Due: Friday, April 18, 2008

Math 360 - Advanced Calculus / Problem Set 12

1) Give complete proofs of the following assertion used in the class:

a) Let A_i, $i = 1, \ldots, k$, and $A := \cup_i A_i$. Then the k-fold cartesian product $A^k := \{(a_1, \ldots, a_k) \mid a_i \in A_i\}$ satisfies: $A^k = \cup_{i_1,\ldots,i_k} A_{i_1} \times \ldots \times A_{i_k}$.

b) In the above context, suppose that each $A_i \subset \mathbb{R}$ is an open interval of the from $A_i = (-\epsilon + x_i, x_i + \epsilon)$. Then $A_{i_1} \times \ldots \times A_{i_k}$ is the open ball centered at $(x_{i_1}, \ldots, x_{i_k})$ and radius ϵ in the $|| \, ||_\infty$ norm.

The completed real line

- Recall the completed real line $\overline{\mathbb{R}}$, its total ordering, its topology, and its (partially) defined addition and multiplication. For $x, y \in \overline{\mathbb{R}}$, let $x * y$ is one of the algebraic operations: $x + y$, $x - y$, $x \cdot y$, $\frac{x}{y}$, provided the corresponding $x * y$ is defined.
- In the sequel, $D \subset \overline{\mathbb{R}}$ is an open subset, $f, g : D \to \overline{\mathbb{R}}$ are maps, and $x_0 \in D$. Further, we denote $D_* := \{x \in D \mid f(x) * g(x) \text{ is defined}\}$, and define $(f * g) : D_* \to \overline{\mathbb{R}}$, $(f * g)(x) := f(x) * g(x)$.

2) Let $(x_n)_n, (y_n)_n$ be convergent sequences in $\overline{\mathbb{R}}$, say $\lim x_n = a$, $\lim y_n = b$. Prove or disprove the following:

a) Suppose that $a * b$ is defined in $\overline{\mathbb{R}}$. Then $x_n * y_n$ is defined for $n >> 0$.

b) Suppose that $x_n * y_n$ is defined in $\overline{\mathbb{R}}$ for all n. Then $(x_n * y_n)_n$ is convergent in $\overline{\mathbb{R}}$, and $\lim x_n * y_n = a * b$.

3) Suppose that $a := \lim_{x \to x_0} f(x)$ and $b := \lim_{x \to x_0} g(x)$ both exist in $\overline{\mathbb{R}}$. Prove or disprove:

a) If $a * b$ is defined in $\overline{\mathbb{R}}$, then $\exists$ neighborhood $U \subset \overline{\mathbb{R}}$ of x_0 s.t. $U \backslash \{x_0\} \subset D_*$.

b) $\lim_{x \to x_0} (f * g)(x)$ exists, and $\lim_{x \to x_0} (f * g)(x) = a * b$.

c) If f and g are continuous on D_*, then $D_* \subseteq \overline{\mathbb{R}}$ is open, and $(f * g) : D_* \to \overline{\mathbb{R}}$ is continuous.

4) Prove or disprove the following:

a) $\lim_{x \to x_0} f(x)$ exists $\Leftrightarrow \forall$ sequences $(a_n)_n$ in D s.t. $a_n \neq x_0$ and $\lim_{n \to \infty} a_n = x_0$, one has: $\lim_{n \to \infty} f(a_n)$ exists.

b) f is continuous in $x_0 \Leftrightarrow \forall$ sequences $(a_n)_n$ in D with $\lim a_n = x_0$, one has: $\lim_{n \to \infty} f(a_n) = f(x_0)$.

5) **Examples**: Prove or disprove the following:

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 1$ has a unique continuous prolongation $\overline{f} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

b) $f : (-1, 1) \to \mathbb{R}$, $f(x) = x/(1 - x^2)$ has a unique prolongation to a homeomorphism $\overline{f} : [-1, 1] \to \overline{\mathbb{R}}$.

c) Consider $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$, $f(x) = 1/x$. Find a maximal subset $D \subset \overline{\mathbb{R}}$ to which f has a continuous prolongation to some continuous map $\overline{f} : D \to \overline{\mathbb{R}}$.

Metric spaces

6) Let $\mathbb{R}^n$, $n = 1, 2, 3$, be endowed with any of the usual three norms: $|| \, ||_1$, $|| \, ||_E$, $|| \, ||_\infty$.

a) What is the diameter of $B(x, r)$ and of $\overline{B}(x, r)$ in every of the norms above?

b) What is the distance between two balls $B(x', r')$ and $B(x'', r'')$?

c) Is the same true in arbitrary metric spaces?

7) Let X, d be a metric space. For $A, B \subset X$, let $\overline{A}, \overline{B}$ be their closures. Prove or disprove the following:

a) $d(A, B) = d(\overline{A}, \overline{B})$.

b) $\delta(A) = \delta(\overline{A})$.

c) $A \cap B = \emptyset \Leftrightarrow d(A, B) > 0$.

d) Is the same true if A and B are compact.

8) In which of the following normed real vector spaces is the open, respectively closed, unit ball compact?

a) $\mathbb{R}^n$ for $n > 5$. Respectively, in all the $\mathfrak{l}_q$ with $q \leq 3$.

b) In all $\mathcal{F}_{\mathrm{b}}(X, \mathbb{R})$, provided the space X is finite. Respectively, in all $\mathcal{C}(X, \mathbb{R})$, provided the space X is compact.