y \in \mathfrak{L}$ which is transcendental over $\mathfrak{l}$, thus its image y' is transcendental over $\mathfrak{k}$. Then we have:

$$a_y y' + 1 = y'' + 1 = \imath''(y + 1) = a_{y+1}\imath'(y + 1) = a_{y+1}(y' + 1)$$

or equivalently, $(a_y - a_{y+1})y' = a_{y+1} - 1$. By the hypothesis on y we deduce: $a_y = 1 = a_{y+1}$. Therefore, $y' = y''$, and so $\imath(y') = \imath''(y)$.

Finally, since the set of all transcendental elements of $\mathfrak{L}|\mathfrak{l}$ generates $\mathfrak{L}$, it follows that $\imath' = \imath''$. □

We now turn our attention to rational projections of a concrete Galois formation $\mathcal{G}_X$ as above. Let $\mathfrak{x} \in \mathfrak{K}$ be an arbitrary non-constant function, and let $\mathfrak{K}_\mathfrak{x}$ be the relative algebraic closure of $\mathfrak{k}(\mathfrak{x})$ in $\mathfrak{K}$. Then $\mathfrak{K}_\mathfrak{x}|\mathfrak{k}$ is a function field in one variable. We endow $\mathfrak{K}_\mathfrak{x}|\mathfrak{k}$ with its unique projective normal model $X_\mathfrak{x} \to \mathfrak{k}$, and consider the pre-Galois formation $\mathcal{G}_{X_\mathfrak{x}}$ defined by the algebraic extension $\mathfrak{K}'_\mathfrak{x}|\mathfrak{K}_\mathfrak{x}$, where $\mathfrak{K}'_\mathfrak{x}$ is the relative algebraic closure of $\mathfrak{K}_\mathfrak{x}$ in $\mathfrak{K}'$.

By the facts explained above, it follows that the canonical inclusion of field extensions $\imath_\mathfrak{x} : \mathfrak{K}'_\mathfrak{x}|\mathfrak{K}_\mathfrak{x} \hookrightarrow \mathfrak{K}'|\mathfrak{K}$ gives rise to a morphism $\Phi_\mathfrak{x} : \mathcal{G}_X \to \mathcal{G}_{X_\mathfrak{x}}$. Clearly, $\Phi_\mathfrak{x} : G'_\mathfrak{K} \to G'_{\mathfrak{K}_\mathfrak{x}}$ is surjective (by construction), but though, Φ viewed as morphism of concrete Galois formations $\mathcal{G}_X \to \mathcal{G}_{X_\mathfrak{x}}$ need not necessarily define $\mathcal{G}_{X_\mathfrak{x}}$ as a quotient of $\mathcal{G}_X$. Nevertheless, since X is normal, the rational map $f_\mathfrak{x} : X \dashrightarrow X_\mathfrak{x}$ defining $\imath$ is defined at all points of co-dimension 1. This means that for every $\mathfrak{v} \in \mathfrak{D}_X$, if $\mathfrak{v}$ is not trivial on $\mathfrak{K}_\mathfrak{x}$, then $f_\mathfrak{x}(\mathfrak{v})$ is a closed point $\mathfrak{w} \in \mathfrak{D}_{X_\mathfrak{x}}$. In particular, Φ defines $\mathcal{G}_{X_\mathfrak{x}}$ as a quotient of $\mathcal{G}_X$ if and only if f_i is surjective. This is the case in particular, if X is itself complete (thus projective).

Convention: In order to simplify the discussion, let us assume from now on that $\mathfrak{k}$ is algebraically closed, and that $X \to \mathfrak{k}$ as well as all the residual varieties $X_\mathbf{v} \to \mathfrak{k}$ defining the residual pre-Galois formations $\mathcal{G}_\mathbf{v}$ of X are projective (and normal).

In particular, since $\mathfrak{k}$ is algebraically closed, $\mathfrak{k}^\times$ is ℓ-divisible. Thus the completion homomorphism $\jmath_{\mathfrak{K}} : \mathfrak{K}^\times \to \widehat{\mathfrak{K}}$ has $\ker(\jmath_{\mathfrak{K}}) = \mathfrak{k}^\times$, and $\jmath_{\mathfrak{K}}(\mathfrak{K}^\times)$ is isomorphic to the true lattice of all principal divisors $\mathcal{H}(X)$ inside the free Abelian group $\mathrm{Div}(X)$.

Second, with $\imath_{\mathfrak{x}} : \mathfrak{K}_{\mathfrak{x}} \hookrightarrow \mathfrak{K}$ as defined above, $\Phi_{\mathfrak{x}} : \mathcal{G}_X \to \mathcal{G}_{\mathfrak{x}}$ defines $\mathcal{G}_{\mathfrak{x}}$ as a 1-dimensional projection of $\mathcal{G}$ (all $\mathfrak{x}$).

REMARK 5.9. We say that a function $\mathfrak{x} \in \mathfrak{K}$ is *general,* if $\Phi_{\mathfrak{x}} : \mathcal{G}_X \to \mathcal{G}_{\mathfrak{x}}$ is a rational quotient of $\mathcal{G}_X$. Then following hold:

1) First, a "general" function $\mathfrak{x}$ —in the sense of Bertini— is also general in the sense above. This follows by the fact that for such functions $\mathfrak{x}$, the corresponding $\mathfrak{K}_{\mathfrak{x}}$ is the rational function field in the variable $\mathfrak{x}$, and moreover, the resulting $\Phi_{\mathfrak{x}} : \mathcal{G} \to \mathcal{G}_{\mathfrak{x}}$ is a rational quotient of $\mathcal{G}$. Indeed, this nothing but the following well known fact: Consider the rational map $f_{\mathfrak{x}} : X \dashrightarrow X_{\mathfrak{x}}$, and further let $f_{\mathfrak{x},\overline{\eta}} : X_{\overline{\eta}} \to \overline{\mathfrak{K}}_{\mathfrak{x}}$ be the geometric generic fiber of $f_{\mathfrak{x}}$. Then under the canonical projection $X_{\overline{\eta}} \to X$ the geometric fundamental group of X is a quotient of the fundamental group of $X_{\overline{\eta}}$.

We denote $\mathcal{L}_{\mathfrak{x}} = \jmath_{\mathfrak{x}}(\mathfrak{K}_{\mathfrak{x}})$ the image of $\mathfrak{K}_{\mathfrak{x}}$ in $\widehat{\mathfrak{K}}_{\mathfrak{x}}$ via the completion morphism $\jmath_{\mathfrak{x}}$. We remark that since $\mathfrak{k}$ is algebraically closed, the set of $\mathfrak{x}$-generating elements is $\mathcal{P}_{\mathfrak{x}} = \{\, \mathfrak{x} - a \mid a \in \mathfrak{k} \,\}$ (all general $\mathfrak{x}$).

Now suppose that $\mathrm{td}(\mathfrak{K}\,|\,\mathfrak{k}) > 1$. Then one has:

2) The Kummer homomorphism $\hat{\varphi}_{\mathfrak{x}} : \widehat{\mathfrak{K}}_{\mathfrak{x}} \to \widehat{\mathfrak{K}}$ maps each $\mathcal{L}_{\mathfrak{x}}$ isomorphically into $\mathcal{L} := \jmath_{\mathfrak{K}}(\mathfrak{K}^{\times})$. Further, it is well known that $\widehat{U}_{\mathcal{G}_X}$ together with the $\mathcal{L}_{\mathfrak{x}}$ (all general $\mathfrak{x}$) generate $\mathcal{L}_X = \widehat{U}_{\mathcal{G}_X} \cdot \mathcal{L}$. Then an easy induction argument shows that $\Phi_{\mathfrak{x}}$ (all general $\mathfrak{x}$) is an ample family of rational quotients of $\mathcal{G}_X$.

3) Moreover, a "Bertini type argument" shows that $\mathcal{L}_X$ endowed with the $\mathcal{L}_{\mathfrak{x}}$ (all general $\mathfrak{x}$) is actually a *field formation* defining $\mathcal{G}_X$.

Therefore we have the following:

PROPOSITION/DEFINITION 5.10. *In the above context —including the Convention—, we consider $\mathcal{G}_X$ as a Galois formation of level $\delta = \mathrm{td}(\mathfrak{K}\,|\,\mathfrak{k})$, and endow it with its canonical true $\widehat{U}_{\mathcal{G}_X}$-lattice $\mathcal{L}_X := \widehat{U}_{\mathcal{G}_X} \cdot \mathcal{L}$. For $\Phi_{\mathfrak{x}} : \mathcal{G}_X \to \mathcal{G}_{X_{\mathfrak{x}}}$* (*all general $\mathfrak{x}$*), *let $\mathcal{L}_{\mathfrak{x}} \subset \widehat{\mathfrak{K}}_{\mathfrak{x}}$ be the image of $\mathfrak{K}_{\mathfrak{x}}^{\times}$ under the completion functor. Then we have:*

(*1*) *$\mathcal{G}_X$ endowed with $\Phi_{\mathfrak{x}}$* (*all general $\mathfrak{x}$*) *is a Galois formation, which we call a* concrete Galois formation.

(*2*) *Further, $\mathcal{L}_X$ endowed with the family $\mathcal{L}_{\mathfrak{x}}$* (*all general $\mathfrak{x}$*) *is a field formation defining $\mathcal{G}_X$.*

A field formation of the type above will be called a level $\delta = \mathrm{td}(\mathfrak{K}\,|\,\mathfrak{k})$ concrete field formation.

We now can announce one of the main results concerning the Galois formations:

THEOREM 5.11. *Let $\{\mathcal{G},(\Phi_i)_i\}$ be a level δ Galois formation, and in the usual notations, $\mathcal{L}$ endowed with $(\mathcal{L}_i)_i$ is a field formation defining $\mathcal{G}$. Let further $\{\mathcal{G}_X,(\Phi_{\mathfrak{x}})_{\mathfrak{x}}\}$ be a concrete Galois formation, and $\mathcal{L}_X$ endowed with $(\mathcal{L}_{\mathfrak{x}})_{\mathfrak{x}}$ be a field formation defining $\mathcal{G}_X$. Then one has:*

(1) Up to conjugation by concrete automorphisms of $\{\mathcal{G}_X,(\Phi_{\mathfrak{x}})_{\mathfrak{x}}\}$, there exists at most one isomorphism of Galois formations $\Phi:\{\mathcal{G},(\Phi_{\mathfrak{x}})_{\mathfrak{x}}\}\to\{\mathcal{G},(\Phi_i)_i\}$.

(2) Suppose that $\{\mathcal{G},(\Phi_i)_i\}$ is a concrete pre-Galois formation as well, say of the form $\{\mathcal{G}_Y,(\Phi_{\mathfrak{y}})_{\mathfrak{y}}\}$, with $Y\to\mathfrak{l}$ a projective normal model of some function field $\mathfrak{L}|\mathfrak{l}$ with $\mathfrak{l}$ algebraically closed. Then every isomorphisms Φ as above is a concrete one. Moreover, the function field embedding $\imath:\mathfrak{L}|\mathfrak{l}\hookrightarrow\mathfrak{K}|\mathfrak{k}$ defining Φ maps $\mathfrak{l}$ isomorphically onto $\mathfrak{k}$, and $\mathfrak{K}\,|\,\imath(\mathfrak{L})$ is purely inseparable.

Proof. Since (1) follows from (2), it suffices to prove assertion (2). Thus let $\Phi:\{\mathcal{G},(\Phi_{\mathfrak{x}})_{\mathfrak{x}}\}\to\{\mathcal{G}_Y,(\Phi_{\mathfrak{y}})_{\mathfrak{y}}\}$ be an isomorphism of (concrete) Galois formations.

For the beginning, let us view Φ as an isomorphism of abstract Galois formations. Then, first, the Kummer morphism $\hat\varphi:\widehat{\mathcal{L}}_{\mathcal{G}_Y}\to\widehat{\mathcal{L}}_{\mathcal{G}_X}$ is actually an isomorphism. Second, if $\mathcal{L}'_{X,(\ell)}$ and $\mathcal{L}'_{Y,(\ell)}$ are properly chosen abstract field formations defining $\mathcal{G}_X$, respectively $\mathcal{G}_Y$, we have: $\hat\varphi$ maps $\mathcal{L}'_{Y,(\ell)}$ isomorphically onto $\mathcal{L}'_{X,(\ell)}$. Since $\mathcal{L}'_{X,(\ell)}$, $\mathcal{L}'_{Y,(\ell)}$ are ℓ-adically equivalent to $\mathcal{L}_{X,(\ell)}$, respectively $\mathcal{L}_{Y,(\ell)}$, we get: After multiplication of $\hat\varphi$ by a properly chosen ℓ-adic unit, we can suppose that $\hat\varphi$ maps $\mathcal{L}_{Y,(\ell)}$ isomorphically onto $\mathcal{L}_{X,(\ell)}$. Second, it follows by Remark, 5.7, 2), that $\hat\varphi$ maps $\widehat{U}_{\mathcal{G}_Y}$ isomorphically onto $\widehat{U}_{\mathcal{G}_X}$.

In orde to simplify notations, let us identify $\widehat{\mathfrak{K}}_{\mathfrak{x}}$ and $\widehat{\mathfrak{L}}_{\mathfrak{y}}$ with their images via $\hat\varphi_{\mathfrak{x}}$, respectively $\hat\varphi_{\mathfrak{y}}$. In this notation, via the isomorphism Φ, one gets a bijection $\mathcal{L}_{\mathfrak{y}}\mapsto\mathcal{L}_{\mathfrak{x}}$, defined by the fact that $\widehat{\mathfrak{K}}_{\mathfrak{x}}=\hat\varphi(\widehat{\mathfrak{L}}_{\mathfrak{y}})$.[4]

- *$\hat\varphi$ maps $\mathfrak{L}_{(\ell)}$ isomorphically onto $\mathfrak{K}_{(\ell)}$.*

Indeed, we have $\mathcal{L}_{\mathfrak{x},(\ell)}=\widehat{\mathfrak{K}}_{\mathfrak{x}}\cap\mathfrak{K}_{(\ell)}$, and correspondingly for $\mathcal{L}_{\mathfrak{y},(\ell)}$. Since Φ is an isomorphism, we have $\hat\varphi\big(\mathcal{L}_{\mathfrak{y},(\ell)}\big)=\mathcal{L}_{\mathfrak{x},(\ell)}$, provided $\hat\varphi\,(\widehat{\mathfrak{L}}_{\mathfrak{y}})=\widehat{\mathfrak{K}}_{\mathfrak{x}}$. Since the set of all general $\mathfrak{x}$ multiplicatively generates $\mathfrak{K}^\times$, it follows that the set of all the elements in all the $\mathcal{L}_{\mathfrak{x},(\ell)}$ multiplicatively generates $\mathfrak{K}_{(\ell)}$. Since the same holds correspondingly for $\mathfrak{L}$, we are done.

Next recall that $\mathcal{L}_{\mathfrak{K}}:=\jmath_{\mathfrak{K}}(\mathfrak{K}^\times)$ and $\mathcal{L}_{\mathfrak{L}}:=\jmath_{\mathfrak{L}}(\mathfrak{L}^\times)$ are true latices in $\mathfrak{K}_{(\ell)}$, respectively $\mathfrak{L}_{(\ell)}$. Since $\mathcal{L}_{\mathfrak{K}}$ is a true lattice in the $\mathbb{Z}_{(\ell)}$-module $\mathfrak{K}_{(\ell)}$, it follows that every $x'\in\mathfrak{K}_{(\ell)}$ is of the form $x'=(m/n)x$ with m/n a positive rational number which is also an ℓ-adic unit, and $x\in\mathcal{L}_{\mathfrak{K}}$. For such an x' and the corresponding x we define their ℓ-adic value as being the maximum over all non-negative integers k such that $\ell^{-k'}x'\in\mathfrak{K}_{(\ell)}$, respectively $\ell^{-k}x\in\mathcal{L}_{\mathfrak{K}}$. It is

[4] N.B., this does not mean yet mean that we have a bijection $\mathfrak{y}\mapsto\mathfrak{x}$.

clear that x and x' have the same ℓ-adic value. We will say that x (and x') is *primitive,* if its ℓ-adic value is 0, or equivalently, if $\mathbb{Z}\,x$ is pure in $\mathcal{L}_{\mathfrak{K}}$.

Now consider some $y \in \mathcal{L}_{\mathfrak{L}}$. Then by discussion above, and taking into account that $\hat{\varphi}$ maps $\mathfrak{L}_{(\ell)}$ isomorphically onto $\mathfrak{K}_{(\ell)}$, we have: There is a unique positive rational number which is also an ℓ-adic unit $\epsilon_y = m_y/n_y$, and a unique element $x \in \mathcal{L}_{\mathfrak{K}}$ such that $\hat{\varphi}\,(y) = (m_y/n_y)x$. Moreover y is primitive iff x is primitive. Thus $\hat{\varphi}$ gives rise to a canonical bijection

$$\tilde{\varphi} : \mathcal{L}_{\mathfrak{L}} \to \mathcal{L}_{\mathfrak{K}}, \quad y \mapsto x := \epsilon_y^{-1}\phi(y)\,.$$

mapping the primitive elements of $\mathcal{L}_{\mathfrak{L}}$ bijectively onto those of $\mathcal{L}_{\mathfrak{K}}$, and respecting the ℓ-adic value of the elements. We say that $\hat{\varphi}$ is *y-normed,* if $\epsilon_y = 1$. And if $y = \jmath_{\mathfrak{L}}(\mathfrak{y})$, we will also say that $\hat{\varphi}$ is *$\mathfrak{y}$-normed.*

Now suppose that $y = \jmath_{\mathfrak{L}}(\mathfrak{y})$ with $\mathfrak{y}$ a general element. Then for $x \in \mathcal{L}_{\mathfrak{K}}$ with $\hat{\varphi}\,(y) = (m_y/n_y)x$, choose some $\mathfrak{x} \in \mathfrak{K}$ such that $\jmath_{\mathfrak{K}}(\mathfrak{x}) = x$. Then we see that $\hat{\varphi}$ maps $\mathfrak{K}_{\mathfrak{y},(\ell)}$ into $\mathfrak{K}_{\mathfrak{x},(\ell)}$. Thus $\mathfrak{x}$ is a general element of $\mathfrak{K}$, and we have $\hat{\varphi}\,(\mathcal{L}_{\mathfrak{y},(\ell)}) = \mathcal{L}_{\mathfrak{x},(\ell)}$. Therefore, $\hat{\varphi}\,(\mathcal{L}_{\mathfrak{y}}) = \epsilon_{\mathfrak{y}} \cdot \mathcal{L}_{\mathfrak{x}}$, and $\tilde{\varphi}(\mathcal{L}_{\mathfrak{y}}) = \mathcal{L}_{\mathfrak{x}}$. Thus we have:

FACT 5.12. In the above context, the following assertions are equivalent:

(i) $\hat{\varphi}$ is $\mathfrak{y}$-normed.

(ii) $\hat{\varphi}$ is y'-normed for some $y' \in \mathcal{L}_{\mathfrak{y}}$.

(iii) $\hat{\varphi}\,(\mathcal{L}_{\mathfrak{y}}) = \mathcal{L}_{\mathfrak{x}}$

Naturally, *a priori,* it might be possible that $\hat{\varphi}$ is not normed with respect to any element $y \in \mathcal{L}_{\mathfrak{L}}$. To "artificially" remedy this, we fix some $\mathfrak{y}$, and replace $\hat{\varphi}$ by its $(1/\epsilon_{\mathfrak{y}})$-multiple, which we again denote by $\hat{\varphi}$. Precisely, without loss of generality, we will suppose from the beginning that:

- *$\hat{\varphi}$ is $\mathfrak{y}$-normed for some fixed $\mathfrak{y}$.*

As we will see below, the above apparently "artificial" way to remedy the lack of normation of $\hat{\varphi}$ is actually not at all artificial, but it has a deep arithmetic meaning.

First let us denote $M_{\mathfrak{K}} := \hat{\varphi}\,(\mathcal{L}_{\mathfrak{L}}) \cap \mathcal{L}_{\mathfrak{K}}$, $M_{\mathfrak{L}} := \hat{\varphi}^{-1}(\mathcal{L}_{\mathfrak{K}}) \cap \mathcal{L}_{\mathfrak{L}}$. Thus $M_{\mathfrak{K}} = \hat{\varphi}\,(M_{\mathfrak{L}})$, and for $u \in \mathfrak{L}^{\times}$ and $t \in \mathfrak{K}^{\times}$ such that $\hat{\varphi} \circ \jmath_{\mathfrak{L}}(u) = \jmath_{\mathfrak{K}}(t)$ we have: $\jmath_{\mathfrak{L}}(u) \in M_{\mathfrak{L}}$ if and only if $\jmath_{\mathfrak{K}}(t) \in M_{\mathfrak{K}}$. If u and t are as above, we will say that *u is a pre-image of t* via (the $\mathfrak{y}$-normed morphism) $\hat{\varphi}$.

LEMMA 1. *Consider any $t \in \mathfrak{K}$ and $u \in \mathfrak{L}$ as above. Then we have the following: $\jmath_{\mathfrak{K}}(\mathfrak{k}\,t + 1) \subseteq M_{\mathfrak{K}}$ and $\jmath_{\mathfrak{L}}(\mathfrak{k}\,u + 1) \in M_{\mathfrak{L}}$.*

Proof of Lemma 1. By symmetry, it is sufficient to prove the assertion for t. In the notaions form Fact above we have: Since $\hat{\varphi}$ is $\mathfrak{y}$-normed, it follows by loc.cit. above that $\mathcal{L}_{\mathfrak{y}} \subset M_{\mathfrak{L}}$ and $\mathcal{L}_{\mathfrak{x}} \subset M_{\mathfrak{K}}$. Then we have: If t and $\mathfrak{x}$ are algebraically $\mathfrak{k}$-dependent, then $\mathfrak{K}_t \subseteq \mathfrak{K}_{\mathfrak{x}}$, as $\mathfrak{K}_{\mathfrak{x}}$ is relatively algebraically closed in $\mathfrak{K}$. As $\hat{\varphi}$ is $\mathfrak{y}$-normed, the assertion of Lemma 1 follows from the Fact above. Therefore, we may/will assume that t and $\mathfrak{x}$ are *algebraically independent over* $\mathfrak{k}$. (Thus by symmetry, u and $\mathfrak{y}$ are also algebraically independent over $\mathfrak{k}$.) For almost all $a \in \mathfrak{k}$, and almost all $b = b_a \in \mathfrak{k}^{\times}$, by the "birational Bertini" we have: both $t'_{a,b} := t/(b\,\mathfrak{x} + a)$ and $t''_{a,b} := t/(b\,\mathfrak{x} + a + 1)$ are general elements of $\mathfrak{K}$. In particular, since both $\jmath_{\mathfrak{K}}(t)$ and $\jmath_{\mathfrak{K}}(b\,\mathfrak{x}+a)$ are elements of $M_{\mathfrak{K}}$, it follows that both $\jmath_{\mathfrak{K}}(t'_{a,b})$ and $\jmath_{\mathfrak{K}}(t''_{a,b})$ are in $M_{\mathfrak{K}}$. Since they are general elements, it follows that $\mathcal{L}_{t'_{a,b}}$ and $\mathcal{L}_{t''_{a,b}}$ are contained in $M_{\mathfrak{K}}$.

In the same way, one can see that for a fixed $c \in \mathfrak{k}^{\times}$, and almost all $a, b \in \mathfrak{k}$ as above, the function $t_{a,b,c} = (ct + b\mathfrak{x} + a + 1)/(t + b\mathfrak{x} + a)$ is a general element of $\mathfrak{K}$ too. On the other hand, we have:

$$t_{a,b,c} = \frac{b\,\mathfrak{x} + a + 1}{b\,\mathfrak{x} + a}\,\frac{c\,t''_{a,b} + 1}{t'_{a,b} + 1}\,.$$

From this we finally deduce: The images via $\jmath_{\mathfrak{K}}$ of both the denominators as well as of the numerators of the fractions above are elements of $M_{\mathfrak{K}}$. Thus finally, $\jmath_{\mathfrak{K}}(t_{a,b,c}) \in M_{\mathfrak{K}}$. Therefore, reasoning as above we have: $\mathcal{L}_{t_{a,b,c}}$ is contained in $M_{\mathfrak{K}}$. In particular, $\jmath_{\mathfrak{K}}(t_{a,b,c} - 1) \in M$. On the other hand, $t_{a,b,c} - 1 = [(c-1)t + 1]/(t + b\mathfrak{x} + a)$. Since $\jmath_{\mathfrak{K}}(t + b\mathfrak{x} + a) \in M_{\mathfrak{K}}$, we deduce that $\jmath_{\mathfrak{K}}\big((c-1)t + 1\big) \in M_{\mathfrak{K}}$ for all $c \in \mathfrak{k}^{\times}$. Equivalently, since $t \in M_{\mathfrak{K}}$ already, we get: $\jmath_{\mathfrak{K}}(\mathfrak{k}\,t + 1) \subseteq M_{\mathfrak{K}}$, as claimed. $\square$

LEMMA 2. *The $\mathfrak{y}$-normed morphism $\hat{\varphi}$ respects co-lineations, i.e., let $u_i \in \mathfrak{L}^{\times}$ and pre-images $t_i \in \mathfrak{K}^{\times}$ be given $(i = 1, 2, 3)$. Then $(u_i)_i$ are linearly $\mathfrak{l}$-(in)dependent if and only if $(t_i)_i$ are linearly $\mathfrak{k}$-(in)dependent.*

Proof of Lemma 2. See e.g. BOGOMOLOV [Bo], second Remark, p.51. $\square$

Now let $\mathfrak{K}_0 = \jmath_{\mathfrak{K}}^{-1}(M_{\mathfrak{K}}) \cup \{0\}$ be the pre-image of $M_{\mathfrak{K}}$ in $\mathfrak{K}$ together with 0 added. We claim that $\mathfrak{K}_0 \subset \mathfrak{K}$ *is a subfield.* Indeed, since $M_{\mathfrak{K}}$ is a subgroup of $\mathcal{L}_{\mathfrak{K}}$, its pre-image $\jmath_{\mathfrak{K}}^{-1}(M_{\mathfrak{K}})$ in $\mathfrak{K}^{\times}$ is a subgroup too. We check that $\mathfrak{K}_0$ is closed with respect to addition. Indeed, for $t, t' \in \mathfrak{K}_0$ we have: $t'' = t'/t \in \mathfrak{K}_0$, and further, $t + t' = t(t'' + 1)$. As by Lemma 1 we have $t'' + 1 \in \mathfrak{K}_0$, we finally get $t + t' = t(t'' + 1) \in \mathfrak{K}_0$. Moreover, we remark that $M_{\mathfrak{K}} = \mathcal{P}(\mathfrak{K}_0)$ is by definition exactly the *projectivization of* $(\mathfrak{K}_0, +)$.

Consider/define correspondingly $\mathfrak{L}_0 = \jmath_{\mathfrak{L}}^{-1}(M_{\mathfrak{L}}) \cup \{0\}$ inside $(\mathfrak{L}, +)$. Then by the definition of $\mathfrak{K}_0$ and $\mathfrak{L}_0$, the Kummer isomorphism $\hat{\varphi} : \widehat{\mathfrak{L}} \to \widehat{\mathfrak{K}}$ defines a bijective mapping

$$\varphi := \widehat{\varphi}|_{\mathcal{P}(\mathfrak{L})} : \mathcal{P}(\mathfrak{L}) = M_{\mathfrak{L}} \to M_{\mathfrak{K}} = \mathcal{P}(\mathfrak{K})\,,$$

which, by Lemma 2, respects co-lineations.Therefore, by the *Fundamental theorem of projective geometries,* see e.g. ARTIN, φ is the projectivisation $\varphi = \mathcal{P}(\phi)$ of some linear $\imath_0$-isomorphism $\phi : (\mathfrak{L}_0, +) \to (\mathfrak{K}_0, +)$, i.e., $\imath_0 : \mathfrak{l} \to \mathfrak{k}$ is an isomorphism of fields, and ϕ is a isomorphism of Abelian groups, such that $\phi(au) = \imath_0(a)\phi(u)$ for all $a \in \mathfrak{l}$ and $u \in \mathfrak{L}_0$. Moreover, ϕ is unique up to composition by homotheties of the form $l_a \circ \phi \circ l_b$ (all $a \in \mathfrak{k}$, $b \in \mathfrak{l}$).

Further, as $\mathfrak{k}^\times = \ker(\jmath_{\mathfrak{K}})$ and $\mathfrak{l}^\times = \ker(\jmath_{\mathfrak{L}})$, it follows that $\phi(\mathfrak{l}) = \mathfrak{k}$. We set $\phi_0 := \big(1/\phi(1)\big)\phi$, and claim that ϕ_0 is a field isomorphism. Indeed, for a fixed $y \in \mathfrak{L}_0$, consider $\phi_y : \mathfrak{L}_0 \to \mathfrak{K}_0$ defined by $\phi_y(u) = \phi_0(y\,u)$. Then ϕ_y is a linear $\imath_0$-isomorphism. Set $x = \phi_0(y)$. Then considering projectivisations, and using the fact that $\varphi = \mathcal{P}(\phi_0)$ is multiplicative, it follows that for all $u \in \mathfrak{L}_0$ we have: $\mathcal{P}(\phi_y)(u) = \mathcal{P}(\phi_0)(yu) = \mathcal{P}(l_x) \circ \mathcal{P}(\phi_0)(u)$, where l_x is the multiplication by x on $\mathfrak{K}_0$. Therefore, there exist $a \in \mathfrak{k}^\times$ and $b \in \mathfrak{l}^\times$ such that $l_b \circ \phi_y \circ l_b = l_x \circ \phi_0$. In other words, $a\,\phi_0(y\,b\,u) = x\,\phi_0(u)$ for all $u \in \mathfrak{L}_0$. Setting $u = 1$, and taking into account that $\phi_0(1) = 1$, we have: $a\,\imath_0(b)\,x = x$. Thus $a\,\imath_0(b) = 1$, the effects of l_a and l_b cancel each other. Hence we have

$$\phi_0(y\,u) = x\,\phi_0(u) = \phi_0(y)\,\phi_0(u), \quad (\text{all } u, y \in \mathfrak{L}_0).$$

Therefore, ϕ_0 is a field isomorphism as claimed.

In order to conclude, we have to show that both $\mathfrak{L}|\mathfrak{L}_0$ and $\mathfrak{K}|\mathfrak{K}_0$ are purely inseparable field extensions. By symmetry, it is sufficient to prove this for $\mathfrak{K}|\mathfrak{K}_0$. Let p be the characteristic exponent of $\mathfrak{k}$. For every general element $\mathfrak{x} \in \mathfrak{K}$, consider the smallest integer $m = m_{\mathfrak{x}}$ such that $\mathfrak{x}^m \in \mathfrak{K}_0$. We claim that m is a power of p. Indeed, by Fact 5.12, for all primitive elements $x_0' \in \mathfrak{K}_{\mathfrak{x}}$ one has: m is the smallest positive integer such that ${x_0'}^m \in \mathfrak{K}_0$. This means that $\mathfrak{K}_{0,\mathfrak{x}} := \mathfrak{K}_{\mathfrak{x}} \cap \mathfrak{K}_0$ is a subfield of $\mathfrak{K}_{\mathfrak{x}}$ such that $\mathfrak{K}_{0,\mathfrak{x}}$ is contained in the m^{th} powers of $\mathfrak{K}_{\mathfrak{x}}$. But then m must be a power of the characteristic exponent of $\mathfrak{k}$. Equivalently, $\mathfrak{K}_{\mathfrak{x}}$ is purely inseparable over $\mathfrak{K}_{0,\mathfrak{x}}$. Finally, since the set of all general elements $\mathfrak{x}$ generates the multiplicative group $\mathfrak{K}^\times$, it follows that for every $x \in \mathfrak{K}$ there exists some p-power p^e such that $x^{p^e} \in \mathfrak{K}_0$.

The Theorem 5.11 is proved.

6. Concluding the Proof of the Theorem (Introduction)

We now consider the context from the Theorem (Introduction). Let k be the constant field of K. Recall that the canonical projection $pr_K : G_K^\ell \to G_k$ as well as the Frobenius Frob_k of k are group theoretically encoded in G_K^ℓ. Further, the same is true correspondingly about L. On the other hand, we see that the "recipe" for deducing pr and Frob are actually invariant under isomorphisms, thus under Φ. Therefore we finally have: K and L have the same field of constants, say k, and Φ is a G_k isomorphism, i.e., $\Phi \circ pr_L = pr_K$.

Further, by Theorem 4.13 it follows that the $\mathbb{Z}_{(\ell)}$-lattice $K_{(\ell)}$ is encoded group theoretically in G^ℓ_K. Correspondingly, the same is true concerning L. Again, by the way $K_{(\ell)}$ is encoded in G^ℓ_K, one immediately sees that the Kummer homomorphism

$$\hat\varphi_K : \widehat{L} \to \widehat{K}$$

maps $L_{(\ell)}$ isomorphically onto $K_{(\ell)}$. Moreover, let $K'|K$ be a finite (Galois) extension of K inside K^ℓ, and via the isomorphism Φ the corresponding finite (Galois) extension $L'|L$. Then using the functoriality of Kummer theory, we have a commutative diagram of the form

$$\begin{array}{ccc} \widehat{L} & \xrightarrow{\hat\varphi_K} & \widehat{K} \\ \Big\downarrow \text{can} & & \Big\downarrow \text{can} \\ \widehat{L'} & \xrightarrow{\hat\varphi_{K'}} & \widehat{K'} \end{array}$$

which induces a corresponding one with $(\ )_{(\ell)}$ in stead of $\widehat{(\ )}$. Suppose $K'|K$ is Galois, and $\Phi' : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(L'|L)$ is the isomorphism induced by Φ. Then for all $\sigma \in \mathrm{Gal}(K'|K)$, and $\tau = \Phi(\sigma)$ we have:

$$\sigma \circ \hat\varphi_{K'} = \hat\varphi_{K'} \circ \tau \quad \text{i.e.,} \quad \Phi(\sigma) = \hat\varphi^{-1}_{K'} \circ \sigma \circ \hat\varphi_{K'}\,.$$

Next let us take limits over the *constant extensions* of K, i.e., ones of the form $K' = Kk'$, with $k'|k$ a finite extension of the constant field k of K, we deduce that a similar result as Theorem 4.13 holds also for $\mathfrak{K} = K\overline{k}$, respectively $\mathfrak{L} = L\overline{k}$. Precisely, as Φ is a G_k isomorphism, it maps $G^\ell_{\mathfrak{K}} \subset G^\ell_K$ isomorphically onto $G^\ell_{\mathfrak{L}} \subset G^\ell_L$. Moreover, the resulting Kummer isomorphism $\hat\varphi_{\mathfrak{K}} : \widehat{\mathfrak{L}} \to \widehat{\mathfrak{K}}$ maps $\mathfrak{L}_{(\ell)}$ isomorphically onto $\mathfrak{K}_{(\ell)}$. We should also remark here that $\hat\varphi_{\mathfrak{K}}$ is a G_k-morphism, thus in particular, $\widehat{K}$ as well as $K_{(\ell)}$ are the G_k-invariants of $\widehat{\mathfrak{K}}$, respectively $\mathfrak{K}_{(\ell)}$. The same correspondingly for L.

Proposition 6.1. *In the above context, the 1-dimensional projections $\Phi_x : G^\ell_{\mathfrak{K}} \to G^\ell_{\mathfrak{K}_x}$ of $G^\ell_{\mathfrak{K}}$ are group theoretically encoded in $G^\ell_{\mathfrak{K}}$, in a way which is invariant under isomorphisms.*

Moreover, if $\Phi_{\mathfrak{x}} : G^\ell_{\mathfrak{K}} \to G^\ell_{\mathfrak{K}_{\mathfrak{x}}}$ is a rational projection defined by a general element $\mathfrak{x}$ of $\mathfrak{K}$, then the corresponding projection $\Phi_{\mathfrak{y}} : G^\ell_{\mathfrak{L}} \to G^\ell_{\mathfrak{L}_{\mathfrak{y}}}$ is a rational projection, and $\mathfrak{y}$ is a general element of $\mathfrak{L}$.

Proof. The first assertion is Bogomolov's Lemma 4.2 from [Bo]. On the other hand, it is clear that the criterion from loc.cit. is invariant under isomorphisms. To prove the last assertion, we remark that the fact Φ_x is rational, as well as the fact that Φ_x corresponds to a rational projection defined by a general element, are both assertions completely described by the inertia structure of $G^{\ell}_{\mathfrak{K}_x}$. On the other hand, the inertia elements in $G^{\ell}_{\mathfrak{K}_x}$ are group theoretically encoded in the Galois theory by the Local theory. $\square$

Conclusion:

We finally apply Theorem 5.11, and deduce that there exist finite pure inseparable extensions $\mathfrak{K}_0|\mathfrak{K}$ and $\mathfrak{L}_0|\mathfrak{L}$ such that $\hat{\varphi}_{\mathfrak{K}_0} : \widehat{\mathfrak{L}_0} \to \widehat{\mathfrak{K}_0}$ is the ℓ-adic completion of a unique field isomorphism $\imath_0 : \mathfrak{K}_0 \to \mathfrak{L}_0$. Let K_0 be the the purely inseparable closure of K in $\mathfrak{K}_0$, and define L_0 correspondingly. Then considering G_k-invariants, it follows that $\imath_0$ maps L_0 isomorphically onto N_0, and $\hat{\varphi}_{K_0} : \widehat{L}_0 \to \widehat{K}_0$ is exactly the ℓ-adic completion of $\imath_K$. Now taking limits over all finite extensions $K'|K_0$ inside $K_0^{\ell} = K^{\ell}K_0$, and the corresponding finite extensions $L'|L_0$ inside L_0^{ℓ}, we finally get an isomorphism $\phi : L^{\mathrm{i}^{\ell}} \to K^{\mathrm{i}^{\ell}}$ defining Φ, i.e., $\Phi(g) = \phi^{-1}\, g\, \phi$ for all $g \in G^{\ell}_K$.

We still have to prove that any two automorphisms ϕ' and ϕ'' both defining Φ differ by a power of Frobenius. Indeed, setting $\phi := \phi'' \circ \phi'^{-1}$ we get: ϕ is an automorphism of $L^{\mathrm{i}^{\ell}}$ which maps L^{i} onto itself, and induces the identity on $G^{\ell}_{L^{\mathrm{i}}}$. We claim that such an automorphism ϕ is a power of Frobenius. Indeed, let v be an arbitrary Zariski prime divisor of $L^{\mathrm{i}^{\ell}}$. Then $w := v \circ \phi$ is also a Zariski prime divisor of $L^{\mathrm{i}^{\ell}}$. Moreover, by the usual formalism, we have $Z_w = \phi^{-1} Z_v\, \phi$ inside $G^{\ell}_{L^{\mathrm{i}}}$. Since the conjugation by ϕ is the identity on $G^{\ell}_{L^{\mathrm{i}}}$, it follows that $Z_w = Z_v$. Thus by Proposition 1.6, (1), it follows that v and w are equivalent valuations on $L^{\mathrm{i}^{\ell}}$. In particular, $v(x) > 0$ if and only if $w(x) > 0$ (all $x \in L^{\mathrm{i}^{\ell}}$). Let $\mathrm{char}(k) = p$. We claim that $y := \phi(x)$ is some p-power of x. First, if y and x are algebraically independent, then there exists a Zariski prime divisor v of $L^{\mathrm{i}^{\ell}}$ such that $v(x) = 1$ and $v(y) = 0$. But $v(y) = v \circ \phi(x) = w(x)$, contradiction! Therefore, $y = \phi(x)$ is always algebraic over $\overline{\mathbb{F}}_p(x)$ inside $L^{\mathrm{i}^{\ell}}$. Now let $f(Y) \in \overline{\mathbb{F}}_p(x)[Y]$ be the minimal polynomial of y over $\overline{\mathbb{F}}_p(x)$ of y. We show that $f(Y)$ is totally inseparable. Indeed, for a "general" constant $a \in \overline{\mathbb{F}}_p$, we replace x by $x_a = x + a$, thus y by $y_a = y + \phi(a)$. Then the minimal polynomial of y_a over $\overline{\mathbb{F}}_p(x)$ is $f_a(Y) := f(Y - a)$, and the following holds: The distinct roots $y_a =: y_1, \dots, y_r$ of $f_a(Y)$ have disjoint zero divisors in $\mathfrak{D}^1_{\mathfrak{K}^{\ell}}$. On the other hand, all these zero divisors are zeros of x_a. Further, by the fact that all the zeros of x_a are as well zeros of $y_a = \phi(x_a)$, it follows that $r = 1$, i.e., $f_a(Y)$ is purely inseparable. Thus $f(Y)$ is purely inseparable too. Thus finally we have: $\phi(x) = x^{p^e}$, where $e = e_x$ depending on x. We next show that $e = e_x$ does not depend on x. Indeed, since ϕ is a field automorphism, we have for all $x, y \in L^{\mathrm{i}^{\ell}}$ the following:

$$x^{e_{x+y}} + y^{e_{x+y}} = (x+y)^{e_{x+y}} = \phi(x+y) = \phi(x) + \phi(y) = x^e + y^{e_y}\,.$$

Considering an arbitrary $y \in L^{\mathrm{i}^{\ell}}$ which is algebraically independent over $\overline{\mathbb{F}}_p(x)$, we therefore must have $e_{x+y} = e_y = e$. Now choosing a transcendence basis $\mathcal{T}$

of L over $\mathbb{F}_p$ (say, which contains x), we see that the restriction of ϕ to $\overline{\mathbb{F}}_p(\mathcal{T})$ is Frob^e. Therefore, $\phi = \text{Frob}^e$ on $L^{\mathrm{i}\ell}$.

The Theorem (Introduction) is proved.

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