Due: Friday, April 4, 2008

Math 360 - Advanced Calculus / Problem Set 10

Homeomorphisms

1) Let $f : X \to Y$ be a continuous function. Show that the following are equivalent:

i) f is a homeomorphism.

ii) f is bijective, and for every open subset $U \subseteq X$ one has: $f(U)$ is open in Y.

iii) f is bijective, and for every closed subset $A \subseteq X$ one has: $f(A)$ is closed in Y.

2) Let $f : X \to Y$ be a bijective continuous function. Suppose that X is compact, and Y is Hausdorff. Show that f is a homeomorphism.

3) Let $I, J \subseteq \mathbb{R}$ are intervals, and $f : I \to J$ be a continuous bijective function. Prove or disprove:

a) $f : I \to J$ is a homeomorphism.

b) I is (half) open/closed $\Leftrightarrow J$ is so.

c) f is strictly monotone.

4) Prove or disprove the following:

a) The open interval $I = (-1, 1)$ is homeomorphic to $\mathbb{R}$.

b) More general, any two open non-empty intervals $I, J \subseteq \mathbb{R}$ are homeomorphic.

c) A subspace $X \subset \mathbb{R}$ is homeomorphic to $\mathbb{R} \Leftrightarrow X$ is an open interval.

5) Prove or disprove the following:

a) The open cube $K = I \times I = \{(x, y) \mid -1 < x, y < 1\}$ is homeomorphic to $\mathbb{R}^2$.

b) Every open cube $I \times J$, with $I, J \subset \mathbb{R}$ open non-empty intervals, is homeomorphic to $\mathbb{R}^2$.

c) What are the generalizations of the assertions above for $\mathbb{R}^n$?

6) Google the term "space-filling curve" and learn about that. Prove or disprove: The interval $I = (-1, 1)$ is not homeomorphic to the open cube $K = I \times I$. How do you explain your answer versus space-filling curves?

Metric spaces

7) Let X', d' and X'', d'' be metric spaces. we set set $X := X' \times X''$. Prove or disprove that the following maps on $X \times X$ are distances:

a) $d_1 : X \times X \to \mathbb{R}$, by $d_1\big((x', x''), (y', y'')\big) = d'(x', y') + d''(x'', y'')$.

b) $d_E : X \times X \to \mathbb{R}$, by $d_E\big((x', x''), (y', y'')\big) = d\sqrt{d'(x', y')^2 + d''(x'', y'')^2}$.

c) $d_{\max} : X \times X \to \mathbb{R}$, by $d_E\big((x', x''), (y', y'')\big) = \max\big(d'(x', y'), d''(x'', y'')\big)$.

8) Answer the following:

a) Draw the unit balls in $\mathbb{R}^n$, for $n = 1, 2, 3$, for the distances d_1, the Euclidean distance d_E, and the distance $d_{\max}$.

b) Make an educated guess: Do the distance map d_1, the Euclidean distance d_E, and the distance $d_{\max}$ define the same topology on $\mathbb{R}^n$?