Due: Friday, April 11, 2008

Math 360 - Advanced Calculus / Problem Set 11

Metric spaces

1) Let X, d be a metric space. Prove or disprove the following:

a) Let $A \subseteq X$ non-empty. Then $x \in \overline{A} \Leftrightarrow \exists\ (x_n)_n$ with $x_n \in A$ and $\lim_{n\to\infty} = x$.

b) Suppose that X is complete. Then A is closed $\Leftrightarrow A$ is complete.

2) Let $I = [0,1]$, and $V \subset \mathcal{C}(I, \mathbb{R})$ be the subset defined by: $f \in V \Leftrightarrow \exists\, \epsilon_f > 0$ s.t. $f(x) = 0$ for $0 \leq x \leq \epsilon_f$. And for $n > 1$ consider $f_n : I \to \mathbb{R}$ defined by $f(x) = 0$ for $0 \leq x \leq \frac{1}{n}$ and $f_n(x) = \frac{n}{n-1}x - \frac{1}{n-1}$ for $\frac{1}{n} \leq x \leq 1$. Show/answer the following:

a) V is a real vector subspace of $\mathcal{C}(I, \mathbb{R})$.

b) $f_n \in V$ for all $n > 1$.

c) $(f_n)_n$ is a Cauchy sequence in the sup-norm $\|\ \|_\infty$.

d) $(f_n)_n$ has no limit in V.

e) $(f_n)_n$ has a limit in $\mathcal{C}(I, \mathbb{R})$. What is $\lim_{n\to\infty} f_n$?

3) Which of the following sets are bounded, respectively connected, respectively compact:

a) $\{(x, y) \in \mathbb{R}^2 \mid x^2 < 3, |y| \leq 2\}$

b) $\{v \in \mathbb{R}^n \mid d_{\|\ \|_E}(v, 0) \leq 10\}$.

c) $A \subset \mathbb{C}$ finite non-empty set.

d) The boundary of a bounded non-empty set $A \subset \mathbb{R}^n$.

e) $A = \{(a, b) \in \mathbb{R}^2 \mid a \in \mathbb{Q}\}$.

f) The unit ball in $\mathfrak{l}_1$.

4) Prove the assertion made in the class: If X is a topological space, and $A \subset X$ is a path-connected subset, then A is connected.

5) Prove or disprove the following assertions:

a) If $A \subset \mathbb{R}^n$ is connected, then $\mathbb{R}^n \backslash A$ is disconnected.

b) If $A \subset \mathbb{R}^n$ is compact, then $\mathbb{R}^n \backslash A$ is not compact.

c) If $A \subset \mathbb{R}^n$ and $\mathbb{R}^n \backslash A$ are both connected, then one of them must be bounded.

e) Every open and connected subset of $\mathbb{R}^n$ is path-connected.

6) Let V endowed with $\|\ \|$ be a normed vector space. For non-empty subsets $A, B \subseteq V$ we denote their sum by $A + B = \{v + w \mid v \in A, b \in B\}$. Prove or disprove the following:

a) A, B are bounded iff $A + B$ is bounded.

b) A, B are compact iff $A + B$ is compact.

c) A, B are connected iff $A + B$ is connected.

Supplement: *The completion of a metric space.*

Let X, d be a metric space, and let $\mathcal{C}(X)$ be the set of all the Cauchy sequences $f : \mathbb{N} \to X$ with values in X. (Recall that for such an $f : \mathbb{N} \to X$, we usually write $(x_n)_n$ with $x_n := f(n)$. In order to simplify notations, we will though prefer to denote a sequence by f, understanding that that is a map from $\mathbb{N}$ to X.)

7) Define $\delta : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}$ by $\delta(f, g) = \lim_{n\to\infty} d\big(f(n), g(n)\big)$. Prove the following:

a) δ is well defined, i.e., $\lim_{n\to\infty} d\big(f(n), g(n)\big)$ exists for all $f, g \in \mathcal{C}(X)$.

b) Show that δ is symmetric, i.e., $\delta(f, g) = \delta(g, f)$ for all $f, g \in \mathcal{C}(X)$.

c) Show that δ is satisfies the triangle inequality, i.e., $\delta(f,h) \le \delta(f,g) + \delta(g,h)$ for all $f,g,h \in \mathcal{C}(X)$.

d) Show that if $\delta\big(f,f'\big) = 0$ and $\delta(g,g') = 0$, then $\delta(f,g) = \delta(f',g')$ for all $f,f',g,g' \in \mathcal{C}(X)$.

8) For every $f \in \mathcal{C}(X)$, define $\hat{f} := \{f' \mid \delta(f,f') = 0\}$. And let $\widehat{X} = \{\hat{f} \mid f \in \mathcal{C}(X)\}$. [This is a set of subsets of $\mathcal{C}(X)$!] Define $\hat{d} : \widehat{X} \times \widehat{X} \to \mathbb{R}$ by $\hat{d}(\hat{f},\hat{g}) := \delta(f,g)$. Using the previous Problem, prove the following:

a) For all f,g one has: If $\hat{f} \cap \hat{g}$ is not empty, then $\hat{f} = \hat{g}$, hence $\delta(f,g) = 0$.

b) The map $\hat{d} : \widehat{X} \times \widehat{X} \to \mathbb{R}$ is well defined, and is a distance map.

c) $\widehat{X}$ is complete w.r.t. the distance map $\hat{d}$.

9) Finally, define $\imath : X \to \widehat{X}$, $x \mapsto f_x$ the constant Cauchy sequence $f_x : \mathbb{N} \to X$, by $f(n) = x$. Prove the following:

a) $\imath$ is an isometric embedding, i.e., $d(x,y) = \hat{d}(\hat{f}_x, \hat{f}_y)$ for all $x,y \in X$.

b) $\imath(X)$ is a dense subset of $\widehat{X}$.

Language: One says that $\imath : X \to \widehat{X}$ is the completion of X, d.