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Pro-ℓ birational anabelian geometry over algebraically closed fields I

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Introduction

Let ℓ be a fixed rational prime number. For every field K of positive characteristic $\neq \ell$ containing the ℓ^{th} roots of unity, let $K^\ell|K$ be a maximal pro-ℓ Galois extension, and $G_K^\ell = \text{Gal}(K^\ell|K)$ its Galois group.

The aim of this paper is to prove a *geometric pro-ℓ version* of Grothendieck's birational anabelian conjecture over algebraic closures of finite fields as follows:

Theorem. *Let $\mathcal{F}$ be the category of all function fields $K|k$, with* $\text{td}(K|k) > 1$ *and k an algebraic closure of some finite field. Then there exists a group theoretic recipe by which we can recover every field $K \in \mathcal{F}$ from G_K^ℓ, up to a pure inseparable extensions. This recipe is invariant under profinite group isomorphisms. In particular, if K and L are in $\mathcal{F}$, there exists a canonical bijection*

$$\text{Isom}^{\text{i}}\,(L^{\text{i}}, K^{\text{i}}) \longrightarrow \text{Out}\,(G_K^\ell, G_L^\ell)\,,$$

where $(\ )^{\text{i}}$ *denotes pure inseparable closure, and* Isom^{i} *means up to Frobenius twists, and* Out *denotes outer isomorphisms of profinite groups.*

Equivalently, if $\Phi : G_K^\ell \to G_L^\ell$ is an isomorphism of profinite groups, then up to a Frobenius twist, there exists a unique field isomorphism $\phi : L^{\text{i}\,\ell} \to K^{\text{i}\,\ell}$ such that $\Phi(g) = \phi^{-1}\, g\, \phi$ for all $g \in G_K^\ell$. In particular, $\phi(L^{\text{i}}) = K^{\text{i}}$.

Remarks.

I first want to mention that this manuscript is motivated by the *program initiated by* Bogomolov [Bo]. And the result above completes that program in the case of function fields over algebraic closures of finite fields. At least at the level of rough ideas, the paper [Bo] was quite inspiring for me... and

* Ideas used in this work originate in part from my visit to the IAS Princeton in 1994 and 1996. I would like to thank Pierre Deligne for several fruitful discussions and suggestions, see also [P2], [P3]. The main technical details of the present manuscript were solved during my visit to the IAS Princeton in 2002. During this last visit I was partially supported by the NSF grant DMS 9729992.

at a first glance there is a lot of similarity of what we are doing here with loc.cit.. Without going into details, we hope nevertheless that the advised reader will realize the essential differences. Compare also with Bogomolov–Tschinkel [B–T2].

1) The Theorem above implies corresponding "full Galois assertions", i.e., the corresponding assertions for the full Galois group G_K in stead of the quotient G^ℓ_K.

Second, the corresponding questions concerning the Galois characterisation of finitely generated fields K with $\mathrm{td}(K) > 1$ in positive characteristic, see [P2], [P3], [P4] can be reduced to the Main Theorem above. Thus the above result *generalises* in a non-trivial way the celebrated results by Neukirch, Ikeda, Iwasawa, Uchida concerning the Galois characterisation global fields.

2) The above result *cannot* be true in the case $\mathrm{td}(K|k) = 1$, as in this case G^ℓ_K is a pro-ℓ free group on countably many generators, thus its structure does not depend on K at all.

Further, because of the reason just mentioned above, one cannot expect a *Hom-form* type result in the above context, as proved by Mochizuki over sub-p-adic fields. Indeed, for any given function field $K|k$ there exist "many" surjective homomorphisms of profinite groups $G_K \to G_{k(t)}$, given say by field k-embeddings $k(t) \hookrightarrow K$ such that $k(t)$ is relatively closed in K; and further, since $G_{k(t)}$ is profinite free, see Harbater [Ha] and Pop [Pop], there are "many" surjective projections $G_{k(t)} \to G_K$. Thus finally, there are many surjective group homomorphisms $G_K \to G_K$ which do not arise in a geometric way...

3) Finally, we remark that in stead of working with the full pro-ℓ Galois group G^ℓ_K, one could work as well with truncations of it, e.g., with ones coming from the central series. This is doable, but the resulting assertions are quite technical, and at the moment maybe too complicated in order to be interesting...

Rough idea of Proof (Comp. with Bogomolov [Bo])

Let $\mathfrak{K}|\mathfrak{k}$ be an extension of fields of characteristic $\neq \ell$, and suppose that $\mathfrak{k}$ is algebraically closed. We consider algebraic extensions $\mathfrak{K}'|\mathfrak{K}$ with the following properties:

i) $\mathfrak{K}'|\mathfrak{K}$ is a pro-ℓ Galois extension.

ii) There exists a Galois sub-extension $\mathfrak{K}_1$ of $\mathfrak{K}'|\mathfrak{K}$ such that $\mathfrak{K}'|\mathfrak{K}_1$ is a maximal pro-ℓ Abelian extension of $\mathfrak{K}_1$.

We remark that the extension $K^\ell|K$ as introduced before the Main Theorem as well as $K^{\ell,\mathrm{ab}}|K$ satisfies the conditions i), ii) above.

To fix notations, let $G'_{\mathfrak{K}} = \mathrm{Gal}(\mathfrak{K}'|\mathfrak{K})$, and $\mathbb{T}_\ell \xrightarrow{\imath} \mathbb{Z}_\ell$ a fixed identification as $G'_{\mathfrak{K}}$-modules. By Kummer Theory there is a functorial isomorphism

$$(*) \qquad \widehat{\delta}_{\mathfrak{K}} : \widehat{\mathfrak{K}} \longrightarrow \mathrm{H}^1\Big(\mathfrak{K}, \mathbb{T}_\ell\Big) \xrightarrow{\imath} \mathrm{Hom}\big(G'_{\mathfrak{K}}, \mathbb{Z}_\ell\big)\,,$$

where $\widehat{\mathfrak{K}}$ denotes the ℓ-adic completion of the multiplicative group $\mathfrak{K}^\times$ of $\mathfrak{K}$. Since $\mathfrak{k}^\times$ is divisible, and $\mathfrak{K}^\times/\mathfrak{k}^\times$ is a free Abelian group, the ℓ-adic completion homomorphism defines an exact sequence

$$1 \to \mathfrak{k}^\times \hookrightarrow \mathfrak{K}^\times \xrightarrow{\jmath_{\mathfrak{K}}} \widehat{\mathfrak{K}}\,.$$

Denote $\mathcal{P}(\mathfrak{K}) = \mathfrak{K}^\times/\mathfrak{k}^\times$ inside $\widehat{\mathfrak{K}}$, and view $\mathcal{P}(\mathfrak{K})$ as the projectivization of the (infinite) dimensional $\mathfrak{k}$-vector space $(\mathfrak{K}, +)$. **Suppose** that we have a Galois theoretic recipe in order to detect: First the *image of* $\mathcal{P}(\mathfrak{K})$ inside the Galois theoretically "known" $\widehat{\mathfrak{K}}$. Second, the *projective lines in* $\mathcal{P}(\mathfrak{K})$. Remark that the multiplication l_x by any non-zero element $x \in \mathfrak{K}$ defines an "automorphisms" of $\mathcal{P}(\mathfrak{K})$ which respects "co-lineations". This automorphism is noting but the translation $l_{\jmath_{\mathfrak{K}}(x)}$ in $\mathcal{P}(\mathfrak{K})$, this time viewed again as the multiplicative group $\mathcal{P}(\mathfrak{K}) = \mathfrak{K}^\times/\mathfrak{k}^\times$. Now **using** the *Fundamental Theorem of projective geometry*, see e.g. Artin [A], it follows that the additive structure of $\mathfrak{K}$ can be deduced from the knowledge of all the projective lines in $\mathcal{P}(\mathfrak{K})$. And finally, since the "multiplications" $l_{\jmath_{\mathfrak{K}}(x)}$ do respect this structure, we finally deduce from this the field structure of $\mathfrak{K}$, as well the field extension $\mathfrak{K}|\mathfrak{k}$.

Now coming back to the case of a function fields $K|k$ as in the Theorem above, the problems we have to tackle are the following: Give *Galois theoretic recipes* in order to detect:

I) $K_{(\ell)} := K^\times \otimes \mathbb{Z}_{(\ell)}$ *inside* $\widehat{K}$.

This almost answers the question about detecting $\mathcal{P}(K)$: We know namely its $\mathbb{Z}_{(\ell)}$-version, but not $\mathcal{P}(K)$ itself.

II) $\mathcal{P}(K)$ *and its projective lines.*

III) Using the functoriality of the construction, show that the recipe for detecting $(K, +, \cdot)$ is invariant under isomorphisms.

Organization of the paper

In the first part we put together the necessary tools for the proof as follows:

a) First, a "pro-ℓ Local theory" similar to the Local Theory form [P1], etc.. It relies on results by Ware [W], Koenigsmann [Ko], see also Bogomolov–Tschinkel [B–T1], Efrat [Ef], Engler–Koenigsmann [E–K]. The aim of this local theory is to recover in a *functorial way* from G^ℓ_K the set $\mathfrak{D}_K$ of all the

Zariski prime divisors of $K|k$, where K is any function field over an algebraic closure k of a finite field such that $\mathrm{td}(K|k) > 1$.

b) Part of the local theory is to show that in the context from above, the whole inertia structure which is significant for us is encoded in G_K^ℓ.

c) Third, a more technical result by which we recover the "geometric sets of prime divisors" of function fields $K|k$ as above. By definition, a geometric set of prime divisors of K is the set of Zariski prime divisors $\mathfrak{D}_X \subset \mathfrak{D}_K$ defined by the Weil prime divisors of some quasi-projective normal model X of K. This result itself relies on de Jong's *theory of alterations* [J].

In the second part of the manuscript, we give a simplified version of a part of the "abstract non-sense" from Part II of [P4] (which follows a suggestion by Deligne [D2]), reminding one in some sense of the abstract class field theory. We define so called *pre-divisorial Galois formations*. The aim of this theory is to lay an axiomatic strategy for the proof of the main result.

The interesting example of pre-divisorial Galois formations are the *geometric* Galois formations, which arise from geometry (and arithmetic).

A very basic results here is Proposition 3.18, which shows that in the case $K|k$ is a function field with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite field, the geometric Galois formations on $G = G_K^{\ell,\mathrm{ab}}$ are group theoretically encoded in G_K^ℓ.

Finally in the last Section we prove the main Theorem announced above. The main tool here is Proposition 4.1, which gives a Galois characterization of the "rational projections".

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1. Local theory and inertia elements

In this section we first recall the main facts concerning the Local theory from [P1], but in a pro-ℓ setting. Our first aim is to give to give a group theoretic recipe for finding information about the *space of Zariski prime divisors* in a functorial way.

Let $\mathfrak{K}|\mathfrak{k}$ be a function field over some base field $\mathfrak{k}$, and $\mathrm{td}(\mathfrak{K}|\mathfrak{k}) = d > 0$. We consider the family of all models $X_i \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$ such that the structure sheaf of X_i is a sheaf of sub-rings of $\mathfrak{K}$ with inclusions as structure morphisms. On the family of all the X_i's there exists a naturally defined domination relation as follows: $X_j \geq X_i$ if there exists a surjective $\mathfrak{k}$-morphism $\varphi_{ji} : X_j \to X_i$ which at the structure sheaf level is defined by inclusions. Let $\mathfrak{Proj}_{\mathfrak{K}}$ be the subfamily of all projective, normal models of the function field $\mathfrak{K}$. The following is well known, see e.g. ZARISKI–SAMUEL [Z–S], Ch.VI, especially §17:

- Every complete model is dominated by some $X_i \in \mathfrak{Proj}_{\mathfrak{K}}$ (Chow Lemma).

- The set $\mathfrak{Proj}_{\mathfrak{K}}$ is increasingly filtered with respect to $\geq$, hence it is a surjective projective system.

- Denote $\mathfrak{R}_{\mathfrak{K}} = \varprojlim_i X_i$ as topological spaces. We will call $\mathfrak{R}_{\mathfrak{K}}$ the *Riemann space* of $\mathfrak{K}$. The points of $\mathfrak{R}_{\mathfrak{K}}$ are in bijection with the space of all valuation $\mathfrak{k}$-rings of $\mathfrak{K}$. For $v = (x_i)_i$ in $\mathfrak{R}_{\mathfrak{K}}$ one has: x_i is the centre of v on X_i in the usual sense, and $\mathcal{O}_v = \cup_i \mathcal{O}_{X_i,x_i}$.

FACT/DEFINITION 1.1. Using e.g. [BOU], Ch.IV, §3, one shows that for a point $v = (x_i)_i$ in $\mathfrak{R}_{\mathfrak{K}}$ the following conditions are equivalent:

i) For i sufficiently large, x_i has co-dimension 1, or equivalently, x_i is the generic point of a prime Weil divisor of X_i. Hence v is the discrete $\mathfrak{k}$-valuation of $\mathfrak{K}$ with valuation ring $\mathcal{O}_{X_i,x_i}$.

ii) $\mathrm{td}(\mathfrak{K}v \mid \mathfrak{k}) = \mathrm{td}(\mathfrak{K}|\mathfrak{k}) - 1$.

We will say that a point $v = (x_i)_i$ in $\mathfrak{R}_{\mathfrak{K}}$ satisfying the above equivalent conditions is a *Zariski prime divisor* of $\mathfrak{K}|\mathfrak{k}$. We denote the space of all Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$ by $\mathfrak{D}_{\mathfrak{K}} \subset \mathfrak{R}_{\mathfrak{K}}$. One has: The space $\mathfrak{D}_{\mathfrak{K}} \subset \mathfrak{R}_{\mathfrak{K}}$ of all Zariski prime divisors of $\mathfrak{K}$ is the union of the spaces $\mathfrak{D}_i$ of prime Weil divisors of all the models X_i of $\mathfrak{K}|\mathfrak{k}$ in $\mathfrak{Proj}_{\mathfrak{K}}$ (if we identify every prime Weil divisor with the discrete valuation on $\mathfrak{K}$ it defines).

More generally, let $\mathfrak{K}'|\mathfrak{K}$ be an arbitrary algebraic extension, and v' a valuation on $\mathfrak{K}'$. Then we will say that v' is a Zariski prime divisor (of $\mathfrak{K}$ or of $\mathfrak{K}'$), if its restriction v to $\mathfrak{K}$ is a Zariski prime divisor of $\mathfrak{K}|\mathfrak{k}$.

A) *On the decomposition group*

Let $\mathfrak{K}'|\mathfrak{K}$ be some Galois field extension as in the Introduction, *Rough idea of proof,* and v is a valuation on $\mathfrak{K}'$ which is trivial on $\mathfrak{k}$. Let Z_v, T_v, and V_v be respectively the decomposition group, the inertia group, and the ramification group of v in $G'_{\mathfrak{K}} := \mathrm{Gal}(\mathfrak{K}'|\mathfrak{K})$. Since $\ell \neq \mathrm{char}(\mathfrak{k})$, and v is trivial on $\mathfrak{k}$, V_v is trivial. We denote by $\mathfrak{K}^T$, and $\mathfrak{K}^Z$ the corresponding fixed fields.

FACT 1.2. The following are well known facts from Hilbert decomposition, and/or ramification theory for general valuations: $\mathfrak{K}'v \mid \mathfrak{K}v$ is a Galois field extension, which also satisfies the properties i), ii), from loc.cit.. Let $G'_{\mathfrak{K}v} = \mathrm{Gal}\,(\mathfrak{K}'v \mid \mathfrak{K}v)$ be its Galois group. One has a canonical exact sequence

$$1 \to T_v \to Z_v \xrightarrow{\pi_v} G_v := G'_{\mathfrak{K}v} \to 1\,.$$

Moreover, $v(\mathfrak{K}^T) = v(\mathfrak{K}^Z) = v\mathfrak{K}$, and $\mathfrak{K}v = \mathfrak{K}^Z v$; and further $v\mathfrak{K}'$ is an ℓ-divisible hull of $v\mathfrak{K}$.

Finally, there exists a pairing $\Psi_{\mathfrak{K}'} : T_v \times v\mathfrak{K}' \to \mu_{\mathfrak{K}'v}$, $(g, vx) \mapsto (gx/x)v$, and the following hold: The left kernel of $\Psi_{\mathfrak{K}'}$ is exactly $V_v = \{1\}$, and the right kernel of $\Psi_{\mathfrak{K}'}$ is $v\mathfrak{K}$. In particular, T_v is Abelian. Further, Ψ is compatible with the action of G_v on T_v via π_v. In particular, this action is via the cyclotomic character, thus trivial, as $\mathfrak{K}v$ contains by hypothesis the algebraically closed field $\mathfrak{k}$. Therefore, we finally have: $Z_v \cong T_v \times G_v$.

1) Let $\mathcal{B} = (vx_i)_i$ be an $\mathbb{F}_\ell$-basis of $v\mathfrak{K} \,/\, \ell$. Then $T_v \cong \mathbb{Z}_\ell^{\mathcal{B}}$, and finally

$$Z_v \;\cong\; T_v \times G'_{\mathfrak{K}v} \;\cong\; \mathbb{Z}_\ell^{\mathcal{B}} \times G'_{\mathfrak{K}v}$$

as profinite groups. In particular, $\mathrm{cd}(Z_v) = \mathrm{cd}(G'_{\mathfrak{K}v}) + |\mathcal{B}|$, where cd denotes the cohomological dimension.

2) Thus finally there exist canonical isomorphisms $\theta^v : \widehat{v\mathfrak{K}} \to \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big)$ and $\theta_v : T_v \to \mathrm{Hom}\big(v\mathfrak{K}, \mathbb{Z}_\ell\big)$.

3) Recalling the notations and remarks from Introduction, one gets a commutative diagram of the form:

$$\begin{array}{ccc}
\mathfrak{K}^\times & \xrightarrow{\;v\;} & v\mathfrak{K} \\
\Big\downarrow{\scriptstyle \jmath_{\mathfrak{K}}} & & \Big\downarrow{\scriptstyle \theta^v} \\
\mathrm{Hom}\big(G'_{\mathfrak{K}}, \mathbb{Z}_\ell\big) & \xrightarrow{\;\jmath^v\;} & \mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big)
\end{array}$$

4) Let $U_v \subset \mathfrak{K}^\times$ denote the v-units, and $U_v^1 = 1 + \mathfrak{m}_v \subset U_v$ the principal v-units. Then U_v^1 becomes ℓ-divisible in $\mathfrak{K}^Z$, and $(\mathfrak{K}v)^\times = U_v/U_v^1$. We have a commutative diagram of the form:

$$\begin{array}{ccc}
U_v & \longrightarrow & \mathfrak{K}v \\
\Big\downarrow{\scriptstyle \jmath_{\mathfrak{K}}} & & \Big\downarrow{\scriptstyle \jmath_{\mathfrak{K}v}} \\
\mathrm{Hom}\big(G'_{\mathfrak{K}}, \mathbb{Z}_\ell\big) & \xrightarrow{\;\jmath_v\;} & \mathrm{Hom}\big(G'_{\mathfrak{K}v}, \mathbb{Z}_\ell\big)
\end{array}$$

B) *Recovering $\mathfrak{D}_K$ from G_K^ℓ*

Definition/Remark 1.3. In the notations from Introduction and the above ones, let $K|k$ be a function field with $\mathrm{td}(K|k) > 0$ and k an algebraic closure of a finite field.

1) The decomposition group $Z_v \subset G_K^\ell$ of some divisorial valuation $v \in \mathfrak{D}_{K^\ell}$ is called a *divisorial subgroup.*

2) A subgroup $Z \subset G_K^\ell$ which is isomorphic to a divisorial subgroup of some function field of transcendence degree equal to $\mathrm{td}(K|k)$ is called a *divisorial like subgroup.*

PROPOSITION 1.4. *Let $K|k$ be a function field with* $\mathrm{td}(K|k) > 1$ *and k an algebraic closure of a finite field. Let $Z_v \subset G^\ell_K$ be a divisorial subgroup, say defined by a divisorial valuation v on K^ℓ. Let T_v be the inertia group of v. Then the following hold:*

(1) *Z_v is self-normalising in G^ℓ_K. Further, if $Z_{v'} \neq Z_v$ is another divisorial subgroup, then one has: $Z_{v'} \cap Z_v = 1$.*

(2) *$T_v \cong \mathbb{Z}_\ell$ as a G^ℓ_{Kv}-module. Further T_v is the unique maximal pro-ℓ Abelian normal subgroup of Z_v. And finally $Z_v \cong T_v \times G^\ell_{Kv} \cong \mathbb{Z}_\ell \times G^\ell_{Kv}$.*

Proof. To (1): Both assertions follow using a result of F. K. SCHMIDT, see e.g., POP [P1], Proposition 1.3; see also the proof of Proposition 1.14 from loc.cit.. Concerning (2), the only non-obvious part is the fact that given any function field $K' = Kv$ over k, the Galois group $G^\ell_{K'}$ has no non-trivial Abelian normal subgroups. This follows from the Hilbertianity of Kv. □

A first content of the local theory is that "morally" the converse of the above Proposition is also true, i.e., if $Z \subset G^\ell_K$ is a divisorial like subgroup, then it comes from a Zariski prime divisor.

PROPOSITION 1.5. *Let $K|k$ be a function field with* $\mathrm{td}(K|k) = d > 1$ *and k an algebraic closure of a finite field. Then one has:*

(1) *For every divisorial like subgroup Z of G^ℓ_K there exists a unique divisorial valuation v of K^ℓ such that $Z \subset Z_v$ and* $\mathrm{char}(Kv) \neq \ell$.

(2) *Moreover, if $T \subset Z$ is the unique maximal Abelian normal subgroup of Z, then $T = Z \cap T_v$, where T_v is the inertia subgroup of Z_v.*

Therefore, the space $\mathfrak{D}_{K^\ell}$ of all Zariski prime divisors of K^ℓ is in bijection with the divisorial subgroups of G^ℓ_K. This bijection is given by $v \mapsto Z_v$.

Proof. The main step in the proof is the following ℓ-Lemma below, which replaces the q-Lemma from [P1], Local theory. After having the ℓ-Lemma, the remaining steps in the proof are similar to (but easier than) the ones from the proof of Theorem 1.16 from [P1], Local theory. We will skip the remaining details. □

THE ℓ-LEMMA (revisited). *In the context of Proposition 1.5, let $Z_0 \cong \mathbb{Z}^d_\ell$ be a closed subgroup of G^ℓ_K. Then there exists a valuation v_0 of K^ℓ such that $Z_0 \subset Z_{v_0}$, and* $\mathrm{char}(Kv) \neq \ell$.

There are several ways to prove the ℓ-Lemma above: First, one could develop the corresponding model theoretic machinery, and proceed as in [P1], Local theory. Second, one can apply the results from WARE [W], and KOENIGSMANN [Ko], more precisely ENGLER–KOENIGSMANN [E–K]; see also EFRAT [Ef]. Or third, apply BOGOMOLOV–TSCHINKEL [B–T1].

C) *Inertia elements of* G_K^ℓ

Let $K|k$ be a function field with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite field. In this subsection we give a description of an important class of inertia elements in G_K^ℓ via divisorial inertia elements.

DEFINITION. 1.6. Let $K'|K$ be a Galois extension, and let denote $G' := \mathrm{Gal}(K'|K)$. For a valuation v of K and a prolongation v' of v to K', let $T_{v'} \subset Z_{v'}$ be its inertia, respectively decomposition group in G'.

1) An element g of G' is called a v-inertia element, if $g \in T_{v'}$, for some prolongation $v'|v$. In general, an element g of G' is called an inertia element, if there exists v such that g is a v-inertia element.

Denote by $\mathfrak{inr}(K')$ is the set of all the inertia elements in G'.

2) An inertia element g of G' is called divisorial inertia element, if g is a v-inertia element to some divisorial valuation v of $\mathfrak{K}$.

Denote by $\mathfrak{div.inr}(K')$ is the set of all divisorial inertia elements of G'.

REMARKS 1.7. Let K_0 be a finitely generated infinite field, e.g. a global field. If $\mathrm{char}(K_0) > 0$, then denote by $K = K_0k$ a maximal constant extension of K_0. (Thus k is an algebraic extension of the constants of K_0, and $K|k$ is a function field as above.) As above, let $K_0'|K_0$ be some Galois extension, and G_0' its Galois group. Thus if K_0 has positive characteristic, then $K' := K_0'k$ is a Galois extension of K, and let $G' \subset G_0'$ is a closed subgroup of G_0'.

a) Let $\mathcal{X} \to \mathbb{Z}$ be a model of K_0. Recall, that an element σ of $\mathrm{Gal}(K_0'|K_0)$ is called a Frobenius element of K_0' (over $\mathcal{X}$), if there exits a regular closed point $x \in \mathcal{X}$, and a decomposition groups $D_x \subset \mathrm{Gal}(K_0'|K_0)$ over x, such that $\sigma \in D_x$ is in the pre-image of the Frobenius at x in D_x. Let $\mathfrak{Frob}(K_0')$ be the set of all the Frobenius elements in $\mathrm{Gal}(K_0'|K_0)$ (for the several models of K_0 as above). By the *Chebotarev Density Theorem,* $\mathfrak{Frob}(K_0')$ is dense in $\mathrm{Gal}(K_0'|K_0)$.

b) In contrast to the situation above, the set $\mathfrak{div.inr}(K_0')$ is in general not dense in $\mathrm{Gal}(K_0'|K_0)$. Namely, if T' is the closed subgroup of $\mathrm{Gal}(K_0'|K_0)$ generated by all the divisorial inertia elements, and if $\mathcal{X}$ is any *complete regular* model of K_0 (if any such models do exist), then $\mathrm{Gal}(K_0'|K_0)\,/\,T' \cong \pi_1(\mathcal{X})$.

This is a totally different situation than that of the Frobenius elements of global fields. But a much more stronger assertion holds, as follows. First, recall that for every valuation v on K one has the following fundamental inequality, see e.g. [BOU], Ch.IV, §10, 3:

$$\mathrm{td}(K|k) \geq \mathrm{rr}(vK) + \mathrm{td}(Kv|k)$$

where $\mathrm{rr}(vK)$ the rational rank of the group vK. A valuation of K is called *defectless,* if the above inequality is an equality.

FACT. The most prominent defectless valuations are the "iterations" of Zariski prime divisors as follows: Let $\delta \leq d = \mathrm{td}(K|k)$ be a fixed positive integer. Define inductively a sequence of function fields K_k endowed with Zariski prime divisors v_k as follows: v_1 is a Zariski prime divisor of $K_1 := K$, and set $K_2 := K_1 v_1$, etc.. Thus inductively: v_k is a Zariski prime divisor of $K_k := K_{k-1}v_{k-1}$ for all $k \leq \delta$. Finally set $v = v_\delta \circ \ldots \circ v_1$ as a valuation on K. Then v is a "generalised" discrete valuation of K with $vK \cong \mathbb{Z}^\delta$ lexicographically ordered, and $\mathrm{td}(Kv|k) = \mathrm{td}(K_\delta v_\delta|k) = \mathrm{td}(K|k) - \delta$. Therefore v is defectless.

THEOREM 1.8. *In the notations from the Definition above, let $K|k$ be a function field with k an algebraically closed base field, $K'|K$ be a Galois sub-extension of $K^\ell \,|\, K$. Then the following hold:*

(1) $\mathfrak{inr}(K')$ *is closed in* $\mathrm{Gal}(K'|K)$.

(2) *The closure of* $\mathfrak{div.inr}(K')$ *in* $\mathrm{Gal}(K'|K)$ *contains all the inertia elements at defectless valuations v which are trivial on k.*

Proof. In both cases we can suppose that $K' = K^\ell$, as both closed subsets and inertia elements are closed under the projection $G^\ell_K \to \mathrm{Gal}(K'|K)$.

To (1): Let $g \neq 1$ lie in the closure of $\mathfrak{inr}(K^\ell)$. Equivalently, for every finite Galois sub-extension $K_i|K$ of $G^\ell_K \,|\, K$, there do exist:

i) A quasi divisorial valuation v_i on K^ℓ.

ii) Some g_i in the inertia group $T_i := T_{v_i}$ of v_i in G^ℓ_K, such that g and g_i have the same restriction to K_i.

We will show that g is a v-inertia element of G^ℓ_K. First, in the notations from above, let ℓ^{n_i} be the common order of (the restriction of) g and g_i on K_i. Further, let K_0 be the fixed field of g in K^ℓ, thus $K^\ell \,|\, K_0$ has Galois group generated by g. By Kummer Theory, there exits some $x \in K_0$ which is not an ℓ^{th} power in $K_0^\times$; and for every $\alpha_n \in K^\ell$ such that $\alpha^{\ell^n} = x$, and $K_n := K_0[\alpha_n]$ is the unique extension of K_0 of degree ℓ^n inside K^ℓ. Moreover, we can suppose that $\alpha_n^\ell = \alpha_{n-1}$, where $\alpha_0 := x$.

In a first approximation, we consider only those K_i which contain x, as this is a co-final set of finite Galois sub-extensions of $K^\ell \,|\, K$. In a second approximation, we consider the subset of those K_i for which $\alpha_{n_i} \in K_i$. We claim that this is also co-final in the set of all finite Galois sub-extensions of $K^\ell \,|\, K$. Indeed, it is sufficient to show that the restriction of g to $K_i[\alpha_{n_i}]$ has order ℓ^{n_i}. But this follows from the fact that the restriction of g to both K_i and $K_{n_i} := K[x, \alpha_{n_i}]$ is ℓ^{n_i}.

Now for a co-final set $K_i|K$ as above, let v_i and g_i with the properties i), ii), above. We claim that $v_i(x)$ is not divisible by ℓ in $v_i K$. Indeed, otherwise let $u \in K$ such that $\ell \cdot vu = vx$. Then $y = x/u^\ell$ is a v_i-unit. Thus in the inertia field K^{T_i} we have: y is an ℓ^{th} power, and hence x is an ℓ^{th} power in K^{T_i}. Therefore, α_1 lies in K^{T_i}. Hence g_i acts trivially on α_1. This contradicts the fact that g acts non-trivially on α_1 ! Therefore, we finally have: x is not a v_i-unit, and $K^\ell = K^{g_i}[(\alpha_n)_n]$.

Now consider and ultrafilter $\mathcal{D}$ on the (index set $I = \{i\}$ of the) family K_i such that for every n, $\mathcal{D}$ contains all the K_i with $n_i \geq n$. By the "general non-standard non-sense" we have the following facts:

a) ${}^*K := K^{\ell^I} / \mathcal{D}$ carries the valuation ${}^*v = \prod v_i / \mathcal{D}$ such that the residual field ${}^*K\,{}^*v$ has characteristic $\neq \ell$.

b) $\mathfrak{L} := \prod K^{g_i} / \mathcal{D}$ has a canonical embedding into ${}^*K$. Furthermore, one has $\mathfrak{L}^\ell = \mathfrak{L}[(\alpha_n)_n]$, and ${}^*v\,x$ is not divisible by ℓ in ${}^*v\,\mathfrak{L}$.

c) Since each v_i is pro-ℓ Henselian on K^{g_i}, it follows that the restriction $\mathfrak{v}$ of ${}^*v$ to $\mathfrak{L}$ is pro-ℓ Henselian on $\mathfrak{L}$.

From this we deduce: $\mathfrak{L} \cap K^\ell = K_0$ is the fixed field of g. Therefore, if v is the restriction of $\mathfrak{v}$ to $K^\ell \subset \mathfrak{L}^\ell$, then v is pro-ℓ Henselian on K_0. And since vx is not divisible by ℓ in vK_0, it follows that $T_v = G^\ell_{K_0}$. Thus g lies in $\mathfrak{inr}(K^\ell)$.

To (2): Let $g \neq 1$ be an inertia element at some v. Generally, in order to show that g is an accumulation point of divisorial inertia elements, w.l.o.g. we can suppose that g is not an ℓ^{th}-power in T_v.

As in the proof of (1), let K_0 be the fixed field of g in K^ℓ, and $(\alpha_n)_n$ the compatible system of roots $\alpha_n^{\ell^n} = x$ of x such that $K_n := K_0[\alpha_n]$ is the unique extension of degree ℓ^n of K_0. By replacing x by its inverse (if necessary), we can suppose that $vx > 0$. Finally consider a co-final set of finite Galois sub-extensions $K_i|K$ of $G^\ell_K \,|\, K$ as in the proof of assertion (1) above, i.e., such that: $x \in K_i$, and the restriction of g to K_i has order ℓ^{n_i}, and $\alpha_{n_i} \in K_i$, where α_{n_i} are as above.

Hypothesis/Remark. For every i, there exist projective models $X_i \to k$ of $K_i^g|k$ such that the centre $\mathfrak{x}_i$ of v is a regular point.

We remark that by Abhyankar's desingularization Theorem, Abhyankar [A], this hypothesis is satisfied if $\mathrm{td}(K|k) \leq 2$. Further, if v is a defectless valuation, the above hypothesis is satisfied by the main result of Knaf–Kuhlmann, [K–K].

For every i consider a projective model $X_i \to k$ of $K_i^g|k$, such that the center $\mathfrak{x}_i$ of v on X_i is a regular point. Equivalently, $\mathcal{O}_v$ dominates the regular local ring $\mathcal{O}_i := \mathcal{O}_{X_i,\mathfrak{x}_i}$ of $\mathfrak{x}_i$. In particular, if $\mathcal{M}_i = \mathcal{O}_i \cap \mathcal{M}_v$ is the maximal ideal of $\mathcal{O}_i$, then $vx > 0$ implies $x \in \mathcal{M}_i$. On the other hand, since $\mathcal{O}_i$ is regular, it is a factorial ring. Let $x = \epsilon\pi_1^{\nu_1} \ldots \pi_r^{\nu_r}$ be its representation as product of powers of non-associate prime elements in $\mathcal{O}_i$.

Claim. There is an exponent ν_m which is not divisible by ℓ.

Indeed, by its choice, x is not an ℓ^{th} power in $\mathcal{O}^g_v$. Further, $\mathcal{O}^g_v$ is pro-ℓ Henselian. As it dominates $\mathcal{O}_i$, it contains a pro-ℓ Henselisation $\mathcal{O}^{\text{h}}_i$ of $\mathcal{O}_i$. Further, since ϵ is unit in $\mathcal{O}_i$, and the residue field $\kappa_i = \mathcal{O}_i/\mathcal{M}_i = k$ is algebraically closed, it follows that ϵ is an ℓ^{th} powers in $\mathcal{O}^{\text{h}}_i$. Therefore, if all the exponents ν_m were divisible by ℓ, then x would be an ℓ^{th} power in $\mathcal{O}^{\text{h}}_i \subset \mathcal{O}^g_v$, contradiction!

Now let ν_{m_i} be an exponent which is not divisible by ℓ. Let v^g_i be the divisorial valuation on K^g_i defined by the prime element π_{m_i}, and let v_i be some prolongation of v^g_i to K^ℓ. Then we have:

a) The value $v_i(x) = v^g_i(x) = \nu_{m_i}$ is not divisible by ℓ.

b) Since $\alpha_i \in K_i$ satisfies $\alpha_i^{\ell^{n_i}} = x$, we have $\ell^{n_i} \cdot v_i(\alpha_i) = v_i(x)$. Thus v_i is totally (tamely) ramified in $K_i|K^g_i$.

In particular, by general decomposition theory, it follows that the restriction of T_{v_i} to K_i contains $\mathrm{Gal}\,(K|K^g_i)$. Thus, there exists $g_i \in T_{v_i}$ such that g_i and g coincide on K_i.

This conclude the proof of assertion (2) of Theorem 2.10. $\square$

2. First consequences of the local theory

Let $K|k$ be a function field over with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite field. Using the Theorem above one can recover from G^ℓ_K certain geometric invariants as follows.

A) *Recovering* $\mathrm{td}(K|k)$.

We first remark that from the generalized Milnor Conjectures it immediately follows that $\mathrm{cd}(G^\ell_K) = \mathrm{td}(K|k)$. But this fact can be deduced from the local theory as follows:

Fact. 2.1. In the above context we have:

1) The following conditions are equivalent, and they are satisfied if an only if $\mathrm{td}(K|k) = 2$:

(i) G^ℓ_{Kv} is pro-ℓ free for some $v \in \mathfrak{D}_{Kv}$.

(ii) G^ℓ_{Kv} is pro-ℓ free for all $v \in \mathfrak{D}_{Kv}$.

2) By induction, we can characterize $d = \mathrm{td}(K|k)$ as follows: The following conditions are equivalent, and they are satisfied if an only if $\mathrm{td}(K|k) = d$:

(j) For some $v \in \mathfrak{D}_K$, the pro-ℓ group G^ℓ_{Kv} has the property characterizing the fact that $\mathrm{td}(Kv|k) = d - 1$.

(jj) G^ℓ_{Kv} has the property characterizing the fact that $\mathrm{td}(Kv) = d - 1$ for all $v \in \mathfrak{D}_v$.

Proof. It is sufficient to prove 1). First assume $\mathrm{td}(K|k) = 2$. Then Kv is a function field in one variable over k. Thus the result follows by the fact that the absolute Galois group of Kv is profinite free, see e.g. HARBATER [Ha], or POP [Po]. Conversely, suppose that G^ℓ_{Kv} is pro-ℓ free for some $v \in \mathfrak{D}_K$. By contradiction, suppose that $\mathrm{td}(K|k > 2$. Since $\mathrm{td}(Kv|k) = \mathrm{td}(K|k) - 1 > 1$, G^ℓ_{Kv} contains divisorial subgroups of the form $Z_w \cong \mathbb{Z}_\ell \times G^\ell_{(Kv)w}$, which have $\mathrm{cd} > 1$. Thus G^ℓ_{Kv} is not pro-ℓ free, contradiction! $\square$

B) *Recovering the residually divisorial inertia*

In the usual hypothesis from above, let v be a Zariski prime divisor, and G^ℓ_{Kv} the residual Galois extension. In order to have a name, we will say that the divisorial inertia elements $\mathfrak{div.inr}(Kv^\ell)$ are *v-residually divisorial inertia* elements of G^ℓ_K.

By the local theory, if $\mathrm{td}(K|k) > 2$, then $\mathrm{td}(Kv|k) > 1$. Thus the Zariski prime divisors of Kv are encoded in G^ℓ_{Kv} as shown in Proposition 1.5 of the Local Theory. In particular, the v-residually divisorial inertia elements of G^ℓ_{Kv} are encoded in G^ℓ_{Kv} (and indirectly, in G^ℓ_K too).

Nevertheless, if $\mathrm{td}(K|k) = 2$, a fact which by the subsection above is encoded in G^ℓ_K, then the residue Galois group G^ℓ_{Kv} is pro-ℓ free. Thus there is no group theoretic way to recognise the inertia elements of G^ℓ_{Kv} from G^ℓ_{Kv} itself, as this group does not depend on Kv at all...

The solution to recovering the divisorial inertia elements in the residual Galois group G^ℓ_{Kv} in general in the following: Let v_0 be a Zariski divisor of Kv, and $T_{v_0} \subset Z_{v_0}$ its inertia, respectively decomposition group in G^ℓ_{Kv}. Further let $w = v_0 \circ v$ the composition of v_0 with v. Thus one has $Kw = (Kv)v_0$ and $0 \to vK \to wK \to v_0(Kv) \to 0$ in a canonical way. Further, via the canonical exact sequence $1 \to T_v \to Z_v \xrightarrow{\pi_v} G^\ell_{Kv} \to 1$ we have: $T_w \subset Z_w$ are exactly the pre-images of $T_{v_0} \subset Z_{v_0}$ via π_v.

On the other hand, T_w consists of inertia elements of G^ℓ_K, thus by Theorem 1.8 it follows that T_w is contained in the topological closure of $\mathfrak{div.inr}(K^\ell)$. And this last set is known, as all the divisorial subgroups Z_v together with their inertia groups T_v are encoded in G^ℓ_K, as shown in Proposition 1.5. Finally, in the "trouble case" $\mathrm{td}(K|k) = 2$, we deduce the following: The image of $\mathfrak{div.inr}(K^\ell) \cap Z_v$ in G^ℓ_{Kv} via π_v consists of exactly all the v-residually inertia elements of G^ℓ_{Kv}. And finally, by general decomposition theory, a v-residually inertia element g_v generates an inertia group of G^ℓ_{Kv} if and only if the pro-cyclic closed subgroup T_{g_v} generated by g_v in G^ℓ_{Kv} is maximal among the pro-cyclic subgroups of G^ℓ_{Kv}. Summarising we the following:

FACT 2.2. In the above context, let $g_v \in G^\ell_{Kv}$ be a given element, and T_{g_v} the pro-cyclic subgroup generated by g_v. Then T_{g_v} is an inertia subgroup of G^ℓ_{Kv} if and only if the following holds:

1) If $\mathrm{td}(K|k) = 2$, then g_v lies in the image of $\mathfrak{div.inr}(K^\ell) \cap Z_v$ in G^ℓ_{Kv}, and T_{g_v} is maximal among the pro-cyclic subgroups of G^ℓ_{Kv}.

2) If $\mathrm{td}(K|k) > 2$, then T_{g_v} is the inertia group of a divisorial subgroup of G^ℓ_{Kv} as resulting from Proposition 1.5 from the Local Theory for G^ℓ_{Kv}.

C) *More about the case* $\mathrm{td}(K|k) = 2$

We first recall some basic facts concerning Galois theory over curves and their function fields over algebraically closed fields as follows. Let $\mathfrak{K}|\mathfrak{k}$ be a function field in one variable over the algebraically closed field $\mathfrak{k}$. Let $X \to \mathfrak{k}$ be the unique complete normal model of $\mathfrak{K}|\mathfrak{k}$, thus $X \to \mathfrak{k}$ is projective and smooth. The closed points $a \in X$ are in one-to-one correspondence with the divisorial valuations v_a of $\mathfrak{K}|\mathfrak{k}$.

Let $\ell \neq \mathrm{char}$ be some prime number, and $G'_{\mathfrak{K}} = \mathrm{Gal}(\mathfrak{K}^{\ell,\mathrm{ab}}|\mathfrak{K})$ be the Galois group of a maximal abelian pro-ℓ Galois extension of $\mathfrak{K}$. Let $T_a = Z_a$ be the inertia/decomposition groups over each $a \in X(\mathfrak{k})$ in G. (N.B., these groups depend only on a, and not on the specific Zariski prime divisor of $\mathfrak{K}^{\ell,\mathrm{ab}}|\mathfrak{k}$ used to define each particular $T_a = Z_a$.)

a) *Canonical inertia generators*

Let $g \geq 0$ be genus of X. It is well known (e.g., using for instance generalized Jacobians, or the specialization theorem of Grothendieck) that there exist elements $\rho_1, \sigma_1, \ldots, \rho_g, \sigma_g$ in $G'_{\mathfrak{K}}$, and "Abelian inertia elements" $\tau_a \in T_a$ (all $a \in X(\mathfrak{k})$) such that one has:

$$G'_{\mathfrak{K}} = G =< \rho_1, \sigma_1, \ldots, \rho_g, \sigma_g, (\tau_a)_a \mid \textstyle\prod_a \tau_a = 1 > .$$

In particular, if $\tau'_a \in T_a$ are such that $\prod_a \tau'_a = 1$ inside G, then there exists a unique $\epsilon \in \mathbb{Z}_\ell$ such that $\tau'_a = \tau_a^\epsilon$ for all $a \in X$.

We will say that $(\tau_a)_{a\in X}$ is a *canonical system of Abelian inertia generators* in $G'_{\mathfrak{K}}$. Such a system is unique up to simultaneous multiplication by an element of $\mathbb{Z}_\ell^\times$.

b) $\pi_1^{\ell,ab}(X)$ *and the genus*

In the context above, let $T'_{\mathfrak{K}}$ be the closed subgroup of $G'_{\mathfrak{K}}$ generated by the inertia subgroups T_a (all a). Then we have a canonical exact sequence of the form:

$$1 \to T'_{\mathfrak{K}} \to G'_{\mathfrak{K}} \to \pi'_1(X) \to 1 .$$

In particular, $\pi'_1(X)$ is the free Abelian pro-ℓ of rank $2g$, where g is the genus of X. Thus the genus of X is encoded in $G^{\ell,\mathrm{ab}}_{\mathfrak{K}}$ endowed via the inertia groups $(T_a)_{a\in X}$.

c) *Detecting* $\mathrm{Div}^0(X)$

For $a \in X$, recall the commutative diagram form section 1, A), 4):

$$
(a) \qquad
\begin{array}{ccc}
\mathfrak{K}^\times & \xrightarrow{\text{can}} & \mathrm{Hom}\big(G'_{\mathfrak{K}}, \mathbb{Z}_\ell\big) \\
\Big\downarrow v_a & & \Big\downarrow \text{res} \\
v_a\mathfrak{K} & \xrightarrow{\jmath^a} & \mathrm{Hom}\big(T_a, \mathbb{Z}_\ell\big)
\end{array}
$$

For every a, let $\gamma_a \in v_a\mathfrak{K}$ be the unique positive generator, thus $\gamma_a = v_v\pi_a$ for every uniformizing parameter π_a at a. Then T_a has a *unique generator* τ_a^0 such that $\jmath^a\,(\gamma_a)(\tau_a^0) = 1 \in \mathbb{Z}_\ell$. In particular, if $x \in \mathfrak{K}^\times$ is arbitrary, and $v_a x = \gamma_x = m_{a,x}\,\gamma_a$ for some integer $m_{a,x}$, then $\jmath_{v_a}(\gamma_x)(\tau'_a) = m_{a,x}$ inside $\mathbb{Z}_\ell$. We will call τ_v^0 the canonical inertia generator of T_a (all a).

It is a well known fact, that $\mathfrak{T}^0 = (\tau_a^0)_a$ is also a canonical system of Abelian inertia generators in $G'_{\mathfrak{K}}$, and $\mathrm{Div}^0(X)$ is *canonically* isomorphic to $\mathfrak{D}^0_{\mathfrak{T}^0} = \{\psi \in \mathrm{Hom}\big(T'_{\mathfrak{K}}, \mathbb{Z}_\ell\big) \mid \psi(\tau_a^0) \in \mathbb{Z} \text{ (all } a),\ \psi(\tau_a^0) = 0 \text{ (almost all } a)\,\}$.

FACT 2.3. Let $\mathfrak{T} = (\tau_a)_a$ be an arbitrary canonical system of Abelian inertia generators, thus $\tau_a^0 = \tau_a^\epsilon$ (all a) for some ℓ-adic unit ϵ. Further denote

$$\mathfrak{D}^0_{\mathfrak{T}} = \{\psi \in \mathrm{Hom}\big(T'_{\mathfrak{K}}, \mathbb{Z}_\ell\big) \mid \psi(\tau_a) \in \mathbb{Z} \text{ (all } a),\ \psi(\tau_a) = 0 \text{ (almost all } a)\,\}.$$

Then $\epsilon \cdot \mathrm{Div}^0(X) = \epsilon \cdot \mathfrak{D}^0_{\mathfrak{T}^0} = \mathfrak{D}^0_{\mathfrak{T}}$ inside $\mathrm{Hom}\big(T^{\ell,\mathrm{ab}}_{\mathfrak{K}}, \mathbb{Z}_\ell\big)$.

Prof. Clear from the discussion above.

FACT 2.4. Now let $K|k$ be a function field with $\mathrm{td}(K|k) = 2$, and k the algebraic closure of a finite field. Let v be a Zariski prime divisor of K, and G^ℓ_{Kv} the corresponding residual Galois group. Then by Fact 2.2 above, the inertia groups in G^ℓ_{Kv} are known. Thus applying the above discussion in the case $\mathfrak{K} = Kv$ and $X = X_v$, we get the following:

1) The inertia groups $(T_a)_{a\in X_v}$ and the canonical generating systems of inertia $\mathfrak{T}_v = (\tau_a)_{a\in X_v}$ can be deduced in a group theoretic way from G^ℓ_K.

2) The exact sequence $1 \to T^{\ell,\mathrm{ab}}_{Kv} \to G^{\ell,\mathrm{ab}}_{Kv} \to \pi_1^{\ell,ab}(X_v) \to 1$, can be deduced in a group theoretic way from G^ℓ_K. Thus also the genus g_v of X_v, as $\pi_1^{\ell,ab}(X_v)$ has $\mathbb{Z}_\ell$-rank equal to $2g_v$.

3) Each canonical generating systems of inertia $\mathfrak{T}_v = (\tau_a)_{a\in X_v}$ defines in a canonical way the subgroup

$$\mathfrak{D}^0_{\mathfrak{T}_v} = \{\psi \in \mathrm{Hom}\big(T^{\ell,\mathrm{ab}}_{Kv}, \mathbb{Z}_\ell\big) \mid \psi(\tau_a) \in \mathbb{Z} \text{ (all } a),\ \psi(\tau_a) = 0 \text{ (almost all } a)\,\}$$

which up to multiplication by an ℓ-adic unit equals the image of $\mathrm{Div}^0(X_v)$ in $\mathrm{Hom}\big(T^{\ell,\mathrm{ab}}_{Kv}, \mathbb{Z}_\ell\big)$ defined canonically via Kummer Theory.

D) *Geometric families of Zariski prime divisors*

First we recall the basic definitions and facts. Let $\mathfrak{K}|\mathfrak{k}$ be a function field over an algebraically closed base field $\mathfrak{k}$.

FACT/DEFINITION 2.5. In the above context, for a quasi-projective normal model $X \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$, let $\mathfrak{D}_X$ be the Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$ it defines. We will say that a subset $\mathfrak{D} \subset \mathfrak{D}_{\mathfrak{K}|\mathfrak{k}}$ is a *geometric set of Zariski prime divisors,* if it satisfies the following equivalent conditions:

(i) For all projective normal models $X \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$ one has: $\mathfrak{D}$ and $\mathfrak{D}_X$ are almost equal.[2]

(ii) There exists quasi-projective normal models $X' \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$ such that the equality $\mathfrak{D} = \mathfrak{D}_{X'}$ holds.

Our aim in this subsection is to recall a criterion for describing the geometric sets $\mathfrak{D}$ of divisorial valuations of $\mathfrak{K}|\mathfrak{k}$, and to show that the criterion is encoded group theoretically in G^{ℓ}_K in the case $K|k$ is a function field with $\mathrm{td}(K|k) > 1$ and k the algebraic closure of a finite field. See [P4], Section 3, for more details for general function fields.

Recall that a *line* on a $\mathfrak{k}$-variety is by definition an integral $\mathfrak{k}$-subvariety $\mathfrak{l} \subseteq X$, which is a curve of geometric genus equal to 0. We denote by X^{line} the union of all the lines on X.

We will say that a variety $X \to \mathfrak{k}$ is very unruly if the set X^{line} is not dense in X. In particular, a curve X is very unruly if and only if its geometric genus g_X is positive.

Further recall that being very unruly is a birational notion. Thus we will say that a function field $\mathfrak{K}|\mathfrak{k}$ with $\mathrm{td}(\mathfrak{K}|\mathfrak{k} = d > 0$ is *very unruly,* if $\mathfrak{K}|\mathfrak{k}$ has models $X \to \mathfrak{k}$ which are very unruly.

Suppose that $d > 1$. We call a Zariski prime divisor of $\mathfrak{K}|\mathfrak{k}$ *very unruly,* if $\mathfrak{K}v \,|\, \mathfrak{k}$ is very unruly. A Zariski prime divisor v of $\mathfrak{K}|\mathfrak{k}$ is very unruly if and only if there exists a normal model $X \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$, and a very unruly prime Weil divisor X_1 of X such that $v = v_{X_1}$.

Now let $\mathfrak{D}$ be a set of Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$. For every finite extension $\mathfrak{K}_i|\mathfrak{K}$, let $\mathfrak{D}_i$ be the prolongation of $\mathfrak{D}$ to $\mathfrak{K}_i$. Thus in particular, $\mathfrak{D}_i$ is a set of Zariski prime divisors of $\mathfrak{K}_i|\mathfrak{k}$. As shown in loc.cit., one has the following:

PROPOSITION 2.6. *In the context above, set $d = \mathrm{td}(\mathfrak{K}|\mathfrak{k})$. A set $\mathfrak{D}$ of Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$ is geometric if and only if the following conditions are satisfied:*

[2] We say that two sets are almost equal, if their symmetric difference is finite.

(i) *There exists a finite ℓ-elementary extension $\mathfrak{K}_0|\mathfrak{K}$ of degree $\leq \ell^d$ such that $\mathfrak{D}_0$ is almost equal to the set of all very unruly prime divisors of $\mathfrak{K}_0$.*

(ii) *If $\mathfrak{K}_2|\mathfrak{K}$ is any ℓ-elementary extension of degree $\leq \ell^d$, and $\mathfrak{K}_1 = \mathfrak{K}_2\mathfrak{K}_0$ is the compositum, then $\mathfrak{D}_1$ is almost equal to the set of all very unruly prime divisors of $\mathfrak{K}_1$.*

From this Proposition one deduces the following inductive procedure on $d = \mathrm{td}(\mathfrak{K}\,|\,\mathfrak{k})$ for deciding whether a given set $\mathfrak{D}$ of Zariski prime divisors is geometric, respectively whether $\mathfrak{K}\,|\,\mathfrak{k}$ is very unruly.

Criterion 2.7. In the above context one has:

1) Case $d = 1$:

$\mathcal{P}^{(1)}_{\mathrm{geom}}(\mathfrak{D})$: A set of Zariski prime divisors $\mathfrak{D}$ of $\mathfrak{K}\,|\,\mathfrak{k}$ is geometric if and only if it is almost equal to the set of all Zariski prime divisors $\mathfrak{D}_{\mathfrak{K}\,|\,\mathfrak{k}}$.

$\mathcal{P}^{(1)}_{\mathrm{v.u.}}(\mathfrak{K}\,|\,\mathfrak{k})$: $\mathfrak{K}\,|\,\mathfrak{k}$ is very unruly if and only if the geometric genus g_X of the complete normal model $X \to \mathfrak{k}$ of $\mathfrak{K}\,|\,\mathfrak{k}$ satisfies $g_X > 0$.

2) Case $d > 1$: Then by induction on d, we already have criteria $\mathcal{P}^{(\delta)}_{\mathrm{geom}}$ and $\mathcal{P}^{(\delta)}_{\mathrm{v.u.}}$ which assure that sets of Zariski prime divisors are geometric, respectively that function fields are very unruly (all $1 \leq \delta < d$). We now make the induction step $d = \mathrm{td}(\mathfrak{K}\,|\,\mathfrak{k})$ as follows:

$\mathcal{P}^{(d)}_{\mathrm{geom}}(\mathfrak{D})$: The criterion given by Proposition 2.6 is satisfied for the set of Zariski prime divisors $\mathfrak{D}$ of $\mathfrak{K}\,|\,\mathfrak{k}$.

Here we remark that given a finite extension $\mathfrak{K}_i|\mathfrak{K}$, the assertion "*$v_i$ is a very unruly Zariski prime divisor of $\mathfrak{K}_i|\,\mathfrak{k}$*" which is essentially used in loc.cit., is actually equivalent to "*$\mathfrak{K}_i v_i|\,\mathfrak{k}$ satisfies $\mathcal{P}^{(d-1)}_{\mathrm{v.u.}}(\mathfrak{K}_i v_i|\,\mathfrak{k})$*".

$\mathcal{P}^{(d)}_{\mathrm{v.u.}}(\mathfrak{K}\,|\,\mathfrak{k})$: There exists a subset $\mathfrak{D} \subset \mathfrak{D}_{\mathfrak{K}\,|\,\mathfrak{k}}$ such that $\mathcal{P}^{(d)}_{\mathrm{geom}}(\mathfrak{D})$ holds; and further, $\mathcal{P}^{(d-1)}_{\mathrm{v.u.}}(\mathfrak{K}v\,|\,\mathfrak{k})$ is true for almost all $\mathfrak{K}v\,|\,\mathfrak{k}$ $(v \in \mathfrak{D})$.

We finally come to case $K|k$ with $\mathrm{td}(K|k) > 1$ and k the algebraic closure of a finite field. We show that Criterion 2.7 can be interpreted in G^{ℓ}_K.

Indeed, by Proposition 1.5, the Zariski prime divisors of K^{ℓ} are in bijection with the divisorial subgroups of G^{ℓ}_K via $v \mapsto Z_v$; thus a bijection $\mathfrak{D}^{\ell} \mapsto \mathcal{Z}_{\mathfrak{D}^{\ell}}$ from sets of Zariski prime divisors $\mathfrak{D}^{\ell} \subset \mathfrak{D}_{K^{\ell}}$ to sets of divisorial subgroups $\mathcal{Z}$ of G^{ℓ}_K. Moreover, a set $\mathfrak{D}^{\ell}$ is the prolongation to K^{ℓ} of a set of Zariski prime divisors $\mathfrak{D} \subset \mathfrak{D}_K$ if and only if $\mathcal{Z}_{\mathfrak{D}^{\ell}}$ is invariant under conjugation in G^{ℓ}_K. If this is the case, then every conjugacy class represents an element of $\mathfrak{D}$.

For $\mathcal{Z}^{\ell}$ a set of divisorial subgroups of G^{ℓ}_K, which is closed under conjugation in G^{ℓ}_K, let $\mathfrak{D}_{\mathcal{Z}^{\ell}}$ be the corresponding set of Zariski prime divisors of K^{ℓ}. For every sub-extension $K_i|K$ of $K^{\ell}|K$, let $\mathfrak{D}^i_{\mathcal{Z}}$ be the restriction of $\mathfrak{D}^{\ell}_{\mathcal{Z}}$ to K_i. Thus $\mathfrak{D}^i_{\mathcal{Z}}$ is exactly the prolongation of $\mathfrak{D}_{\mathcal{Z}}$ to K_i.

Now suppose that $d = 2$. Let v be some Zariski prime divisor of $K|k$. Let further $X_v \to k$ be the complete smooth model of $Kv|k$. Then by Fact 2.4, the inertia groups $T_a \subset G^\ell_{Kv}$ (all $a \in X_v$), thus $\pi_1(X_v)$ and the genus g_{X_v} of X_v can be recovered from G^ℓ_K. The same is true if we replace $K|k$ by some finite sub-extension $K_i|K$ of $K^\ell|K$.

Therefore, by induction on $d = \mathrm{td}(K|k) > 0$, we have the following Galois translation of the Criterion above.

Gal-Criterion 2.8. In the above context one has:

1) Case $d = 2$:

$\mathrm{Gal}\,\mathcal{P}^{(2)}_{\mathrm{geom}}(\mathfrak{D}_{\mathcal{Z}})$ is said to be satisfied, if the Galois theoretic translation of $\mathcal{P}^{(2)}_{\mathrm{geom}}(\mathfrak{D}_{\mathcal{Z}})$ given above holds.

$\mathrm{Gal}\,\mathcal{P}^{(2)}_{\mathrm{v.u.}}(K|k)$ is said to be satisfied, if for some $\mathfrak{D}_{\mathcal{Z}}$ as above one has: $\mathrm{Gal}\,\mathcal{P}^{(2)}_{\mathrm{geom}}(\mathfrak{D}_{\mathcal{Z}})$ holds, and $g_{X_v} > 0$ for almost all $v \in \mathfrak{D}_{\mathcal{Z}}$.

2) Case $d > 1$:

By induction on d, we already have the Galois translation $\mathrm{Gal}\,\mathcal{P}^{(\delta)}_{\mathrm{geom}}$ and $\mathrm{Gal}\,\mathcal{P}^{(\delta)}_{\mathrm{v.u.}}$ of the criteria $\mathcal{P}^{(\delta)}_{\mathrm{geom}}$ and $\mathcal{P}^{(\delta)}_{\mathrm{v.u.}}$ (all $1 \le \delta < d$). We now make the induction step $d = \mathrm{td}(K|k)$ as follows:

$\mathrm{Gal}\,\mathcal{P}^{(d)}_{\mathrm{geom}}(\mathfrak{D}_{\mathcal{Z}})$ is said to be satisfied, if (the Galois translation of) the criterion given by Proposition 2.6 is satisfied.

Here we remark that given a finite extension $K_i|K$, the assertion "*v_i is a very unruly Zariski prime divisor of $K_i|k$*," which is essentially used in loc.cit., is actually equivalent to $\mathrm{Gal}\,\mathcal{P}^{(\delta)}_{\mathrm{v.u.}}(K_i v_i \,|\, k)$

$\mathrm{Gal}\,\mathcal{P}^{(d)}_{\mathrm{v.u.}}(K|k)$ is said to be satisfied if and only if for some set $\mathfrak{D}_{\mathcal{Z}}$ as above satisfying $\mathrm{Gal}\,\mathcal{P}^{(d)}_{\mathrm{geom}}(\mathfrak{D}_{\mathcal{Z}})$ one has: For almost all conjugacy classes of decomposition groups Z_v, the condition $\mathrm{Gal}\,\mathcal{P}^{(d-1)}_{\mathrm{v.u.}}(Kv|k)$ is satisfied.

3. Abstract pro-ℓ Galois theory

In this section we develop a "pro-ℓ Galois theory", which in some sense has a flavor similar to one of the abstract class field theory. The final aim of this theory is to provide a machinery for recovering function fields (over algebraically closed base fields) from their pro-ℓ Galois theory in an axiomatic way. The material here is a simplified version of the corresponding part from Pop [P4].

A) *Axioms and definitions*

The context is the following: Let ℓ be a fixed prime number, and $\mathcal{Z}$ a quotient of $\mathbb{Z}_\ell$.

Definition. A *pre-divisorial level $\delta \geq 0$* (pro-ℓ) *Galois formation* $\mathcal{G} = \big(G, \mathfrak{Z}\big)$ is defined by induction on δ to be an Abelian pro-ℓ group G endowed with extra structure as follows:

Axiom I) The level $\delta = 0$:

A level $\delta = 0$ pre-divisorial Galois formation is simply $\mathcal{G} = (G, \emptyset)$.

We will say that $\mathcal{G}$ *has level $\delta > 0$,* if the following Axioms II, III, are inductively satisfied.

Axiom II) Decomposition structure:

$\mathfrak{Z} = (Z_v)_v$ is a family of closed subgroups of G. We call Z_v the decomposition group at v, and suppose that each Z_v is endowed with a subgroup T_v such that:

i) *$T_v \cong \mathcal{Z}$ as an abstract pro-ℓ groups, and $T_v \cap T_w = \{1\}$ for all $v \neq w$.*

For every co-finite subset $\mathfrak{U}_i$ of valuations v, let $T_{\mathfrak{U}_i}$ be the closed subgroup of G generated by the T_v ($v \in \mathfrak{U}_i$). A system $(\mathfrak{U}_i)_i$ of such $\mathfrak{U}_i$ is called co-final, if every finite set of valuations is contained in the complement of some $\mathfrak{U}_i$.

ii) *There exist co-final systems $(\mathfrak{U}_i)_i$ with $T_v \cap T_{\mathfrak{U}_i} = \{1\}$ (all i, and $v \notin \mathfrak{U}_i$).*

Axiom III) Induction:

Every $G_v := Z_v/T_v$ caries itself the structure of a pre-divisorial Galois formation of level $(\delta - 1)$.

Convention. Let $\mathcal{G} = (G, \mathfrak{Z})$ be a level $\delta \geq 0$ pre-divisorial Galois formation. In order to have a uniform notation, we enlarge the index set v (which in the case $\delta = 0$ is empty) by a new symbol, which we denote v_*, by setting $Z_{v_*} = G$ and $T_{v_*} = \{1\}$. We will say that v_* is the "trivial valuation". Thus G is the decomposition group of the trivial valuation, and its "inertia group" is the trivial group. In particular, the residue Galois group at v_* is $G_{v_*} = G$.

Definition/Remark 3.1. Let $\mathcal{G} = (G, \mathfrak{Z})$ be a pre-divisorial Galois formation of some level $\delta \geq 0$. Consider any α such that $0 \leq \alpha \leq \delta$.

1) By induction on δ it is easy to see that one can view $\mathcal{G} = (G, \mathfrak{Z})$ canonically as a pre-divisorial Galois formation of level α.

2) We define inductively the system $\mathcal{G}^{(\alpha)} = (\mathcal{G}_{\alpha,i})_i$ of the *α-residual pre-divisorial Galois formations* of $\mathcal{G}$ as follows: First, if $\alpha = 0$, then this system consists of $\mathcal{G}$ only. In general, if $0 < \alpha \leq \delta$, we recall that every residue group G_v carries the structure of a pre-divisorial Galois formation of level $(\delta - 1)$; thus of level $(\alpha - 1)$ by remark 1) above. Let $\mathcal{G}_v$ be this pre-divisorial Galois formation. Then by induction, the system of the $(\alpha - 1)$-residual pre-divisorial Galois formations $\mathcal{G}_v^{(\alpha-1)}$ of each $\mathcal{G}_v$ are defined. We then set $\mathcal{G}^{(\alpha)} = \big(\mathcal{G}_v^{(\alpha-1)}\big)_v$.

We remark that the "correct" notation for the system of the α-residual pre-divisorial Galois formations $(\mathcal{G}_{\alpha,i})_i$ is to index it by multi-indices of length α of the form $\mathbf{v} = (v_{i_1} \dots v_{i_\alpha})$, where v_{i_1} is a valuation of G, v_{i_2} is a valuation of G_{v_1}, etc..

3) For every multi-index $\mathbf{v}$ as above, let $G_{\mathbf{v}}$ be the profinite group on which the $\mathbf{v}$-residual pre-divisorial Galois formation $\mathcal{G}_{\mathbf{v}}$ is based. Then we will call $G_{\mathbf{v}}$ a $\mathbf{v}$-*residual group* of $\mathcal{G}$. One can further elaborate here as follows: Given a multi-index $\mathbf{v} = (v_{i_1} \dots v_{i_\alpha})$, we can define inductively the following:

a) The $\mathbf{v}$-decomposition group $Z_{\mathbf{v}}$ of G, as being —inductively on α— the pre-image of $Z_{(v_{i_2} \dots v_{i_\alpha})} \subseteq G_{v_{i_1}}$ in $Z_{v_{i_1}}$ via $Z_{v_{i_1}} \to G_{v_{i_1}}$.

b) The inertia group $T_{\mathbf{v}}$ of $Z_{\mathbf{v}}$, as being the kernel of $Z_{\mathbf{v}} \to G_{\mathbf{v}}$. From the definition it follows that $T_{\mathbf{v}} \cong \mathcal{Z}^\alpha$.

Definition/Remark 3.2. Let $\mathcal{G} = \big(G, (Z_v)_v\big)$ be a pre-divisorial Galois formation of some level $\delta' \geq 0$.

1) We denote $\widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{Hom}\big(G, \mathcal{Z}\big)$ and call it the (ℓ-adic completion of the) *pre-divisorial pre-field formation defining* $\mathcal{G}$.

From now on suppose that $\delta' > 0$.

2) Let $T \subseteq G$ be the closed subgroup generated by all the inertia groups T_v (all v). We set $\pi_{1,\mathcal{G}} := G/T$ and call it the abstract fundamental group of $\mathcal{G}$. One has a canonical exact sequence

$$1 \to T \to G \to \pi_{1,\mathcal{G}} := G/T \to 1$$

Taking $\mathcal{Z}$-Homs, we get an exact sequence of the form

$$0 \to \widehat{U}_{\mathcal{G}} := \mathrm{Hom}\big(\pi_{1,\mathcal{G}}, \mathcal{Z}\big) \xrightarrow{\text{can}} \widehat{\mathcal{L}}_{\mathcal{G}} := \mathrm{Hom}\big(G, \mathcal{Z}\big) \xrightarrow{\jmath^{\mathcal{G}}} \mathrm{Hom}\big(T, \mathcal{Z}\big).$$

We will call $\widehat{U}_{\mathcal{G}} = \mathrm{Hom}\big(\pi_{1,\mathcal{G}}, \mathcal{Z}\big)$ the *unramified part* of $\widehat{\mathcal{L}}_{\mathcal{G}}$. And if no confusion is possible, we will identify $\widehat{U}_{\mathcal{G}}$ with its image in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

3) We now have a closer look at the structure of $\widehat{\mathcal{L}}_{\mathcal{G}}$. For an arbitrary v we have inclusions $T_v \hookrightarrow Z_v \hookrightarrow G$. Thus we can/will consider/denote restriction maps as follows:

$$\jmath^v : \widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{Hom}\big(G, \mathcal{Z}\big) \xrightarrow{\mathrm{res}_{Z_v}} \mathrm{Hom}\big(Z_v, \mathcal{Z}\big) \xrightarrow{\mathrm{res}_v} \mathrm{Hom}\big(T_v, \mathcal{Z}\big).$$

We set $\widehat{U}_v = \ker(\jmath^v)$ and call it the v-*units* in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Thus the unramified part of $\widehat{\mathcal{L}}_{\mathcal{G}}$ is exactly $\widehat{U}_{\mathcal{G}} = \cap_v \ker(\jmath^v)$.

We further denote $\mathcal{L}_{\mathcal{G},\mathrm{fin}} = \{\, x \in \widehat{\mathcal{L}}_{\mathcal{G}} \mid \jmath^v(x) = 0 \text{ for almost all } v \,\}$. We remark that by Axiom II, ii), $\mathcal{L}_{\mathcal{G},\mathrm{fin}}$ is dense in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Indeed, for a co-final system $(\mathfrak{U}_i)_i$ as at loc.cit., denote $G_i = G/T_{\mathfrak{U}_i}$ and $T_i = T/T_{\mathfrak{U}_i}$. We have a canonical exact sequence

$$1 \to T_i \to G_i \to \pi_{1,\mathcal{G}} \to 1\,,$$

and T_i is generated by the images $T_{v,i}$ of T_v in G_i ($v \notin \mathfrak{U}_i$). Clearly, the image of the inflation map

$$\inf_i : \mathrm{Hom}\big(G_i, \mathcal{Z}\big) \to \mathrm{Hom}\big(G, \mathcal{Z}\big)$$

is exactly $\Delta_i := \{x \in \widehat{\mathcal{L}}_{\mathcal{G}} \mid \jmath^v(x) = 0 \text{ for all } v \in \mathfrak{U}_i\}$. Finally, taking inductive limits over the co-final system $(\mathfrak{U}_i)_i$, the density assertion follows.

A closed submodule $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ is said to have *finite co-rank,* if $\Delta \subset \mathcal{L}_{\mathcal{G},\mathrm{fin}}$, and $\Delta / \widehat{U}_{\mathcal{G}}$ is a finite $\mathcal{Z}$-module (or equivalently, Δ is contained in $\ker(\jmath^v)$ for almost all v). Clearly, the sum of two finite co-rank submodules of $\widehat{\mathcal{L}}_{\mathcal{G}}$ is again of finite co-rank. Thus the set of such submodules is inductive. And one has:

$$\mathcal{L}_{\mathcal{G},\mathrm{fin}} = \cup_\Delta \,(\text{all finite co-rank } \Delta)$$

4) By the discussion above, the family $(\jmath^v)_v$ gives rise canonically to a continuous homomorphism $\widehat{\oplus_v}\,\jmath^v$ of ℓ-adically complete $\mathbb{Z}_\ell$-modules

$$\widehat{\oplus_v}\,\jmath^v : \widehat{\mathcal{L}}_{\mathcal{G}} = \mathrm{Hom}\big(G, \mathcal{Z}\big) \to \mathrm{Hom}\big(T, \mathcal{Z}\big) \hookrightarrow \widehat{\oplus_v}\,\mathrm{Hom}\big(T_v, \mathcal{Z}\big)\,.$$

We will identify $\mathrm{Hom}\big(T, \mathcal{Z}\big)$ with its image inside $\widehat{\oplus_v}\,\mathrm{Hom}\big(T_v, \mathcal{Z}\big)$. Therefore, $\jmath^{\mathcal{G}} = \widehat{\oplus_v}\,\jmath^v$ on $\widehat{\mathcal{L}}_{\mathcal{G}}$.

We will denote $\widehat{\mathrm{Div}}_{\mathcal{G}} := \widehat{\oplus_v}\,\mathrm{Hom}\big(T_v, \mathcal{Z}\big)$ and call it the (ℓ-adic completion of the) *abstract divisor group* of $\mathcal{G}$. We will say that the image of $\widehat{\mathcal{L}}_{\mathcal{G}}$ in $\widehat{\mathrm{Div}}_{\mathcal{G}}$ is the *divisorial quotient* (or the divisorial part) of $\widehat{\mathcal{L}}_{\mathcal{G}}$.

We further set $\widehat{\mathfrak{Cl}}_{\mathcal{G}} = \mathrm{coker}(\jmath^{\mathcal{G}})$, and call it the ($\ell$-adic completion of the) *abstract divisor class group* of $\mathcal{G}$. Therefore, we finally have a canonical exact sequence

$$0 \to \widehat{U}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\mathcal{G}}} \widehat{\mathrm{Div}}_{\mathcal{G}} \xrightarrow{\mathrm{can}} \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0\,.$$

5) We say that $\mathcal{G}$ is *complete curve like* if the following holds: There exist generators τ_v of T_v such that $\prod_v \tau_v = 1$, and this is the only pro-relation satisfied by the system of elements $\mathfrak{T} = (\tau_v)_v$.

Further consider $0 \leq \delta < \delta'$. We say that $\mathcal{G}$ is *δ-residually complete curve like* if all the δ-residual pre-divisorial Galois formations $\mathcal{G}_{\mathbf{v}}$ are residually complete curve like. In particular, "0-residually complete curve like" is the same as "residually complete curve like".

6) Consider the exact sequence $1 \to T_v \to Z_v \to G_v := Z_v/T_v \to 1$ given by Axiom II, i), (all v). Let $\inf_v : \mathrm{Hom}\big(G_v, \mathcal{Z}\big) \to \mathrm{Hom}\big(Z_v, \mathcal{Z}\big)$ be the resulting inflation homomorphism.

Since $T_v = \ker(Z_v \to G_v)$, it follows that $\mathrm{res}_{Z_v}(\widehat{U}_v)$ is the image of the inflation map infl_v. Therefore there exists a canonical continuous homomorphism, which we call the *v-reduction homomorphism:*

$$\jmath_v : \widehat{U}_v \to \mathrm{Hom}\big(G_v, \mathcal{Z}\big) = \widehat{\mathcal{L}}_{\mathcal{G}_v}$$

where $\widehat{\mathcal{L}}_{\mathcal{G}_v}$ is the ℓ-adic completion of the abstract v-residual field, i.e., the one attached to the v-residual Galois formation $\mathcal{G}_v$.

7) Next let v be arbitrary, and $\mathcal{G}_v$ be the corresponding residual Galois formation. To $\mathcal{G}_v$ we have the corresponding exact sequence as defined for $\mathcal{G}$ at point 3) above:

$$0 \to \widehat{U}_{\mathcal{G}_v} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}_v} \xrightarrow{\jmath^{\mathcal{G}_v}} \widehat{\mathrm{Div}}_{\mathcal{G}_v}$$

We will say that $\mathcal{G}$ is *ample,* if in the notations from point 3) above the following hold:

i) The canonical projection $\widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{(\jmath^v)_v} \prod_{v \notin \mathfrak{U}_i} \mathrm{Hom}\big(T_v, \mathcal{Z}\big)$ is surjective.

ii) There exists v such that:

a) $\Delta_i \subseteq \widehat{U}_v$ and $\jmath_v(\Delta_i) \subseteq \mathcal{L}_{\mathcal{G}_v, \mathrm{fin}}$.

b) $\ker\big(\,\Delta_i \xrightarrow{\jmath_v} \widehat{\mathcal{L}}_{\mathcal{G}_v} \xrightarrow{\jmath^{\mathcal{G}_v}} \widehat{\mathrm{Div}}_{\mathcal{G}_v}\,\big) \subseteq \widehat{U}_{\mathcal{G}}$.

Next consider $0 \le \delta < \delta'$. We say that $\mathcal{G}$ is *ample up to level* δ, if either $\delta = 0$, or by induction on δ the following hold: First, if $\delta = 1$, then $\mathcal{G}$ is ample in the sense defined above. Second, if $\delta > 1$, then $\mathcal{G}_v$ is ample up to level $(\delta - 1)$ for all v.

Before going to the next Subsection, let us remark that Condition a) and b) for a co-final system $(\mathfrak{U}_i)_i$ implies the corresponding assertions for every co-finite subset of valuations U, respectively for every finite co-rank module Δ (and this fact is obvious).

B) *Abstract divisor groups*

Convention. In order to avoid too technical formulations, we will suppose from now —if not explicitly otherwise stated— that $\mathcal{Z} = \mathbb{Z}_\ell$. In particular, $\mathbb{Z} \subset \mathbb{Z}_{(\ell)}$ are subgroups/subrings of $\mathcal{Z}$.

Definition 3.3.

1) Let M be an arbitrary $\mathcal{Z}$-module. We say that subsets M_1, M_2 of M are *ℓ-adically equivalent,* if there exists an ℓ-adic unit $\epsilon \in \mathcal{Z}$ such that $M_2 = \epsilon \cdot M_1$ inside M. Correspondingly, given systems $S_1 = (x_i)_i$ and $S_2 = (y_i)_i$ of elements of M, we will say that S_1 and S_2 are ℓ-adically equivalent, if there exists an ℓ-adic unit $\epsilon \in \mathcal{Z}$ such that $x_i = \epsilon\, y_i$ (all i).

2) Let M be an arbitrary ℓ-adically complete module. We will say that a $\mathbb{Z}_{(\ell)}$-submodule $\mathcal{M}_{(\ell)} \subseteq M$ of M is a $\mathbb{Z}_{(\ell)}$-*lattice* in M, (for short, a lattice) if $\mathcal{M}_{(\ell)}$ is a free $\mathbb{Z}_{(\ell)}$-module, and it is ℓ-adically dense in M, and it satisfies the following equivalent conditions:

a) $M/\ell = \mathcal{M}_{(\ell)}/\ell$

b) $\mathcal{M}_{(\ell)}$ has a $\mathbb{Z}_{(\ell)}$-basis $\mathfrak{B}$ which is ℓ-adically independent in M.

c) The condition b) above is satisfied for every $\mathbb{Z}_{(\ell)}$-basis of $\mathcal{M}_{(\ell)}$.

More general, let $N \subseteq \mathcal{M}_{(\ell)} \subseteq M$ be $\mathbb{Z}_{(\ell)}$-submodules of M such that M/N is again ℓ-adically complete. We will say that $\mathcal{M}_{(\ell)}$ is an *N-lattice* in M, if $N \subseteq \mathcal{M}_{(\ell)}$, and $\mathcal{M}_{(\ell)}/N$ is a lattice in M/N.

4) Finally, in the context from 3) above, a *true lattice* in M is a free Abelian subgroup $\mathcal{M}$ of M such that $\mathcal{M}_{(\ell)} := \mathcal{M} \otimes \mathbb{Z}_{(\ell)}$ is a lattice in M in the sense of 3) above. And we will say that a $\mathbb{Z}$-submodule $\mathcal{M} \subseteq M$ containing N is a *true N-lattice* in M, if $\mathcal{M}/N$ is a true lattice in M/N.

Construction 3.4. Let $\mathcal{G} = \big(G, (Z_v)_v\big)$ be a pre-divisorial Galois formation which is both *ample up to level δ* and *δ-residually complete curve like* ($\delta \geq 0$ given). Recall the last exact sequence from point 3) from Definition/Remark 3.2:

$$0 \to \widehat{U}_{\mathcal{G}} \hookrightarrow \widehat{\mathcal{L}}_{\mathcal{G}} \xrightarrow{\jmath^{\mathcal{G}}} \widehat{\mathrm{Div}}_{\mathcal{G}} \xrightarrow{\mathrm{can}} \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0\,.$$

The aim of this subsection is to describe the ℓ-adic equivalence class of a lattice $\mathrm{Div}_{\mathcal{G}}$ in $\widehat{\mathrm{Div}}_{\mathcal{G}}$, which —in the case it exists— will be called the *abstract divisor group* of $\mathcal{G}$. By construction, this will be equivalent to giving the equivalence class of a $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$, which will then be exactly the pre-image of $\mathrm{Div}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

The case $\delta = 0$, i.e., $\mathcal{G}$ complete curve like.

In the notations from Definition/Remark 3.2, 5) above, let $\mathfrak{T} = (\tau_v)_v$ be the system of generators of the groups T_v as there. Let as call such a system a *distinguished system of inertia generators.* We remark that any two distinguished system of inertia generators are *strictly ℓ-adically equivalent.* Indeed, if $\tau'_v \in T_v$ is another generator of T_v, then $\tau'_v = \tau_v^{-\epsilon_v}$ for some ℓ-adic units $\epsilon_v \in \mathcal{Z}$. Therefore, if $\mathfrak{T}' = (\tau'_v)_v$ does also satisfy condition ii) from Definition/Remark 3.2, 5), then we have also $\prod_v \tau_v^{-\epsilon_v} = 1$. By the uniqueness of the relation $\prod_v \tau_v = 1$, it follows that $\epsilon_v = \epsilon$ for some fixed ℓ-adic unit $\epsilon \in \mathcal{Z}$, as claimed.

Now let $\mathfrak{T} = (\tau_v)_v$ be a distinguished system generators of T. Further let $\mathcal{F}_{\mathfrak{T}}$ be the Abelian pro-ℓ free group on the system $\mathfrak{T}$ (written multiplicatively). Then one has a canonical exact sequence of pro-ℓ groups

$$1 \to \tau^{\mathcal{Z}} \to \mathcal{F}_{\mathfrak{T}} \to T \to 1\,,$$

where $\tau = \prod_v \tau_v$ in $\mathcal{F}_{\mathfrak{T}}$ is the pro-ℓ sum of the generators τ_v (all v). Taking ℓ-adically continuous Homs we get an exact sequence

$$0 \to \mathrm{Hom}\big(T, \mathcal{Z}\big) \to \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}\big) \to \mathrm{Hom}\big(\tau^{\mathcal{Z}}, \mathcal{Z}\big) \to 0.$$

Remark that $\mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}\big) \cong \widehat{\mathrm{Div}}_{\mathcal{G}}$ in a canonical way, and the last homomorphism is simply the summation: $\varphi \mapsto \big(\tau \mapsto \sum_v \varphi(\tau_v)\big)$. Thus $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}\big)$ consists of all the homomorphisms $\varphi \in \mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$ with trivial "trace".

Consider the system $\mathfrak{B} = (\varphi_v)_v$ of all the functionals $\varphi_v \in \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}\big)$ defined by $\varphi_v(\tau_w) = 1$ if $v = w$, and 0, otherwise (all v, w). We denote by

$$\mathrm{Div}_{\mathfrak{T},(\ell)} = <\mathfrak{B}>_{(\ell)} \subset \mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$$

the $\mathbb{Z}_{(\ell)}$-submodule of $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$ generated by $\mathfrak{B}$. Then $\mathfrak{B}$ *is an ℓ-adic basis* of $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$, i.e., $\mathrm{Div}_{\mathfrak{T},(\ell)}$ is ℓ-adically dense in $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$, and there are no non-trivial ℓ-adic relations between the elements of $\mathfrak{B}$. We will say that $\mathfrak{B} = (\varphi_v)_v$ is the "dual basis" to $\mathfrak{T}$. We next set

$$\mathrm{Div}^0_{\mathfrak{T},\ell} := \{\textstyle\sum_v a_v\,\varphi_v \in \mathrm{Div}_{\mathfrak{T},(\ell)} \mid \sum_v a_v = 0\} = \mathrm{Div}_{\mathfrak{T},(\ell)} \cap \mathrm{Hom}\big(T, \mathcal{Z}\big).$$

Clearly, if φ_{v_0} is fixed, then the system $e_v = \varphi_v - \varphi_{v_0}$ (all $v \neq v_0$) is an ℓ-adically independent $\mathbb{Z}_{(\ell)}$-basis of $\mathrm{Div}^0_{\mathfrak{T},(\ell)}$.

Thus finally, $\mathrm{Div}_{\mathfrak{T},(\ell)}$ is a lattice in $\mathrm{Hom}\big(T, \mathcal{Z}\big)$, and $\mathrm{Div}^0_{\mathfrak{T},(\ell)}$ is a lattice in $\mathrm{Hom}\big(T, \mathcal{Z}\big)$.

The dependence of $\mathrm{Div}_{\mathfrak{T},(\ell)}$ on $\mathfrak{T} = (\tau_v)_v$ is as follows. Let $\mathfrak{T}' = (\tau'_v)_v = \mathfrak{T}^\epsilon$ be another distinguished system of inertia generators. If $\mathfrak{B}' = (\varphi'_v)_v$ is the dual basis to $\mathfrak{T}'$, then $\epsilon \cdot \mathfrak{B}' = \mathfrak{B}$. Thus $\mathfrak{B}$ and $\mathfrak{B}'$ are ℓ-adically equivalent, and we have: $\mathrm{Div}_{\mathfrak{T},(\ell)} = \epsilon \cdot \mathrm{Div}_{\mathfrak{T}',(\ell)}$ and $\mathrm{Div}^0_{\mathfrak{T},(\ell)} = \epsilon \cdot \mathrm{Div}^0_{\mathfrak{T}',(\ell)}$.

Therefore, all the subgroups of $\mathrm{Hom}\big(T^{\mathrm{ab}}, \mathcal{Z}\big)$ the form $\mathrm{Div}_{\mathfrak{T},(\ell)}$, respectively of the form $\mathrm{Div}^0_{\mathfrak{T},(\ell)}$, are ℓ-adically equivalent (all distinguished $\mathfrak{T}$). Hence the ℓ-adic equivalence classes of $\mathrm{Div}_{\mathfrak{T},(\ell)}$ and $\mathrm{Div}^0_{\mathfrak{T},(\ell)}$ do not depend on $\mathfrak{T}$, but only on $\mathcal{G}$.

Fact 3.5. In the above context, denote by $\mathcal{L}_{\mathfrak{T},(\ell)}$ the pre-image of $\mathrm{Div}^0_{\mathfrak{T},(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Further consider all the finite co-rank submodules Δ of $\widehat{\mathcal{L}}_{\mathcal{G}}$ containing $\widehat{U}_{\mathcal{G}}$. Then the following assertions are equivalent:

(i) $\mathcal{L}_{\mathfrak{T},(\ell)}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

(ii) $\Delta \cap \mathcal{L}_{\mathfrak{T},(\ell)}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in Δ (all Δ as above).

Moreover, if (i), (i), are satisfied, then $\jmath^v(\mathcal{L}_{\mathfrak{T},(\ell)}) = \mathbb{Z}_{(\ell)}\varphi_v$ (all v).

Proof. Clear. □

Definition 3.6. In the context of Fact above, suppose that the equivalent conditions (i), (ii) are satisfied. Then we define $\mathrm{Div}_{\mathcal{G}}$ to be any of the lattices $\mathrm{Div}_{\mathfrak{T},(\ell)} \subset \widehat{\mathrm{Div}}_{\mathcal{G}}$, and call it an *abstract divisor group* of $\mathcal{G}$.

We further say that $\mathrm{Div}^0_{\mathfrak{T},(\ell)}$ is the divisor group of degree 0 in $\mathrm{Div}_{\mathfrak{T},(\ell)}$. And remark that any two abstract divisor groups $\mathrm{Div}_{\mathcal{G}}$ and $\mathrm{Div}'_{\mathcal{G}}$ are ℓ-equivalent latices in $\widehat{\mathrm{Div}}_{\mathcal{G}}$, and the same is true for $\mathrm{Div}^0_{\mathcal{G}}$ and $\mathrm{Div}'^0_{\mathcal{G}}$.

The case: $\delta > 0$.

We begin by mimicking the construction from the case $\delta = 0$, and by induction on δ conclude the construction. Thus let $\mathfrak{T} = (\tau_v)_v$ be any system generators for the inertia groups T_v (all v). Further let $\mathcal{F}_{\mathfrak{T}}$ be the Abelian pro-ℓ free group on the system $\mathfrak{T}$ (written multiplicatively). Then T is a quotient $\mathcal{F}_{\mathfrak{T}} \to T \to 1$. Thus taking ℓ-adic Homs we get an exact sequence

$$0 \to \mathrm{Hom}\big(T, \mathcal{Z}\big) \to \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}\big)\,,$$

and remark that $\mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}\big) \cong \widehat{\mathrm{Div}}_{\mathcal{G}}$ in a canonical way. Next consider the system $\mathfrak{B} = (\varphi_v)_v$ of all the functionals $\varphi_v \in \mathrm{Hom}\big(\mathcal{F}_{\mathfrak{T}}, \mathcal{Z}\big)$ defined by $\varphi_v(\tau_w) = 1$ if $v = w$, and 0, otherwise (all v, w). We denote by

$$\mathrm{Div}_{\mathfrak{T},(\ell)} = <\mathfrak{B}>_{(\ell)} \subset \mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$$

the $\mathbb{Z}_{(\ell)}$-submodule of $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$ generated by $\mathfrak{B}$. Then $\mathfrak{B}$ *is an ℓ-adic basis* of $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$, i.e., $\mathrm{Div}_{\mathfrak{T},(\ell)}$ is ℓ-adically dense in $\mathrm{Hom}\big(\mathcal{F}, \mathcal{Z}\big)$, and there are no non-trivial ℓ-adic relations between the elements of $\mathfrak{B}$. We will call $\mathfrak{B} = (\varphi_v)_v$ the "dual basis" to $\mathfrak{T}$. Thus finally, $\mathrm{Div}_{\mathfrak{T},(\ell)}$ is a lattice in $\mathrm{Hom}\big(T, \mathcal{Z}\big)$.

Now let $\mathfrak{T}' = (\tau'_v)_v$ be another system of inertia generators. And suppose that for some $\epsilon \in \mathcal{Z}$ we have $\mathfrak{T}' = \mathfrak{T}^{\epsilon}$. If $\mathfrak{B} = (\varphi'_v)_v$ is the dual basis to $\mathfrak{T}'$, then $\epsilon\,\varphi'_v = \varphi_v$ inside $\mathrm{Hom}\big(T, \mathcal{Z}\big)$. Thus $\epsilon \cdot \mathfrak{B}' = \mathfrak{B}$. In other words, $\mathfrak{B}$ and $\mathfrak{B}'$ are ℓ-adically equivalent. And we have: $\mathrm{Div}_{\mathfrak{T},(\ell)} = \epsilon \cdot \mathrm{Div}_{\mathfrak{T}',(\ell)}$.

Finally, we fix notations as follows:

For a system of inertia generators $\mathfrak{T}$ as above, and the corresponding lattice $\mathrm{Div}_{\mathfrak{T},(\ell)}$ in $\widehat{\mathrm{Div}}_{\mathcal{G}}$, let $\mathcal{L}_{\mathfrak{T},(\ell)}$ be its pre-image in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Next, for every v we denote by $\mathcal{G}_v$ the corresponding v-residual pre-divisorial Galois formation, etc.. In particular, $\mathcal{G}_v$ is both $(\delta-1)$-residually complete curve like, and ample up to level $(\delta - 1)$. Now let Δ be a finite co-rank $\mathcal{Z}$-submodule of $\widehat{\mathcal{L}}_{\mathcal{G}}$ such that $\Delta \cap \widehat{U}_{\mathcal{G}} = 1$. For every v such that $\Delta \subset \widehat{U}_v$, we set $\Delta_v = \jmath_v(\Delta)$. Since $\mathcal{G}$ is *ample up to level* δ, there exists some v such that:

$(*)$ $\Delta \subset \widehat{U}_v$, and Δ_v has finite co-rank in $\widehat{\mathcal{L}}_{\mathcal{G}_v}$, and $\Delta_v \cap \widehat{U}_{\mathcal{G}_v} = 1$. In particular, Δ and Δ_v have the same $\mathcal{Z}$-rank.

FACT 3.7. For every v and the corresponding $\mathcal{G}_v$, let a $\widehat{U}_{\mathcal{G}_v}$-lattice $\mathcal{L}_{v,(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}_v}$ be given. Then up to ℓ-adic equivalence, there exits at most one $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ such that for every finite co-rank $\mathcal{Z}$-module $\Delta \subset \widehat{\mathcal{L}}_{\mathcal{G}}$ with $\Delta \cap \widehat{U}_{\mathcal{G}} = 1$ and v as at $(*)$ above the following hold:

i) $\mathcal{L}_{\Delta} := \Delta \cap \mathcal{L}_{(\ell)}$, $\mathcal{L}_{\Delta_v} := \Delta_v \cap \mathcal{L}_{v,(\ell)}$ are lattices in Δ, respectively Δ_v.

ii) The lattices $\jmath_v(\mathcal{L}_{\Delta})$ and $\mathcal{L}_{\Delta_v}$ are ℓ-adically equivalent in Δ_v.

Moreover, if the $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{(\ell)}$ exists, then its ℓ-adic equivalence class does depend only on the ℓ-adic equivalence classes of the $\widehat{U}_{\mathcal{G}_v}$-lattices $\mathcal{L}_{v,(\ell)}$ (all v).

Proof. Let $\mathcal{L}_{(\ell)}, \mathcal{L}'_{(\ell)}$ be $\widehat{U}_{\mathcal{G}}$-lattices in $\widehat{\mathcal{L}}_{\mathcal{G}}$ satisfying i), ii), above. For $\Delta \neq 0$ as in Fact above, set $\mathcal{L}'_{\Delta} = \Delta \cap \mathcal{L}'_{(\ell)}$. Then by hypothesis ii) it follows that both lattices $\jmath_v(\mathcal{L}_{\Delta})$ and $\jmath_v(\mathcal{L}'_{\Delta})$ are ℓ-adically equivalent to the lattice $\mathcal{L}_{\Delta_v}$ inside Δ_v. Thus they are equivalent. After replacing $\mathcal{L}'_{(\ell)}$ by some multiple $\epsilon \cdot \mathcal{L}'_{(\ell)}$ with $\epsilon \in \mathbb{Z}_{\ell}^{\times}$, we can suppose that $\mathcal{L}_{\Delta} = \mathcal{L}'_{\Delta} \neq 0$. Now consider some arbitrary Δ' containing Δ. In the notations from above -correspondingly- it follows that $\jmath_{v'}(\mathcal{L}_{\Delta'})$ and $\jmath_{v'}(\mathcal{L}'_{\Delta'})$ are ℓ-adically equivalent to the lattice $\mathcal{L}_{\Delta'_{v'}}$ inside $\Delta'_{v'}$. Since both contain $\jmath_{v'}(\mathcal{L}_{\Delta}) = \jmath_{v'}(\mathcal{L}'_{\Delta})$, we therefore must have $\jmath_{v'}(\mathcal{L}_{\Delta'}) = \jmath_{v'}(\mathcal{L}'_{\Delta'})$. Therefore, $\mathcal{L}_{\Delta'} = \mathcal{L}'_{\Delta'}$. Thus finally, as Δ' was arbitrary, it follows $\mathcal{L}_{(\ell)} = \mathcal{L}'_{(\ell)}$. □

Definition 3.8. Let $\mathcal{G}$ be a pre-divisorial Galois formation which is both δ-residual complete curve like and ample up to level $\delta > 0$. We define an abstract divisor group $\mathrm{Div}_{\mathcal{G}} \subset \widehat{\mathrm{Div}}_{\mathcal{G}}$ —if it exists— inductively as follows: First, suppose that for all v, an abstract divisor group $\mathrm{Div}_{\mathcal{G}_v}$ for $\mathcal{G}_v$ does exist. In particular, we have by definition, see hypothesis i) below: $\mathrm{Div}_{\mathcal{G}_v}$ is a lattice in $\widehat{\mathrm{Div}}_{\mathcal{G}_v}$, and its pre-image $\mathcal{L}_{v,(\ell)}$ via $\jmath^{\mathcal{G}_v}$ is a $\widehat{U}_{\mathcal{G}_v}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}_v}$.

Then an *abstract divisor group* of $\mathcal{G}$ —if it exists— is any lattice of the form $\mathrm{Div}_{\mathcal{G}} = \mathrm{Div}_{\mathfrak{T},(\ell)}$ in $\widehat{\mathrm{Div}}_{\mathcal{G}}$ such that its pre-image $\mathcal{L}_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ satisfies the following conditions:

j) $\mathcal{L}_{(\ell)}$ is a $\widehat{U}_{\mathcal{G}}$-lattice in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

jj) $\mathcal{L}_{(\ell)}$ satisfies the conditions i), ii) from Fact 3.7.

Remark/Fact 3.9. Let $\mathcal{G}$ be a pre-divisorial Galois formation which is both δ-residual complete curve like and ample up to level $\delta > 0$. Suppose that abstract divisor groups $\mathrm{Div}_{\mathcal{G}} = \mathrm{Div}_{\mathfrak{T},(\ell)}$ for $\mathcal{G}$ do exist. For such an abstract divisor group $\mathrm{Div}_{\mathcal{G}}$, let $\mathcal{L}_{(\ell)}$ be its pre-image in $\widehat{\mathcal{L}}_{\mathcal{G}}$. Then for all v one has:

$$\jmath^v(\mathcal{L}_{(\ell)}) = \mathbb{Z}_{(\ell)}\,\varphi_v\,.$$

Indeed, by condition i) of the ampleness, see Definition/Remark 3.2, 7), it follows that $\jmath^v(\widehat{\mathcal{L}}_{\mathcal{G}}) = \mathbb{Z}_{\ell}\,\varphi_v$. Further, since $\mathcal{L}_{(\ell)}$ is ℓ-adically dense in $\widehat{\mathcal{L}}_{\mathcal{G}}$, it follows that $\jmath^v(\mathcal{L}_{(\ell)})$ is dense in $\mathbb{Z}_{\ell}\,\varphi_v$. Thus the assertion.

2) In particular, the $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{(\ell)}$ determines $\mathrm{Div}_{\mathcal{G}}$, as being the additive subgroup

$$\mathrm{Div}_{\mathcal{G}} = \oplus_v\, \jmath^v(\mathcal{L}_{(\ell)})$$

Therefore, giving an abstract divisor group $\mathrm{Div}_{\mathcal{G}}$, is equivalent to giving a $\widehat{U}_{\mathcal{G}}$-lattice $\mathcal{L}_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ such that:

j) $\mathcal{L}_{(\ell)}$ satisfies the conditions i), ii) from Fact 3.7 with respect to the pre-images $\mathcal{L}_{v,(\ell)}$ of some abstract divisor groups $\mathrm{Div}_{\mathcal{G}_v}$ (all v).

jj) $\jmath^v(\mathcal{L}_{(\ell)}) \cong \mathbb{Z}_{(\ell)}$ (all v).

3) Finally, for an abstract divisor group $\mathrm{Div}_{\mathcal{G}}$ for $\mathcal{G}$, and its pre-image $\mathcal{L}_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$, we set $\mathfrak{Cl}_{\mathcal{L}_{(\ell)}} = \mathrm{Div}_{\mathcal{G}}/\jmath^{\mathcal{G}}(\mathcal{L}_{(\ell)})$, and call it the abstract ideal class group of $\mathcal{L}_{(\ell)}$. Thus one has a commutative diagram of the form

$$(*)\qquad \begin{array}{ccccccc} 0 \to \widehat{U}_{\mathcal{G}} & \hookrightarrow & \mathcal{L}_{(\ell)} & \xrightarrow{\jmath^{\mathcal{G}}} & \mathrm{Div}_{\mathcal{L}_{(\ell)}} & \xrightarrow{\mathrm{can}} & \mathfrak{Cl}_{\mathcal{L}_{(\ell)}} \to 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \jmath_{\mathfrak{Cl}}} \\ 0 \to \widehat{U}_{\mathcal{G}} & \hookrightarrow & \widehat{\mathcal{L}}_{\mathcal{G}} & \xrightarrow{\jmath^{\mathcal{G}}} & \widehat{\mathrm{Div}}_{\mathcal{G}} & \xrightarrow{\mathrm{can}} & \widehat{\mathfrak{Cl}}_{\mathcal{G}} \to 0 \end{array}$$

where the first three vertical morphisms are the canonical inclusions, and the last one is the ℓ-adic completion homomorphism.

PROPOSITION 3.10. *Let $\mathcal{G}$ be a pre-divisorial Galois formation which is both δ-residual complete curve like and ample up to level $\delta > 0$. Then any two abstract divisor groups $\mathrm{Div}_{\mathcal{G}}$ and $\mathrm{Div}'_{\mathcal{G}}$ for $\mathcal{G}$ are ℓ-adically equivalent as lattices in $\widehat{\mathrm{Div}}_{\mathcal{G}}$. Or equivalently, their pre-images $\mathcal{L}_{(\ell)}$ and $\mathcal{L}'_{(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$ are ℓ-adically equivalent $\widehat{U}_{\mathcal{G}}$-lattices in $\widehat{\mathcal{L}}_{\mathcal{G}}$.*

Proof. We prove this assertion by induction on δ. For $\delta = 0$, the uniqueness is already shown, see Fact 3.5, and Definition in case $\delta = 0$. Now suppose that $\delta > 0$. Let $\mathrm{Div}_{\mathcal{G}_v}$ and $\mathrm{Div}'_{\mathcal{G}_v}$ be abstract divisor groups for $\mathcal{G}$ used for the definition of $\mathrm{Div}_{\mathcal{G}}$, respectively $\mathrm{Div}'_{\mathcal{G}}$ (all v). By the induction hypothesis, $\mathrm{Div}_{\mathcal{G}_v}$ and $\mathrm{Div}'_{\mathcal{G}_v}$ are ℓ-adically equivalent. Thus their pre-images $\mathcal{L}_{v,(\ell)}$ and $\mathcal{L}'_{v,(\ell)}$ in $\widehat{\mathcal{L}}_{\mathcal{G},v}$ are ℓ-adically equivalent $\widehat{U}_{\mathcal{G},v}$-lattices. Therefore, by Fact 3.7, the lattices $\mathcal{L}_{(\ell)}$ and $\mathcal{L}'_{(\ell)}$ (which are the pre-images of $\mathrm{Div}_{\mathcal{G}}$ respectively $\mathrm{Div}'_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$) are ℓ-adically equivalent. Finally, use 2) above to conclude. □

DEFINITION 3.11. A level $\delta > 0$ *divisorial Galois formation* is by definition every pre-divisorial Galois formation $\mathcal{G}$ which is both $(\delta - 1)$-residually complete curve like and ample up to level $(\delta - 1)$ and has abstract divisor groups $\mathrm{Div}_{\mathcal{G}}$. If this is the case, we will denote by $\mathcal{L}_{(\ell)}$ the pre-image of $\mathrm{Div}_{\mathcal{G}}$ in $\widehat{\mathcal{L}}_{\mathcal{G}}$, and call it a *divisorial $\widehat{U}_{\mathcal{G}}$-lattice* in $\widehat{\mathcal{L}}_{\mathcal{G}}$.

C) *Example: Geometric Galois formation*

Let ℓ be a prime number. Let $\mathfrak{K}|\mathfrak{k}$ be a field extension with $\mathfrak{k}$ an algebraically closed field, $\mathrm{char}(\mathfrak{k}) \neq \ell$. Consider a maximal Abelian pro-ℓ field extension $\mathfrak{K}' = \mathfrak{K}^{\ell\mathrm{ab}}$ of $\mathfrak{K}$, and set $G = \mathrm{Gal}(\mathfrak{K}'|\mathfrak{K})$. Then we get:

FACT 3.12. $\mathcal{G} = \big(G, \emptyset\big)$ is a level $\delta = 0$ pre-divisorial Galois formation.

Nevertheless, in the case $\mathfrak{K}|\mathfrak{k}$ is a *function field* with $d = \mathrm{td}(\mathfrak{K}|\mathfrak{k}) > 0$, we can refine the pre-Galois formation above by making it into a pre-divisorial Galois formation of level δ for every $0 \leq \delta \leq d$ as follows: First, let us endow every function field $\mathfrak{K}|\mathfrak{k}$ as above with a geometric set $\mathfrak{D} = \mathfrak{D}_X$ of Zariski prime divisors of $\mathfrak{K}|\mathfrak{k}$. Here $X \to \mathfrak{k}$ is a quasi-projective normal model of $\mathfrak{K}|\mathfrak{k}$, and $\mathfrak{D}_X$ is its set of Weil prime divisors. Second, since $\mathfrak{K}'|\mathfrak{K}$ is Abelian, if v' is a prolongation of $v \in \mathfrak{D}$ to K', then $T_{v'} \subset Z_{v'}$ do not depent on v', but on v.

Then one can endow G with the family $(Z_v)_v$ of the decomposition groups of all the $v \in \mathfrak{D}$. Clearly, if $T_v \subset Z_v$ is the inertia group at v, then $T_v \cong \mathbb{Z}_\ell$ is pro-ℓ cyclic. Further, if $v \neq w$, then $T_v \cap T_w = \{1\}$. Finally, let U_i be a basis of (small enough) affine open subsets in X, and for every i, set $\mathfrak{U}_i = U_i \cap \mathfrak{D}$. Then by general facts about geometric fundamental groups, it follows that $(\mathfrak{U}_i)_i$ is a co-final system in the sense of Axiom II), ii), from Subsection A). Thus finally, we have a pre-divisorial Galois formation supported by G as follows:

$$\mathcal{G}_{\mathfrak{D}} := \big(G, (Z_v)_v\big) =: \mathcal{G}_X$$

Moreover, for every $v \in \mathfrak{D}$ we have: The residue field $\mathfrak{K}v$ is again a function field over $\mathfrak{k}$ with $\mathrm{td}(\mathfrak{K}v \mid \mathfrak{k}) = (d-1)$. Further, by general decomposition theory, the residue field extension $\mathfrak{K}'v \mid \mathfrak{K}v$ is a maximal Abelian pro-ℓ extension of $\mathfrak{K}v$.

Thus if $d > 1$, each such function field $\mathfrak{K}v|\mathfrak{k}$ comes equipped with a geometric set of Zariski prime divisors $\mathfrak{D}_v = \mathfrak{D}_{X_v}$, where X_v a quasi-projective normal model of $\mathfrak{K}v \mid \mathfrak{k}$.

Now consider any δ with $0 \leq \delta \leq d$. Then by induction on the transcendence degree over $\mathfrak{k}$, we can suppose that each G_v endowed with the set of decomposition groups defined by $\mathfrak{D}_v$ is a pre-divisorial Galois formation $\mathcal{G}_{\mathfrak{D}_v}$ of level $(\delta - 1)$. Thus we have the following:

Proposition/Definition 3.13. *Let $\mathcal{G}_{\mathfrak{D}} = \big(G, (Z_v)_v\big)$ be as constructed above. Then $\mathcal{G}_{\mathfrak{D}}$ is in a canonical way a level δ pre-divisorial Galois formation, for every δ satisfying $0 \leq \delta \leq \mathrm{td}(\mathfrak{K}|\mathfrak{k})$.*

A pre-divisorial Galois formation of the form $\mathcal{G}_{\mathfrak{D}}$ will be called a geometric Galois formation *of level δ.*

Remarks 3.14. Let $\mathcal{G}_X = \big(G, (Z_v)_v\big)$ be a geometric Galois formation as constructed/defined above. Then $\widehat{\mathfrak{K}} = \mathrm{Hom}\big(G'_{\mathfrak{K}}, \mathcal{Z}\big) = \widehat{\mathcal{L}}_{\mathcal{G}_X}$ by Kummer Theory. In order to compute $\widehat{U}_{\mathcal{G}_X}$, $\widehat{\mathrm{Div}}_{\mathcal{G}_X}$, and $\widehat{\mathfrak{Cl}}_{\mathcal{G}_X}$, we do the following: First, let $\mathcal{H}(X)$ denote the group of principal divisors on X, and consider the canonical exact sequence $0 \to \mathcal{H}(X) \longrightarrow \mathrm{Div}(X) \xrightarrow{pr} \mathfrak{Cl}(X) \to 0$.

1) Passing to ℓ-adic completions, we get an exact sequence of ℓ-adic complete groups of the form: $0 \to \mathcal{T}_{\ell,X} \longrightarrow \widehat{\mathcal{H}}(X) \longrightarrow \widehat{\mathrm{Div}}(X) \longrightarrow \widehat{\mathfrak{Cl}}(X) \to 0$, where $\mathcal{T}_{\ell,X} = \varprojlim_{\ell^n} \mathfrak{Cl}(X)$ (with multiplication as homomorphisms), and the last three objects the corresponding ℓ-adic completions.

2) On the other hand one has $\mathrm{Div}(X) = \oplus_{v \in \mathfrak{D}_X} v\mathfrak{K}$. For every v, consider the commutative diagram from subsection A), 3). We get a commutative

diagram of the following form:

$$\begin{array}{ccccccc}
\widehat{\mathfrak{K}} & & \to \widehat{\mathrm{Div}}(X) = \widehat{\oplus_v}\, v\mathfrak{K} & \to & \widehat{\mathfrak{Cl}}\,(X) & \to & 0 \\
\big\downarrow{\scriptstyle\hat\delta} & & \big\downarrow{\scriptstyle\oplus\theta^v} & & \big\downarrow & & \\
\mathrm{Hom}\big(G^{\ell}_{\mathfrak{K}}, \mathbb{Z}_\ell\big) & \xrightarrow{\jmath^{\mathcal{G}_X}} & \widehat{\oplus_v}\,\mathrm{Hom}\big(T_v, \mathbb{Z}_\ell\big) & \xrightarrow{\mathrm{can}} & \widehat{\mathcal{P}}_X & \to & 0
\end{array}$$

where the vertical maps are isomorphisms, and $\widehat{\mathcal{P}}_X$ is simply the quotient of the middle group by the second. We remark/recall that $\oplus\theta^v$ is defined as follows: For every v, let $\gamma_v = 1 \cdot v$ be the unique positive generator of $v\mathfrak{K} = \mathbb{Z}v$. Then there exists a unique generator of the arithmetical inertia $\tau_v \in \mathfrak{T}_v$ such that $\jmath_v(\gamma_v)\big(\tau_v\big) = 1$. Hence, the commutativity of the diagram above follows directly from the definition of the homomorphisms.

3) From this we deduce: $\widehat{\mathrm{Div}}_{\mathcal{G}_X} = \widehat{\mathrm{Div}}(X)$ and $\widehat{\mathfrak{Cl}}_{\mathcal{G}_X} = \widehat{\mathfrak{Cl}}(X)$.

4) Concerning $\widehat{U}_{\mathcal{G}_X}$, we recall that $\widehat{U}_{\mathcal{G}_X} = \mathrm{Hom}\big(\pi_1^{\ell,\mathrm{ab}}(X), \mathbb{Z}_\ell\big)$, see Definition/Remark 4.2, 2). Let $T \subset G$ be the closed subgroup generated by the decomposition groups of all the divisorial valuations of $\mathfrak{K}$. We set $\pi_{1,\mathfrak{K}}^{\ell,\mathrm{ab}} = G/T$, and call it the *fundamental group to* $\mathfrak{K}$, as it is a birational invariant of $\mathfrak{K}$. It equals the fundamental group of any complete regular model $X_0 \to \mathfrak{k}$ of $\mathfrak{K}|\mathfrak{k}$, if any such do exist...

We will say that a normal model $X \to \mathfrak{k}$ for $\mathfrak{K}|\mathfrak{k}$ is *regular complete like* if $\pi_1^{\ell,\mathrm{ab}}(X) = \pi_{1,\mathfrak{K}}^{\ell,\mathrm{ab}}$. From the structure theorem for (the Abelian pro-ℓ quotient of the) fundamental groups of a normal curve it follows that such a curve is regular complete like if and only if it is a complete one.

Proposition 3.15. *Let $\mathfrak{K}|\mathfrak{k}$ and a geometric Galois formation $\mathcal{G}_X$ be as introduced/defined in Proposition above. Consider some $\delta \le \mathrm{td}(\mathfrak{K}|\mathfrak{k})$, and let us view $\mathcal{G}_X$ as a pre-divisorial Galois formation of level δ. We denote by $\mathcal{G}_{X_\mathbf{v}}$ the residual Galois formations of $\mathcal{G}_X$ with $\mathbf{v}$ multi-indices of length $\le \delta$. In particular, $X_\mathbf{v} \to \mathfrak{k}$ are quasi-projective normal varieties. Then one has:*

(1) *$\mathcal{G}_X$ is δ-residually complete curve like if (and only if) $\delta = \mathrm{td}(\mathfrak{K}|\mathfrak{k}) - 1$, and all the δ-residual varieties $X_\mathbf{v}$ are complete normal curves.*

(2) *Suppose that $\delta < \mathrm{td}(\mathfrak{K}|\mathfrak{k})$. Further suppose that $X_\mathbf{v}$ is regular complete like (all $\mathbf{v}$ as above). Then $\mathcal{G}_X$ is ample up to level δ.*

(3) *Suppose that $\mathcal{G}_X$ is δ-residually curve like, and ample up to level δ. Then $\mathrm{Div}(X)_{(\ell)} := \mathrm{Div}(X) \otimes \mathbb{Z}_{(\ell)}$ inside $\widehat{\mathrm{Div}}(X)$ is an abstract divisor group of $\mathcal{G}_X$. We will denote by $\mathcal{L}_{X,(\ell)}$ its pre-image in $\widehat{\mathfrak{K}}$.*

Thus for every geometric Galois formation $\mathcal{G}_X$ as above, $\mathrm{Div}(X)_{(\ell)}$ is an abstract divisor group of $\mathcal{G}_X$, and $\mathcal{L}_{X,(\ell)}$ is a divisorial $\widehat{U}_{\mathcal{G}}$-lattice of $\mathcal{G}_X$.

Proof. To (1): Clear. It is nevertheless more/quite difficult to prove the "only if" part of (1), which we will not directly use, thus omit the proof here...

To (2): We make induction on $d = \mathrm{td}(\mathfrak{K}|\mathfrak{k})$. In the notations from Definition/Remarks 3.2, 3), let $\Delta \subset \widehat{\mathfrak{K}}$ be a co-finite rank submodule such that $\Delta \cap \widehat{U}_{\mathcal{G}_X} = 0$. Then there exists an open affine subset $X' \subset X$ such that for all $v \in \mathfrak{D}_{X'}$ one has: $\Delta \subset \widehat{U}_v$. In particular, the canonical projection $\pi_1(X') \to \pi_1(X)$ gives rise to an embedding $\widehat{U}(X) \hookrightarrow \widehat{U}(X')$, and $\Delta \subset \widehat{U}(X')$. Using DE JONGS's alterations, and the inclusion-norm maps, we can suppose that X is a smooth $\mathfrak{k}$-variety. Finally, let $X_1, \dots, X_n$ be the finitely many Weil prime divisors in $X \backslash X'$, and $S := \{x_1, \dots, x_n\}$ their generic points. Thus the inertia groups at these finitely many points generate $T_{X',X} := \ker\big(\pi_1(X') \to \pi_1(X)\big)$.

Now let $\imath : X \to \mathbb{P}^N_{\mathfrak{k}}$ be some $\mathfrak{k}$-embedding. For a general hyper-plane $H \subset \mathbb{P}^N_{\mathfrak{k}}$, we set $Y = H \cap X$, $Y' = H \cap X'$, $Y_i = X_i \cap H$ (all i). Then by Bertini, each Y_i is a prime divisor of Y. Let $T := \{y_1, \dots, y_n\}$ is the set of their generic points of the Y_i (all i). Then by Bertini we have:

i) The canonical projections $\pi_1(Y) \to \pi_1(X)$ and $\pi_1(Y') \to \pi_1(X')$ are surjective.

ii) Set $T_{Y',Y} := \ker\big(\pi_1(Y') \to \pi_1(Y)\big)$. Then $T_{Y',Y}$ is generated by the inertia groups at all the y_i (all i).

iii) Finally, $T_{Y',Y}$ is mapped surjectively onto $T_{X',X}$ under the projection $\pi_1(Y') \to \pi_1(X')$ from i) above.

This is now exactly the translation of the fact that for the Zariski prime divisor v of $\mathfrak{K}$ defined by the Weil prime divisor $Y = H \cap X$ of X, the assertion from Definition/Remarks 3.2, 3), holds for Δ at v.

To (3): It follows immediately from (1) and (2) above. $\square$

D) *The case k is the algebraic closure of a finite field*

In this subsection we will discuss the case $K|k$ is a function field with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite field. Let $G^\ell_K = \mathrm{Gal}(K^\ell|K)$ be as usually defined. The aim of this subsection is to show that the system of all the geometric Galois formations $\mathcal{G}_X$ are actually group theoretically encoded in G^ℓ_K. Moreover, the extra information concerning such a pre-divisorial Galois formation, e.g., δ-residually complete curve like, and/or ample up to level δ, is also encoded in the G^ℓ_K. First some general preparation as follows:

FACT 3.16. Let $\mathfrak{K}|\mathfrak{k}$ be some field extension with $\mathfrak{k}$ algebraically closed, $\mathrm{char}(\mathfrak{K}) \neq \ell$. Let $\mathfrak{K}^\ell$ and $G^\ell_{\mathfrak{K}}$ as usually define. Denote by

$$G^\ell_{\mathfrak{K}} \to G^{\ell,\mathrm{ab}}_{\mathfrak{K}} =: G$$

the canonical projection. For every subgroup $(\cdot)^\ell$ of $G^\ell_{\mathfrak{K}}$ we will denote by $(\cdot)^{\ell,\mathrm{ab}}$ the image of $(\cdot)$ in $G^{\ell,\mathrm{ab}}_{\mathfrak{K}}$. By general decomposition theory we have:

1) For every valuation v^ℓ of $\mathfrak{K}^\ell$, and its inertia, respectively decomposition groups $T^\ell_v \subset Z^\ell_v$ in $G^\ell_{\mathfrak{K}}$ one has: $T^{\ell,\mathrm{ab}}_v$ and $Z^{\ell,\mathrm{ab}}_v$ are exactly the inertia, respectively decomposition groups of v^ℓ in $G^{\ell,\mathrm{ab}}_{\mathfrak{K}}$.

Second, the groups of the form $T^{\ell,\mathrm{ab}}_v$ and $Z^{\ell,\mathrm{ab}}_v$ are exactly all the inertia, respectively decomposition groups in $G^{\ell,\mathrm{ab}}_{\mathfrak{K}}$.

2) Further remark that $T^{\ell,\mathrm{ab}}_v$ and $Z^{\ell,\mathrm{ab}}_v$ do depend only on the restriction v of v^ℓ on $\mathfrak{K}$, and not on v^ℓ itself (which is one of the prolongations of v to $\mathfrak{K}^\ell$).

3) $T^\ell_v \to T^{\ell,\mathrm{ab}}_v$ is an isomorphism, and second, $G^{\ell,\mathrm{ab}}_{Kv}$ is canonically isomorphic to $Z^{\ell,\mathrm{ab}}_v / T^{\ell,\mathrm{ab}}_v$.

4) The Theorem of F. K. Schmidt from Pop [P], Proposition 1.3, holds for the Galois extension $\mathfrak{K}^{\ell,\mathrm{ab}}|\mathfrak{K}$. In particular, if two valuations v and w equal their cores respectively, then $Z^{\ell,\mathrm{ab}}_v \cap Z^{\ell,\mathrm{ab}}_w \neq \{1\}$ if and only if $v = w$.

Fact 3.17. Let $K|k$ be a function field with k the algebraic closure of a finite field. *In order to simplify notations* let us denote $G := G^{\ell,\mathrm{ab}}_K$, and for every valuation v^ℓ and $T^\ell_v \subset Z^\ell_v$ as above, set $T_v = T^{\ell,\mathrm{ab}}_v$ and $Z_v = T^{\ell,\mathrm{ab}}_v$. Then we have the following:

1) $d = \mathrm{td}(K|k)$ is encoded in G^ℓ_K, by Section 2, A).

2) The divisorial inertia groups T_v and the divisorial decomposition groups Z_v in G are known, by Section 1.

The same is true for all the defectless inertia elements, by Section 1), Theorem 1.11.

For given such subgroups Z_v and Z_w we have: $Z_v \cap Z_w$ is non-trivial if and only if $v = w$.

3) If $\pi^\ell_{1,K} = G^\ell_K / T^\ell_K$ is the fundamental group to K, then $\pi_{1,K} := \pi^{\ell,\mathrm{ab}}$ is also encoded in G^ℓ_K, by 2) above.

4) Furthermore, the geometric sets of divisorial decomposition groups, say $\mathcal{Z} = \{Z_v\}_v$ of G are encoded in $G^\ell_K \to G$, see Section 2, C).

5) Let $\mathfrak{D} = \mathfrak{D}_X$ be a geometric set of Zariski prime divisors, say defined by a set $\mathcal{Z} = \{Z_v\}_v$ of decomposition groups. Then the fact that X is complete regular like is encoded in G^ℓ_K and the set $\mathcal{Z} = \{Z_v\}_v$.

Therefore we finally have the following:

Proposition 3.18. *Let $K|k$ be a function field with* $\mathrm{td}(K|k) > 1$, *and k the algebraic closure of a finite field. Then in the notations from above we have the following: The geometric Galois formations $\mathcal{G}_{\mathfrak{D}}$, say $\mathfrak{D} = \mathfrak{D}_X$, of some level $\delta \leq$* $\mathrm{td}(K|k)$ *are encoded in G^ℓ_K along the lines indicated above.*

Moreover, the fact that such a geometric Galois formation $\mathcal{G}_{\mathfrak{D}}$ is δ-residually complete curve like and/or that all residual geometric Galois formations $\mathcal{G}_{\mathbf{v}}$

are regular complete like (*all* $\mathbf{v}$) *is also encoded in the group theoretic information carried by* $\mathcal{G}$.

Let $\mathcal{G}_{\mathfrak{D}}$ *be a geometric Galois formation which is* δ*-residually complete curve like and residually geometric complete like up to level* δ*. Then one has:* $\delta = \mathrm{td}(K|k)-1$*, and viewing* $\mathcal{G}_{\mathfrak{D}}$ *as a pre-divisorial Galois formation, we have:* $\mathrm{Div}(X)_{(\ell)}$ *is a representative for its divisorial lattices, and its pre-image* $\mathcal{L}_{X,(\ell)}$ *in* $\widehat{K}$ *is a representative for the divisorial* $\widehat{U}_X$*-lattices in* $\widehat{K}$.

Proof. Let $\mathcal{G} = \big(G, (Z_v)_v\big)$ be a pre-divisorial Galois formation of level δ on $G = G_K^{\ell,\mathrm{ab}}$. Then $\mathcal{G}$ is a geometric Galois formation of level $\delta \le \mathrm{td}(K|k) =: d$ if an only if by induction the following hold:

i) Z_v is a divisorial decomposition group as given by the Fact above for all "abstract valuations" v, and T_v is the inertia subgroup of Z_v.

ii) The set $\mathfrak{D}$ of all the valuations v of $\mathcal{G}$ is a geometric set of Zariski prime divisors.

(iii) By induction on d, each residual pre-divisorial Galois formation $\mathcal{G}_v$ on $G_v = G^\ell_{Kv}$ is a geometric Galois formation of level $\delta - 1$. Remark that in the limit case when $d = 2$, all the inertia groups $T_a \subset G^\ell_{Kv}$ are encoded in G^ℓ_K, see Fact 2.4, thus "known".

Now suppose that $\mathcal{G} = \big(G^\ell_K, (Z_v)_v\big)$ a geometric Galois formation of level $\delta \le \mathrm{td}(K|k)$ is given. Then $\mathcal{G}$ is δ*-residually complete curve like* if and only if $\delta = d-1$; and further for all the δ-residual Galois formations $\mathcal{G}_{\mathbf{v}} = \big(G_{\mathbf{v}}, (Z_{\mathbf{v}})_{\mathbf{v}}\big)$ one has: $(Z_{\mathbf{v}})_{\mathbf{v}}$ is exactly the family of all the inertia groups of $G_{\mathbf{v}}$. Remark that by induction on the length of the multi-index $\mathbf{v}'$, and Fact 2.4, we know all the geometric inertia elements in $G_{\mathbf{v}'}$. And that for a multi-index $\mathbf{v}$ of length δ, a subgroup of $Z_{\mathbf{v}}$ is inertia group if and only if it is maximal among the pro-cyclic subgroups generated by inertia elements of $G_{\mathbf{v}}$.

Finally, let $\mathcal{G} = \big(G^\ell_K, (Z_v)_v\big)$ be a geometric Galois formation of level $\delta < d$ be given. Let $\mathcal{G}_{\mathbf{v}} = \big(G_{\mathbf{v}}, (Z_{\mathbf{v}})_{\mathbf{v}}\big)$ be the $\mathbf{v}$-residual geometric Galois formation to some multi-index $\mathbf{v}$ of length $\le \delta$. Denote by $\pi_{1,K\mathbf{v}}$ the fundamental group attached to the residual function field $K\mathbf{v}|k$. Then by the Fact above, the fact that the normal model $X_{\mathbf{v}} \to k$ of $K\mathbf{v}|k$ is complete regular like if and only if $\pi_{1,K\mathbf{v}} = \pi_{1,\mathcal{G}_{\mathbf{v}}}$. Thus $\mathcal{G}$ is *ample up to level* δ if and only if $\pi_{1,K\mathbf{v}} = \pi_{1,\mathcal{G}_{\mathbf{v}}}$ (all multi-indeces $\mathbf{v}$ of length $\le \delta$). $\square$

4. Concluding the Proof of the Theorem (Introduction)

A) *Detecting rational projections*

First some definitions: Let $\mathfrak{K}|\mathfrak{k}$ be an arbitrary function field, say with $\mathfrak{k}$ algebraically closed. For every non-constant function $x \in \mathfrak{K}$, let $\mathfrak{K}_x$ be the relative algebraic closure of $\mathfrak{k}(x)$ in $\mathfrak{K}$. Then $\mathfrak{K}_x|\mathfrak{k}$ is a function field in one variable. For a given Galois extension $\mathfrak{K}'|\mathfrak{K}$, the relative algebraic closure of $\mathfrak{K}_x$ in $\mathfrak{K}'$ will be denoted by $\mathfrak{K}'_x|\mathfrak{K}_x$. And the inclusion $\mathfrak{K}'_x \hookrightarrow \mathfrak{K}'$ gives rise to a surjection projection

$$p'_x : G'_{\mathfrak{K}} \to G'_{\mathfrak{K}_x}$$

which we then call the 1-*dimensional projection defined by or attached to* $x \in \mathfrak{K}$ (for the Galois group $G'_{\mathfrak{K}}$). We will say that a 1-dimensional projection p'_x is a *rational projection,* if K_x is a rational function field.

Our aim in this section is to show that in the case $K|k$ with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite field, the rational projections of $G_K^{\ell,\mathrm{ab}}$ are group theoretically encoded in G_K^{ℓ}.

The strategy is as follows, compare with Bogomolov [Bo], Lemma 4.2. [But note also that in loc.cit. one gives/has a recipe for detecting the projections p_x *under the hypothesis that* $\jmath(K^{\times})_{(\ell)}$ *inside* $\widehat{K}$ *is known...* whereas we cannot work under this hypothesis: We are actually *looking for detecting* $\jmath(K^{\times})_{(\ell)}$ *inside* $\widehat{K}$.] First, for $n = \ell^e$, $\ell \neq \mathrm{char}$, consider the cup product:

$$\psi_n : \mathfrak{K}^{\times}\!/n \times \mathfrak{K}^{\times}\!/n = \mathrm{H}^1(G^{\ell}_{\mathfrak{K}}, \mathbb{Z}/n) \times \mathrm{H}^1(G^{\ell}_{\mathfrak{K}}, \mathbb{Z}/n) \to \mathrm{H}^2(G^{\ell}_{\mathfrak{K}}, \mathbb{Z}/n) = {}_n\mathrm{Br}\,(\mathfrak{K}).$$

Now Bogomolov's idea from loc.cit. is to use the following fact: *Let* x, y *be non-constant in* $\mathfrak{K}$. *Then* $\mathfrak{K}_x = \mathfrak{K}_y$ *if and only if* $\psi_n(x, y) = 0$ *for all* n. This means that if we know $\jmath_{\mathfrak{K}}(\mathfrak{K}^{\times})$ inside $\widehat{\mathfrak{K}}$, then we know $\jmath_{\mathfrak{K}}(\mathfrak{K}^{\times}_x)$ too. Now Kummer theory gives an ℓ-adic perfect pairing $<\,,\,> : \widehat{\mathfrak{K}} \times G^{\ell,\mathrm{ab}}_{\mathfrak{K}} \to \mathbb{Z}_{\ell}$. Thus the kernel of the projection $p_x : G^{\ell,\mathrm{ab}}_{\mathfrak{K}} \to G^{\ell,\mathrm{ab}}_{\mathfrak{K}_x}$, is nothing but the orthogonal complement of $\jmath_{\mathfrak{K}}(\mathfrak{K}^{\times}_x)$:

$$\mathcal{N}_x := \ker(p_x) = \{\sigma \in G^{\ell,\mathrm{ab}}_{\mathfrak{K}} \mid <x', \sigma> = 0 \text{ for all } x' \in \jmath_{\mathfrak{K}}(\mathfrak{K}^{\times}_x)\,\}.$$

And this gives a description of all the kernels $\mathcal{N}_x$ of projections p_x. We will nevertheless need a modified version of the fact above as follows:

Fact 4.1. Let $T^{\ell}_{\mathfrak{K}}$ be the subgroup of $G^{\ell}_{\mathfrak{K}}$ generated by all the inertia subgroups of all Zariski prime divisors. For $n = \ell^e$ consider

$$\Psi_n : \mathfrak{K}^{\times}/n \times \mathfrak{K}^{\times}/n \xrightarrow{\psi_n} {}_n\mathrm{Br}\,(\mathfrak{K}) \xrightarrow{\mathrm{res}} \mathrm{H}^2(T^{\ell}_{\mathfrak{K}}, \mathbb{Z}/n)\,.$$

Then $\mathfrak{K}_x = \mathfrak{K}_y$ if and only if $\Psi_n(x, y) = 0$ for all n.

Proof. The implication "$\Rightarrow$" follows by Tsen's Theorem, as $\mathfrak{k}$ is algebraically closed. To prove "$\Leftarrow$", suppose that x, y are algebraically independent over $\mathfrak{k}$. Let $t_1 := x, t_2 := y, \dots, t_d$ be a transcendence base of $\mathfrak{K}|\mathfrak{k}$. Set $\Lambda_0 = \mathfrak{k}((t_1))\dots((t_d))$, thus $G^\ell_{\Lambda_0} = T_{\Lambda_0} \cong \mathbb{Z}^d_\ell$, and $t_1 \cup t_2$ has order n in $\mathrm{H}^2(\Lambda_0, \mathbb{Z}/n)$. Now let $\Lambda = \Lambda_0\mathfrak{K}$ be some compositum of Λ_0 and $\mathfrak{K}$ over $\mathfrak{K}_0 := \mathfrak{k}(t_1, \dots, t_d)$. Then image of the canonical projection $G^\ell_\Lambda \to G^\ell_{\Lambda_0}$ is open, say of index δ. Therefore, the order of $t_1 \cup t_2$ in $\mathrm{H}^2(\Lambda, \mathbb{Z}/n)$ is at least n/δ. Finally, since the image of $G^\ell_\Lambda = T^\ell_\Lambda \to G^\ell_{\mathfrak{K}}$ is contained in $T^\ell_{\mathfrak{K}}$, it follows that for $n = \ell^e$ large enough we have: $x \cup y = t_1 \cup t_2 \neq 0$ in $\mathrm{H}^2(T^\ell_{\mathfrak{K}}, \mathbb{Z}/n)$. Thus $\Psi_n(x, y) \neq 0$. $\square$

Now come back to the situation $K|k$ is a function field with $\mathrm{td}(K|k) > 1$, and k the algebraic closure of a finite field.

Let $\mathcal{G}_X$ be a geometric Galois formation, which is complete curve like and ample up to level $\delta = \mathrm{td}(K|k) - 1$. Further let $\mathrm{Div}_{X,(\ell)}$ be an abstract divisor group for $\mathcal{G}_X$ and $\mathcal{L}_{X,(\ell)}$ its pre-image in $\widehat{K}$. Then after multiplying $\mathrm{Div}_{X,(\ell)}$ by an ℓ-adic unit ϵ, we can suppose that $\mathrm{Div}(X)_{(\ell)} = \mathrm{Div}_{X,(\ell)}$. In particular, the divisorial $\widehat{U}_X$-lattice $\mathcal{L}_{(\ell)}$ contains $K^\times_{(\ell)} := \jmath_K(K^\times) \otimes \mathbb{Z}_{(\ell)}$ inside $\widehat{K}$. We now have the following

Lemma 1. *In the above context, $\mathcal{L}_{X,(\ell)} \,/\, \big(\widehat{U}_X \cdot K^\times_{(\ell)}\big)$ is a torsion group.*

Proof. First remark that it is sufficient to prove the assertion of Lemma 1 after replacing X by a complete model, say $X' \to k$, and further after replacing X' by any projective model $X'' \to k$ which dominates X'. (Indeed, such changes just replace $\widehat{U}_X$ by smaller subgroups, and $\mathcal{L}_{X,(\ell)}$ by possibly bigger ones.) Furthermore, since $\mathcal{L}_{X,(\ell)}$ is the pre-image of $\mathrm{Div}_{X,(\ell)}$ in $\widehat{K}$, the contention of Lemma 1 is equivalent to the following: $\jmath^{\mathcal{G}_X}(\mathcal{L}_{X,(\ell)})/\mathrm{div}(K^\times)_{(\ell)}$ is a torsion group. We have a commutative diagram of the form

$$(*)\qquad \begin{array}{ccccccccc} 0 & \to & \mathrm{div}(K^\times)_{(\ell)} & \hookrightarrow & \mathrm{Div}(X)_{(\ell)} & \xrightarrow{\text{can}} & \mathfrak{Cl}(X)_{(\ell)} & \to & 0 \\ & & \big\downarrow{\scriptstyle\text{incl}} & & \big\downarrow{\scriptstyle\text{id}} & & \big\downarrow{\scriptstyle\text{can}} & & \\ 0 & \to & \jmath^{\mathcal{G}_X}(\mathcal{L}_{X,(\ell)}) & \hookrightarrow & \mathrm{Div}(X)_{(\ell)} & \xrightarrow{\text{can}} & \mathfrak{Cl}_{X,(\ell)} & \to & 0 \end{array}$$

where the first vertical map is an inclusion. So by the Snake Lemma it follows that $\jmath^{\mathcal{G}_X}(\mathcal{L}_{X,(\ell)})/\mathrm{div}(K^\times)_{(\ell)} \cong \ker\big(\mathfrak{Cl}(X)_{(\ell)} \to \mathfrak{Cl}_{X,(\ell)}\big)$. Thus we have to show that this last group is a torsion one.

Now since k is the algebraic closure of a finite field, one has an exact sequence of the form:

$$(*)\qquad 0 \to (\text{torsion group})_{(\ell)} \to \mathfrak{Cl}(X)_{(\ell)} \to \mathrm{NS}(X)_{(\ell)} \to 0$$

where $\mathrm{NS}(X)$ is a finitely generated Abelian group. (Actually, if X is a projective smooth, the assertions above are all well known. If not, then consider a smooth alteration $\tilde{X}$ of X, and conclude by using the inclusion-norm map.)

And remark that the $\mathbb{Q}$-rank of $\mathrm{NS}(X)_{(\ell)}$ is the same as its $\mathbb{Q}_\ell$-rank. And further, that this last rank is the same as the $\mathbb{Q}_\ell$ -rank of $\widehat{\mathfrak{Cl}}_{X,(\ell)}$. Thus finally equal to the $\mathbb{Q}$-rank of $\mathfrak{Cl}_{X,(\ell)}$. Therefore, $\ker\big(\mathfrak{Cl}(X)_{(\ell)} \to \mathfrak{Cl}_{X,(\ell)}\big)$ is a torsion group, as contented. □

Corollary. *For every $\tilde{x} \in \mathcal{L}_{X,(\ell)}$ there exists some $u \in \widehat{U}_X$ and $x \in K^\times_{(\ell)}$ such that $x^m = u \cdot x$ for some integer $m > 0$.*

In other words we have the following: $\mathcal{L}_{X,(\ell)}$ is exactly the relative divisible hull of $\widehat{U}_X \cdot K^\times_{(\ell)}$ in $\widehat{K}$, i.e., one has:

$$\mathcal{L}_{X,(\ell)} = \{\tilde{x} \in \widehat{K} \mid \exists\, m \text{ such that } \tilde{x}^m \in \widehat{U}_X \cdot K^\times_{(\ell)}\,\}$$

Notations/Remark.

We set $\mathcal{K}'_{X,(\ell)} = \widehat{U}_X \cdot K^\times_{(\ell)}$. Further, for $x \in K^\times_{(\ell)}$ we denote:

1) $\mathcal{K}'_{x,(\ell)} = \widehat{U}_X \cdot \jmath_K(K_x)_{(\ell)}$.

2) $\mathcal{L}'_{x,(\ell)}$ the divisible hull of $\mathcal{K}'_{x,(\ell)}$ in $\mathcal{L}_{X,(\ell)}$.

For $\tilde{x} \in \mathcal{L}_{X,(\ell)}$ let us consider some $u \in \widehat{U}_X$ and $x \in \jmath_K(K^\times)_{(\ell)}$ such that $\tilde{x}^m = u \cdot x$ for some $m > 0$. Remark that since $\jmath_K(K_x)_{(\ell)} \cap \widehat{U}_X = \{1\}$, the presentation above is unique in the following sense: If $\tilde{x}^m = u \cdot x$ and $\tilde{x}^{m'} = u' \cdot x'$, then $u^{m'} = u'^m$ and $x^{m'} = x'^m$.

For $\tilde{x}$ and $\tilde{x}^m = u \cdot x$ as above but with $x \in \jmath_K(K^\times)$, we denote:

3) $\mathcal{K}'_{\tilde{x},(\ell)} := \mathcal{K}'_{x,(\ell)}$, and $\mathcal{L}'_{\tilde{x},(\ell)} := \mathcal{L}'_{x,(\ell)}$.

Remark that both $\mathcal{K}'_{\tilde{x},(\ell)}$ and $\mathcal{L}'_{\tilde{x},(\ell)}$ depend only on $\tilde{x}$, and not the representation $\tilde{x}^m = u \cdot x$.

N.B., for all $\tilde{x}$ there exists presentations as above with $x \in \jmath_K(K^\times)$.

Lemma 2. *In the above notations for all $\tilde{x} \in \mathcal{L}_{X,(\ell)}$ one has:*

$$\tilde{x}^\perp := \{\, \tilde{y} \in \mathcal{L}_{X,(\ell)} \mid \Psi_n(\tilde{x}, \tilde{y}) = 0 \textit{ for all } n = \ell^e \,\} = \mathcal{L}'_{x,(\ell)}$$

Proof. For $\tilde{y} \in \tilde{x}^\perp$, consider presentations $\tilde{x}^m = u \cdot x$ and $\tilde{y}^{m'} = u' \cdot y$ as usual. Then we have: $0 = \Psi_n(\tilde{x}^m, \tilde{y}^{m'}) = \Psi_n(x, y)$, as $\widehat{U}_X$ is contained in both the right and the left kernel of Ψ_n for all n. Thus $\Psi_n(x, y) = 0$ for all n. By Fact above, it follows that $K_x = K_y$. Equivalently we have $\mathcal{K}'_{x,(\ell)} = \mathcal{K}'_{y,(\ell)}$, thus also $\mathcal{L}'_{x,(\ell)} = \mathcal{L}'_{y,(\ell)}$. Hence by the definition of $\mathcal{L}'_{x,(\ell)}$ it follows that $\tilde{y} \in \mathcal{L}'_{x,(\ell)}$. Conversely, we show that $\mathcal{L}'_{x,(\ell)}$ is contained in $\tilde{x}^\perp$. Indeed, each $\tilde{y} \in \mathcal{L}'_{x,(\ell)}$, is of the form $\tilde{y}^{m'} = u' \cdot y$ for some integer $m' > 0$, and some $u' \in \widehat{U}_X$, $y \in \mathcal{K}'_{x,(\ell)}$. Thus $0 = \Psi_n(x, y) = \Psi_n(u\,x, u'\,y) = m\,m'\Psi_n(\tilde{x}, \tilde{y})$ for all n. Thus $\Psi_n(\tilde{x}, \tilde{y}) = 0$ for all n, and finally $\tilde{y} \in \tilde{x}^{\perp}$. □

Thus Lemma 2 above shows that the following first two sets are equal, and in a bijective correspondences with the third one:

$$\{\, \tilde{x}^{\perp} \mid \tilde{x} \in \mathcal{L}_{X,(\ell)} \backslash \widehat{U}_X \,\} = \{\, \mathcal{L}'_{x,(\ell)} \mid x \in \mathcal{K}'_{X,(\ell)} \backslash \widehat{U}_X \,\} \cong \{\, \mathcal{K}'_{x,(\ell)} \mid x \in K \backslash k \,\}$$

Now let some $\mathcal{L}'_{\tilde{x},(\ell)}$ be given. By the above correspondence, there exists a unique subfield K_x of K such that $\mathcal{L}'_{\tilde{x},(\ell)} = \mathcal{L}'_{x,(\ell)}$, or equivalently, that $\mathcal{L}'_{\tilde{x},(\ell)}$ is the relative divisible hull of $\widehat{U}_X \cdot K_{x,(\ell)}$. To simplify notations, let us set

$$G := G_K^{\ell,\mathrm{ab}}, \quad G_x := G_{K_x}^{\ell,\mathrm{ab}}, \quad \text{and} \quad p_x : G \to G_x$$

the resulting canonical projection. And let $\hat{\varphi}_x : \widehat{K}_x \to \widehat{K}$ be the embedding defined by p_x via Kummer theory. By ℓ-adic Kummer theory, the knowledge of p_x is equivalent to the knowledge of $\hat{\varphi}_x$. We consider the unique complete smooth model $X_x \to k$ of $K_x|k$, and the corresponding geometric Galois formation $\mathcal{G}_x$, etc.. We set $\mathcal{K}_{x,(\ell)} = \widehat{U}_x \cdot K_{x,(\ell)}$ inside $\widehat{K}_x$. Then we have:

$$\mathcal{K}_{x,(\ell)} := \hat{\varphi}_x(\widehat{U}_x \cdot K_{x,(\ell)}) = \hat{\varphi}_x(\widehat{K}_x) \cap \mathcal{K}'_{x,(\ell)} = \hat{\varphi}_x(\widehat{K}_x) \cap \mathcal{L}'_{x,(\ell)}$$

We next indicate how to identify $\mathcal{K}_{x,(\ell)}$ inside $\mathcal{L}'_{x,(\ell)}$. Equivalently, by the discussion at the beginning of this Subsection, this gives then a recipe for detecting the 1-dimensional projection p_x.

First, let v be a Zariski prime divisor of K, and $\jmath^v : \widehat{K} \to \mathbb{Z}_\ell$ as introduced/defined in Section 3, Definition/Remark 3.2, 3). Since $\jmath^v$ is trivial on $\widehat{U}_X$, it follows that $\jmath^v$ is trivial on $\mathcal{L}_{\tilde{x},(\ell)}$ if and only if v is trivial on K_x.

Next let $\jmath_v : \widehat{U}_v \to \widehat{Kv}$ be the reduction map attached to a Zariski prime divisor v, see Section 3, Definition/Remark 3.2, 6). Then the restriction of $\jmath_v$ to the subgroup $K^{\times}_{(\ell)}$ coincides with the reduction map defined/introduced in Section 1, A), Definition/Remark 2.4, 3). In particular, since $\mathrm{td}(K_x|k) = 1$, one has: $\jmath^v$ is not trivial on K_x if an only if $\jmath_v$ is trivial on K_x.

Lemma 3. *In the above context we have:*

$$\mathcal{K}_{x,(\ell)} = \{\, \tilde{y} \in \mathcal{L}'_{x,(\ell)} \mid \jmath_v(\tilde{y}) = 1 \textit{ for all } v \textit{ with } \jmath^v(\mathcal{L}'_{x,(\ell)}) \neq 0 \,\}$$

Proof. The direct inclusion is clear by the discussion above. In order to prove the converse, we view the regular field extension $K|K_x$ as the generic fiber of a flat family $X \to S$, where $X \to k$ is a projective normal model of $K|k$, and $S \to k$ is a projective normal model of $K_x|k$. By the Specialisation

Theorem for the fundamental group, it follows that for a "general point" $s \in S$, we have a canonical exact sequence

$$\pi_1(X_s) \to \pi_1(X) \to \pi_1(S) \to 1\,.$$

Now let v be the Zariski prime divisor of K defined by the generic point of the special fiber X_s. Then the translation of the above assertion via Kummer Theory means the following: The reduction homomorphism $\jmath_v$ defines an exact sequence of the form:

$$1 \to \widehat{U}_x \to \widehat{U}_X \xrightarrow{\jmath_v} \widehat{U}_{X_v} \subset \widehat{Kv}\,.$$

In other words, $\tilde{y} \notin \widehat{U}_x$ implies $\jmath_v(\tilde{y}) \neq 1$. □

PROPOSITION 4.2. *In the above notations, consider the perfect ℓ-adic pairing given by Kummer theory $<\,,\,>: \widehat{K} \times G_K^{\ell,\mathrm{ab}} \to \mathbb{Z}_\ell$.*

(1) *The kernels of 1-dimensional projections $p_x^{\ell,\mathrm{ab}} : G \to G_x$ are exactly the closed subgroups $\mathcal{N}_x$ of G of the form:*

$$\mathcal{N}_x = \{\sigma \in G \mid <\tilde{y}\,,\sigma> = 0 \textit{ for all } \tilde{y} \in \mathcal{K}_{x,(\ell)}\,\}$$

Let $p_x : G \to G/\mathcal{N}_x =: G_x$ be a 1-dimensional projection. Then one has:

(2) *A procyclic subgroup T_{v_x} of G_x is an inertia group if and only if there exits an inertia group T_v of G such that $p_x(T_v)$ has finite index in T_{v_x}.*

3) *Finally, $p_x : G \to G_x$ is a rational projection if and only if the inertia groups T_{v_x} (all v_x) generate G_x.*

Clearly, all the facts above are group theoretically encoded in G_K^ℓ.

Proof. Clear by the observations above and the characterization of the 1-dimensional projections via $K_{(\ell)}$ mentioned above. □

B) *The proof of Theorem (Introduction)*

By Proposition 3.18 it follows that the geometric Galois formations over $K|k$ are encoded in G_K^ℓ. And it is clear that the group theoretic description of the geometric Galois formations is preserved under isomorphisms. Therefore, every isomorphism

$$\Phi : G_K^\ell \to G_L^\ell$$

defines a bijection from the set of all geometric Galois formations over $K|k$ onto the set of all geometric Galois formations over $L|l$. Further, if $\mathcal{G}_Y$ corresponds to $\mathcal{G}_X$ under this bijection, then Φ gives rise to an isomorphism of divisorial Galois formations

$$\Phi : \mathcal{G}_X \to \mathcal{G}_Y\,.$$

Now suppose that X is complete regular like. Since this fact is encoded group theoretically in G_K^ℓ and the projection $G_K^\ell \to G := G_K^{\ell,\mathrm{ab}}$, it follows that the corresponding Y is complete regular like too. Consider the canonical commutative diagram

$$(*)_{(\ell)} \qquad \begin{array}{ccccccccc} 0 \to & \widehat{U}_Y & \to & \mathcal{L}_{Y,(\ell)} & \to & \mathrm{Div}(Y)_{(\ell)} & \to & \mathfrak{Cl}_{Y,(\ell)} & \to 0 \\ & \downarrow{\scriptstyle\hat\psi} & & \downarrow{\scriptstyle\hat\psi} & & \downarrow & & \downarrow & \\ 0 \to & \widehat{U}_X & \to & \mathcal{L}_{X,(\ell)} & \to & \mathrm{Div}(X)_{(\ell)} & \to & \mathfrak{Cl}_{X,(\ell)} & \to 0 \end{array}$$

where the vertical map $\hat\psi$ is such a multiple $\hat\psi = \epsilon\,\hat\varphi$ of the Kummer homomorphism $\hat\varphi : \widehat{L} \to \widehat{K}$ that one has: $\psi\big(\mathcal{L}_{Y,(\ell)}\big) = \mathcal{L}_{X,(\ell)}$.

We next remark that by Proposition 4.2 above, Φ also defines a bijection from the set of all the rational projections of $\mathcal{G}_X$ onto the set of all the rational projections of $\mathcal{G}_Y$.

On the other hand, $K_{(\ell)}$ is encoded in G^ℓ_K as follows: The multiplicative group $K^\times$ is generated by the set of all the $K^\times_x$ with K_x a rational function field. In other words, if $(p_x)_x$ is the family of all the rational projections as above, then $K_{(\ell)} \subset \widehat{K}$ is generated by all the images $\mathcal{K}_{x,(\ell)}$. Therefore, in order to detect the ℓ-adic equivalence class of the lattice $K_{(\ell)}$ inside $\widehat{K}$, one can proceed as follows: Choose a divisorial lattice $\mathcal{L}_{K,(\ell)}$ in $\widehat{K}$. For every rational projection p_x, consider the unique divisorial lattice $\mathcal{L}_{K_x,(\ell)}$ in $\widehat{K}_x$ whose image $\mathcal{K}_{x,(\ell)}$ under the completion morphism $\hat\varphi_x : \widehat{K}_x \to \widehat{K}$ is contained in $\mathcal{L}_{K,(\ell)}$. Then the subgroup $\mathcal{L}^0_{K,(\ell)}$ of $\mathcal{L}_{K,(\ell)}$ which is generated by all the $\mathcal{K}_{x,(\ell)}$ is the unique lattice in $\mathcal{L}_{K,(\ell)}$ which is equivalent to $K_{(\ell)}$.

Since the description of $\mathcal{L}^0_{K,(\ell)}$ above is given in pure group theoretic terms, it is clear that the Kummer homomorphism $\hat\varphi_K : \widehat{L} \to \widehat{K}$ maps $L_{(\ell)}$ isomorphically onto a lattice $\mathcal{L}^0_{K,(\ell)}$ in $\widehat{K}$ which is ℓ-adically equivalent to $K_{(\ell)}$. Thus after multiplying $\hat\varphi$ by a properly chosen ℓ-adic unit (unique modulo rational ℓ-adic units), we can suppose that $\hat\varphi$ maps $L_{(\ell)}$ isomorphically on $K_{(\ell)}$.

We finally apply Pop [P4], Theorem 5.11, and deduce that there exist finite pure inseparable extensions $K_0|K$ and $L_0|L$ such that $\hat\varphi_{K_0} : \widehat{L_0} \to \widehat{K_0}$ is the ℓ-adic completion of a unique field isomorphism $\imath_0 : K_0 \to L_0$.

Moreover, let $K'|K$ be a finite (Galois) extension of K inside K^ℓ, and via the isomorphism Φ the corresponding finite (Galois) extension $L'|L$. Then using the functoriality of Kummer theory, we have a commutative diagram of the form

$$\begin{array}{ccc} \widehat{L} & \xrightarrow{\hat\varphi_K} & \widehat{K} \\ \downarrow{\scriptstyle\mathrm{can}} & & \downarrow{\scriptstyle\mathrm{can}} \\ \widehat{L'} & \xrightarrow{\hat\varphi_{K'}} & \widehat{K'} \end{array}$$

Suppose $K'|K$ is Galois, and $\Phi' : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(L'|L)$ is the isomorphism induced by Φ. Then for all $\sigma \in \mathrm{Gal}(K'|K)$, and $\tau = \Phi(\sigma)$ we have:

$$\sigma \circ \hat{\varphi}_{K'} = \hat{\varphi}_{K'} \circ \tau \quad \text{i.e.,} \quad \Phi(\sigma) = \hat{\varphi}_{K'}^{-1} \circ \sigma \circ \hat{\varphi}_{K'} \,.$$

So if $\hat{\varphi}_K$ maps $L_{(\ell)}$ into $K_{(\ell)}$, then $\hat{\varphi}_{K'}$ automatically maps $L'_{(\ell)}$ into $K'_{(\ell)}$, i.e., it is not necessary to "re-norm" $\hat{\varphi}_{K'}$ in order to have $\hat{\varphi}_{K'}(L'_{(\ell)}) = K'_{(\ell)}$. Thus the above commutative diagram induces a corresponding one with $K_{(\ell)}$ in stead of $\widehat{K}$, etc..

Conclusion:

Now taking limits over all finite extensions $K'|K_0$ inside $K_0^\ell = K^\ell K_0$, and the corresponding finite extensions $L'|L_0$ inside L_0^ℓ, we finally get an isomorphism $\phi : L^{\mathrm{i}^\ell} \to K^{\mathrm{i}^\ell}$ defining Φ, i.e., $\Phi(g) = \phi^{-1}\, g\, \phi$ for all $g \in G_K^\ell$.

We still have to prove that any two automorphisms ϕ' and ϕ'' both defining Φ differ by a power of Frobenius. Indeed, setting $\phi := \phi'' \circ {\phi'}^{-1}$ we get: ϕ is an automorphism of L^{i^ℓ} which maps L^{i} onto itself, and induces the identity on $G_{L^{\mathrm{i}}}^\ell$. We claim that such an automorphism ϕ is a power of Frobenius. Indeed, let v be an arbitrary Zariski prime divisor of L^{i^ℓ}. Then $w := v \circ \phi$ is also a Zariski prime divisor of L^{i^ℓ}. Moreover, by the usual formalism, we have $Z_w = \phi^{-1} Z_v\, \phi$ inside $G_{L^{\mathrm{i}}}^\ell$. Since the conjugation by ϕ is the identity on $G_{L^{\mathrm{i}}}^\ell$, it follows that $Z_w = Z_v$. Thus by Proposition 1.4, (1), it follows that v and w are equivalent valuations on L^{i^ℓ}. In particular, $v(x) > 0$ if and only if $w(x) > 0$ (all $x \in L^{\mathrm{i}^\ell}$). Let $\mathrm{char}(k) = p$. We claim that $y := \phi(x)$ is some p-power of x. First, if y and x are algebraically independent, then there exists a Zariski prime divisor v of L^{i^ℓ} such that $v(x) = 1$ and $v(y) = 0$. But $v(y) = v \circ \phi(x) = w(x)$, contradiction! Therefore, $y = \phi(x)$ is always algebraic integer over $k[x]$, and vice-versa: x is algebraic integer over $k[y]$. Now let $f(X, Y) \in k[X, Y]$ be the minimal polynomial polynomial relation between x and y over k. We claim that $f(X, Y)$ is pure inseparable in Y (and thus by symmetry, also in X). Let namely $y_a =: y_1, \dots, y_r$ be the distinct roots of $f(x, Y)$ over $k(x)$. The for a "general $a \in k$ we have: If $x_a = x + a$, then $y_a =: \phi(x_a) = y + \phi(a)$. Thus x_a and y_a have the same zeros in $\mathfrak{D}^1_{K^\ell}$. And clearly, the minimal polynomial polynomial relation satisfied by x_a and y_a over k is $f_a(X, Y) = f\big(X - a, Y - \phi(a)\big)$. Therefore, for a general $a \in k$, the polynomial $f_a(0, Y)$ has s distinct roots $b_1, \dots, b_s$. And the specializations $(x, y) \mapsto (a, b_i)$ give rise to s different places v_i of $k(x)^\ell$ all of which are zeros of x_a. But then these places must also be zeros of y_a. Thus $0 = y_a(b_i) = \phi(a) - b_i$ implies $b_i = \phi(a)$, thus $s = 1$. Thus finally $f(X, Y)$ is purely inseparable in Y, and by symmetry also in X.

Thus finally we have: $\phi(x) = a_x\, x^{n_x}$, where $a_x \in k$ and $n_x = p^{e_x}$ depend on x. We next show that $a_x = 1$ and that $n_x = p^e$ do not depend on x. Indeed, since ϕ is a field automorphism, we have $\phi(x+y) = \phi(x) + \phi(y)$ for all x, y from $L^{\mathrm{i}\ell}$. Or equivalently: $a_{x+y}(x+y)^{n_{x+y}} = a_x\, x^{n_x} + a_y y^{n_y}$. Considering several arbitrary $y \in L^{\mathrm{i}\ell}$ such that x and y are algebraically independent over k, we therefore must have $n_{x+y} = n_y = n_x = p^e$ and $a_{x+y} = a_y = a_x = a$, thus independent of x and y. Since $a_1 = 1$, we have $a = 1$. Finally, choosing a transcendence basis $\mathcal{T}$ of L over k (say, which contains x), we see that the restriction of ϕ to $k(\mathcal{T})$ is Frob^e. Therefore, $\phi = \mathrm{Frob}^e$ on $L^{\mathrm{i}\ell}$.

The Theorem (Introduction) is proved.

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