Due: Friday, Feb 15, 2008

Math 360 - Advanced Calculus / Problem Set 4

Real numbers

Recall that given non-empty subsets $X, Y \subset \mathbb{R}$, we defined $X + Y := \{x + y \mid x \in X, y \in Y\}$, and call $X + Y$ "the sum" of X and Y; and $X \cdot Y := \{xy \mid x \in X, y \in Y\}$, and call $X \cdot Y$ "the product" of X and Y.

1) Unswer the questions below:

a) Describe the following sums of sets: $[0,1] + \mathbb{Z}$, $(0,\infty) + (-\infty, 0)$, $[a,b] + \mathbb{Q}$, $[-2,1] + [0,3)$, where $[a,b]$ and $[a,b)$, etc., denote intervals.

b) The same problem for $X + Y$ replaced by $X \cdot Y$.

2) Let $I \subset \mathbb{R}$ be the set of the irrational numbers. Prove or disprove:

a) $a \in I \quad \Leftrightarrow \quad \frac{1}{2}a - 3 \in I$; $\; b + 1 \in I \quad \Leftrightarrow \quad b^2 - 1 \notin \mathbb{Q}$.

b) If $\alpha^5 + 1 = \frac{1}{3}$, then $\alpha \in I$.

c) If $y + z \in I$ and $yz \in I$, then $y, z \in I$.

3) Prove or disprove the following:

a) X and Y are both bounded $\Leftrightarrow$ $X + Y$ is bounded.

b) $\sup X$ and $\sup Y$ do both exist $\Leftrightarrow$ $\sup(X + Y)$ does exist. What is the relation between these numbers, if they all exist.

c) $\max X$ and $\max Y$ do both exist $\Leftrightarrow$ $\max(X+Y)$ does exist. What is the relation between these numbers, if they all exist.

d) X and Y are (bounded) intervals $\Leftrightarrow$ $X + Y$ is a (bounded) interval.

4) The questions as at Problem 2 above for $X + Y$ replaced by $X \cdot Y$.

Sequences

In the problems 5), 6), 7) find the values of a for which the resulting sequence $(x_n)_n$ is:

a) Monotone.

b) Bounded.

c) Cauchy, respectively convergent.

5) For a given (rational, or real) number a, define the sequence $(x_n)_n$ by: $x_0 := a$, and $x_{n+1} = x_n^2 - x_n + 1$ for all $n \geq 0$.

6) For a given (rational, or real) number a, define the sequence $(x_n)_n$ by: $x_0 = a$, and $x_{n+1} = \sqrt{|a^2 - x_n^2|}$ for all $n \geq 0$.

7) For a given (rational, or real) number a, define the sequence $(x_n)_n$ by: $x_0 = a$, and $x_{n+1} = \sqrt{|2 - x_n^2|}$ for all $n \geq 0$.

8) Recall the following definitions:

- The *harmonic series* is the symbol $\sum_n \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$
- The *alternating harmonic series* is the symbol $\sum_n (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \dots$

a) What could be the meaning of the above symbols?

b) Define $\sigma_n := \sum_{k=1}^{n} \frac{1}{k}$. Prove or disprove: $(\sigma_n)_n$ is a Cauchy sequence.

c) Define $\rho_n := \sum_{k=1}^{n} (-1)^k \frac{1}{k}$. Prove or disprove: $(\rho_n)_n$ is a Cauchy sequence.