Due: Friday, Jan 25, 2008

Math 360 - Advanced Calculus / Problem Set 1

Sets and Maps

1) Let $f : X \to Y$, $g : Y \to Z$ be a maps of sets.

a) Show that if f and g are injective, then so is $g \circ f$.
Formulate the converse assertion to a), and prove or disprove it.

b) Show that if f and g are surjective, then so is $g \circ f$.
Formulate the converse assertion to b), and prove or disprove it.

c) Show that if f and g are bijective, then so is $g \circ f$.
Formulate the converse assertion to c), and prove or disprove it.

Recall that for a set X we denote $\mathcal{P}(X) = \{A \mid A \subseteq X\}$, the set of all the subsets of X, and that $\mathcal{P}(X)$ is called the *power set* of X. For every $A \in \mathcal{P}(X)$, we define a map $\chi_A : X \to \{0, 1\}$ by $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$. The function χ_A is called the *characteristic function* of A.

2) Prove or disprove the following:

a) Let $\mathcal{F}_X$ be the set of all the functions from X to $\{0, 1\}$. Then the map $\phi : \mathcal{P}(X) \to \mathcal{F}_X$ defined by $A \mapsto \chi_A$ is bijective.

b) The characteristic functions have the property: For all $A, B \subseteq X$ one has: $\chi_{A\cap B} = \chi_A \cdot \chi_B$, and $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$, where "+" and "−" and "·" are the usual operations of integers.

3) Let X, Y be arbitrary sets, and $f : X \to Y$ be a mapping of sets. Prove or disprove each of the following assertions:

a) $\forall\ A_1, A_2 \subseteq X$ one has: $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, respectively $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

b) f is injective $\Leftrightarrow \forall\ A_1, A_2 \subseteq X$ one has: If $A_1 \neq A_2$, then $f(A_1) \neq f(A_2)$.

c) f is surjective $\Leftrightarrow \forall\ B \in \mathcal{P}(Y)\ \exists\ A \in \mathcal{P}(X)$ such that $f(A) = B$.

4) Let X, Y be arbitrary sets, and $f : X \to Y$ be a mapping of sets. Prove or disprove each of the following assertions:

a) $\forall\ B_1, B_2 \subseteq Y$ one has: $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, resp. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

b) f is injective $\Leftrightarrow \forall\ B_1, B_2 \subseteq Y$ one has: If $B_1 \neq B_2$, then $f^{-1}(B_1) \neq f^{-1}(B_2)$.

c) f is surjective $\Leftrightarrow \forall\ A \in \mathcal{P}(X)\ \exists\ B \in \mathcal{P}(Y)$ such that $f^{-1}(B) = A$.

Recall that to every set X we attach the symbol $|X|$, called the *cardinality* of X. (Intuitively, $|X|$ is a kind of set theoretic "size" of X.) We say by definition that $|X| \leq |Y|$ if there exist injective maps $f : X \to Y$. And we will say that $|X| < |Y|$ if the following holds: $|X| \leq |Y|$ and $|Y| \not\leq |X|$.

5) Prove or disprove the following:

a) If $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$. And if $|X| < |Y|$ and $|Y| < |Z|$, then $|X| < |Z|$.

*b) For every set X one has: $|X| < |\mathcal{P}(X)|$.

c) $|X| \leq |Y| \Leftrightarrow$ there exist surjective maps $g : Y \to X$.

*d) And $|X| \leq |Y|$ and $|Y| \leq |X| \Leftrightarrow$ there exist bijective maps $f : X \to Y$.

*6) Prove the following **characterization of finite sets**: X is finite $\Leftrightarrow$ Every injective map $f : X \to X$ is surjective (hence bijective) $\Leftrightarrow$ Every surjective map $g : X \to X$ is injective (hence bijective).