Due: Friday, Feb 8, 2008

Math 360 - Advanced Calculus / Problem Set 3

Numbers and algebraic structures

1) Let $\mathbb{N}, +, \cdot, \leq$ be the set of natural numbers endowed with the addition, multiplication, and the ordering.

a) Prove the cancellation laws: $\forall\, k, m, n \in \mathbb{N}$ one has: $k + m = k + n \Rightarrow m = n$, and $k\,m = k\,n \Rightarrow m = n$ provided $k \neq 0$.

b) Prove the compatibility of $+$ and $\cdot$ with $\leq$ in $\mathbb{N}$: $\forall\, k, m, n \in \mathbb{N}$ one has: $k + m \leq k + n \Leftrightarrow m \leq n$, and $k\,m \leq k\,n \Rightarrow m \leq n$ provided $k > 0$.

2) Consider the sets $\mathbb{Z}[\sqrt{3}] := \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$, and $\mathbb{Q}[\sqrt{3}] := \{\alpha + \beta\sqrt{3} \mid \alpha, \beta \in \mathbb{Q}\}$, endowed with the usual addition $+$ and multiplication $\cdot$ of numbers, i.e., $(a + b\sqrt{3}) + (a' + b'\sqrt{3}) = (a + a') + (b + b')\sqrt{3}$, and $(a + b\sqrt{3})\,(a' + b'\sqrt{3}) = (aa' + 3bb') + (ab' + a'b)\sqrt{3}$.

a) Show that $\mathbb{Z}[\sqrt{3}], +, \cdot$ is a ring.

b) Which elements $a + b\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ are invertible w.r.t. multiplication?

c) Show that $\forall\, a, b, a', b' \in \mathbb{Q}$ one has: $a + b\sqrt{3} = a' + b'\sqrt{3} \quad \Leftrightarrow \quad a = a'$ and $b' = b'$.

d) Show that $\mathbb{Q}[\sqrt{3}], +, \cdot$ is a field.

3) The same questions concerning $\mathbb{Z}[\boldsymbol{i}] := \{a + b\boldsymbol{i} \mid a, b \in \mathbb{Z}\}$, and $\mathbb{Q}[\boldsymbol{i}] := \{a + b\boldsymbol{i} \mid a, b \in \mathbb{Q}\}$, where $\boldsymbol{i}^2 = -1$.

Language: The ring $\mathbb{Z}[\boldsymbol{i}]$ is a very famous one, and is called the *ring of Gaussian integers.*

In the sequel we consider a sets endowed with composition laws, like the "$+$" or the "$\cdot$" of abstract rings or fields.

4) Let $A = \mathbb{Z}$ or $A = \mathbb{Q}$, or more general, A is any ring R or field K.

a) Endow A with the composition law $\boldsymbol{+}$ defined by $a \boldsymbol{+} b := \alpha\, a + \beta\, b + \gamma$, where $+$ and $\cdot$ are the usual addition and multiplication in A, and $\alpha, \beta, \gamma \in A$. Find the values of $\alpha, \beta, \gamma \in A$ for which $\boldsymbol{+}$ is associative, resp. commutative, resp. has a neutral element, resp. every element $x \in A$ has an inverse.

b) Same questions as above, but for the composition law $a \bullet b := t\,a\,b + u\,a + v\,c + w$, where $+$ and $\cdot$ are the usual addition and multiplication in A, and $t, u, v, w \in A$.

c) In the context above, find all the $\alpha, \beta, \gamma, t, u, v, w$ such that $A, \boldsymbol{+}, \bullet$ is a ring, respectively, a field.

5) Let A be an arbitrary set endowed with a composition law $*$. Prove the following:

a) There exists at most one neutral element $e \in A$ for $*$.

b) Suppose that $*$ is associative. Then every $x \in A$ has at most one inverse w.r.t. $*$.

6) Let $R, +, \cdot$ be a commutative ring with zero element 0_R and neutral element 1_R. For $x \in R$ and $n \in \mathbb{N}$, $a \in \mathbb{Z}$ we defined $a\,x \in R$ and $x^n \in R$. Prove the following computation rules:

a) $x\,0_R = 0_R$ for all $x \in R$. In particular, since $0_R \neq 1_R$, we get: 0_R is not inverible w.r.t. the multiplication.

b) $(-a)x = -(ax) = a(-x)$, $(a+b)x = ax + bx$, $a(x+y) = ax + ay$, $a(bx) = (ab)x$, for all $a, b \in \mathbb{Z}$, $x, y \in R$.

c) $(xy)^n = x^n y^n$, $x^{n+m} = x^n x^m$, $(a^m)^n = a^{m\,n}$, for all $m, n \in \mathbb{N}$ and $x, y \in R$.

d) If R is a field, and x^{-1} denotes the multiplicative inverse of $x \neq 0_R$, then $(x^{-1})^n = (x^n)^{-1}$, for all $x \neq 0_R$ and $n \in \mathbb{N}$.