Due: Friday, Feb 29, 2008

Math 360 - Advanced Calculus / Problem Set 6

Topology of $\mathbb{R}$

Recall that a subset $B \subset \mathbb{R}$ is closed $\overset{\text{Def}}{\Longleftrightarrow}$ $\mathcal{C}_{\mathbb{R}} B := \mathbb{R} \backslash B$ is open in $\mathbb{R}$. We showed in class that if $U_1, U_2 \subseteq \mathbb{R}$ are open, then $U_1 \cap U_2$ is open, and that if $U_\alpha \subseteq \mathbb{R}$, $\alpha \in \Lambda$, is an arbitrary set of open subsets, then $\cup_\alpha U_\alpha$ is open in $\mathbb{R}$.

1) Using the above two facts, prove the following assertions from the class:

a) If $U_1, \dots, U_n$, $n \geq 1$ are open subsets of $\mathbb{R}$, then $\cap_k U_k$, $1 \leq k \leq n$, is open.

b) If $B_1, \dots, b_n$, $n \geq 1$, are closed, then $\cup_k B_k$, $1 \leq k \leq n$, is closed in $\mathbb{R}$.

c) The only subsets of $\mathbb{R}$ which are both closed and open are $\emptyset \subset \mathbb{R}$ and $\mathbb{R} \subseteq \mathbb{R}$.

2) For $x \in \mathbb{R}$, and $\epsilon \in \mathbb{R}$, $\epsilon > 0$, the ϵ-*neighborhood* of x is the open interval $I_{x,\epsilon} := (-\epsilon + x, x + \epsilon)$. Show the following:

a) $U \subset \mathbb{R}$ is open $\Leftrightarrow \forall\, x \in U\ \exists\, \epsilon > 0$ s.t. $I_{x,\epsilon} \subseteq U$.

b) Let $B \subset \mathbb{R}$ be a closed subset, and $(x_n)_n$ a convergent sequence with $x_n \in B$. Then $\lim x_n \in B$.

3) Let $I \subset \mathbb{R}$ be a non-empty interval of extremities $a \leq b$, which means that I could be open, half open, or closed.

a) Prove or disprove: $\overset{\circ}{I} = (a, b)$ and $\overline{I} = [a, b]$

b) Set $X := I \cap \mathbb{Q}$. What is the interior of X, respectively the closure of X in $\mathbb{R}$?

c) The same questions as at b) above for X the set of irrational numbers from I.

4) Consider the sets $X \subset \mathbb{R}$ below, and find their interior and their closures:

a) $X = \{-2\} \cup (-1, 1) \cup \{\frac{2n}{n+1} \mid n \in \mathbb{N}\}$.

b) $X = \cup_{n \geq 0} (\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}]$.

Complex numbers

5) Recall that we set $\mathbb{C} = \{a + b\boldsymbol{i} \mid, a, b \in \mathbb{R}\}$, and $\mathbb{C}' = \{(a, b) \mid, a, b \in \mathbb{R}\}$ with addition laws and and multiplication laws as defined in the class. Show that those composition laws are associative, commutative, and the corresponding multiplication is distributive w.r.t. the corresponding multiplication.

6) Recall that we identified $\mathbb{C}$ and $\mathbb{C}'$ via the bijective map $\varphi : \mathbb{C} \to \mathbb{C}'$, $a + b\boldsymbol{i} \mapsto (a, b)$. Prove the φ is compatible with addition and multiplication.

7) Recall the complex conjugation map, defined by $\sigma : \mathbb{C} \to \mathbb{C}$, $z = a + b\boldsymbol{i} \mapsto \overline{z} := \sigma(z) = a - b\boldsymbol{i}$. Prove the following assertions from the class:

a) σ is an isomorphism of fields, i.e., σ is bijective and is compatible with addition and multiplication.

b) The absolute value map $|\ | : \mathbb{C} \to \mathbb{R}$, $z \mapsto \sqrt{z\overline{z}}$, satisfies: $|zz'| = |z|\,|z'|$, and $|z + z'| \leq |z| + |z'|$.

8) Find the trigonometric representation of the following complex numbers:

a) $z_1 = \frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}$, and $z_2 = z_1^2$, $z_3 = z_1^3$, and more general, $z_n := z_1^n$.

b) $z_1 = 1 + \boldsymbol{i}$, and $z_2 = z_1^2$, $z_3 = z_1^3$, and more general, $z_n := z_1^n$.

c) z a real number, respectively z an imaginary number.