

Math 360 - Advanced Calculus / Problem Set 5

Sequences

1) Finish the proof of the assertion from the class: *Every monotone bounded sequence of real numbers $(x_n)_n$ is convergent.* More precise:

- a) If $(x_n)_n$ is increasing, then $\lim_{n \rightarrow \infty} x_n = \sup \{x_n \mid n \in \mathbb{N}\}$.
 b) If $(x_n)_n$ is decreasing, then $\lim_{n \rightarrow \infty} x_n = \inf \{x_n \mid n \in \mathbb{N}\}$.

2) Let $(x_n)_n$ be a sequence of real numbers. Show the following:

- a) $x_n \rightarrow a \Leftrightarrow \forall n_0 \in \mathbb{N} \exists N_0 \in \mathbb{N}$ such that $|x_n - a| < \frac{1}{n_0}$ for all $n > N_0$.
 a) x_n is Cauchy $\Leftrightarrow \forall n_0 \in \mathbb{N} \exists N_0 \in \mathbb{N}$ such that $|x_n - x_m| < \frac{1}{n_0}$ for all $m, n > N_0$.

3) Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers s.t. $y_n \leq x_n$ for $n \gg 0$. Prove the following:

- a) **Comparison principle:** If $(x_n)_n$ and $(y_n)_n$ are convergent, then $\lim y_n \leq \lim x_n$.
 Is the same true if one replaces “ \leq ” by “ $<$ ”?
 b) **“Sandwich” principle:** If $(z_n)_n$ is a sequence s.t. $y_n \leq z_n \leq x_n$ for $n \gg 0$, and $\lim y_n = \lim x_n$, then $(z_n)_n$ is convergent, and $\lim z_n = \lim x_n$.

4) Consider a non-empty subset $X \subset \mathbb{R}$. Prove or disprove the following:

- a) X is bounded \Leftrightarrow Every sequence $(x_n)_n$ with $x_n \in X$ has a Cauchy sub-sequence.
 b) X has upper bounds \Leftrightarrow Every increasing sequence $(x_n)_n$ with $x_n \in X$ is Cauchy.

5) **Cauchy sequences and real numbers.** Let $X \subset \mathbb{R}$ be a subset which contains \mathbb{Q} and has the property that every Cauchy sequence $(x_n)_n$ with $x_n \in \mathbb{Q}$ has a limit in X . Prove that $X = \mathbb{R}$.

Can you give an interpretation of the above assertion?

6) **Orders of growth/magnitude.** Using the fact that $\frac{1}{n} \rightarrow 0$, prove the following facts:

- a) For $a \in \mathbb{R}$, $a > 1$, and all $k \in \mathbb{N}$ one has: $\lim_{n \rightarrow \infty} n^k/a^n = 0$.
 b) For $a \in \mathbb{R}$, $a > 1$, and all $k \in \mathbb{N}$ one has: $\lim_{n \rightarrow \infty} (\log_a n)^k/n = 0$.
 b) Show that $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$.

[Hint: Prove first that the corresponding sequences are monotone for $n \gg 0$, etc.]

Language: One says that “exponential growth is much faster than polynomial growth”, respectively “polynomial growth is much faster than logarithmic growth”.

7) **Decimal representation of real numbers.** In the sequel we consider decimal representations and generalization of these as follows. Let $k > 1$ be a fixed natural number. A rational number $a \in \mathbb{Q}$ is called *k-ary*, if its denominator is a power of k . Answer the following:

- a) What are the 10-ary rational numbers? What are the 2-ary rational numbers?
 b) The set R_k of all the *k-ary* rational numbers is closed w.r.t. addition and multiplication, hence R_k is a subring of $(\mathbb{Q}, +, \cdot)$. Is R_k a field?
 c) Explain what should be the *k-ary representation* of a real number. Is such a representation unique, i.e., is “the *k-ary representation*” a mathematically correct formulation?

*8) Show that \mathbb{R} is not countable, more precisely, that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

[Hint: Use the fact that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$, and that one has a surjection $\iota : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$, $N \mapsto \sum_{i \in N} \frac{1}{2^i}$, etc.]