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Math 621 (Number Theory II) / Problem Set 1

1) Let K be a field endowed with a non-trivial n.a. absolute value $|\ |$, say with valuation ring $(R, \mathfrak{m})$. Let $(\widehat{R}, \widehat{\mathfrak{m}})$ be the corresponding completions, and $K^{\rm s}$ a separable closure of K, and set $R' = \widehat{R} \cap K^{\rm s}$, $\mathfrak{m}' = \widehat{\mathfrak{m}} \cap K^{\rm s}$. Show that $(R', \mathfrak{m}')$ has the universal property of a Henselisation of $(R, \mathfrak{m})$.

2) Let K be a field complete with respect to an absolute value $|\ |$. Complete the details of the proofs of the following assertions from the Lecture: Let V be a finite dimensional v.s. over K. Then *all K-norms on V are equivalent.*

3) **Teichmüller system of representatives:** Let $(K, |\ |)$ be a complete discrete valued field, and $(R, \mathfrak{m})$ the corresponding valuation ring. Suppose that $\kappa = R/\mathfrak{m}$ is a perfect field with $\mathrm{char}(\kappa) = p > 0$. Thus for $\overline{a} \in \kappa$, and every $n \geq 1$, there exists a unique $\overline{a}_n \in \kappa$ s.t. $\overline{a}_n^{p^n} = \overline{a}$. And let $a_n \in R$ be some pre-image of $\overline{a}_n$. Show the following:

a) The sequence $(a_n^{p^n})_n$ is convergent in R, say to some $a \in R$, and a is a pre-image of $\overline{a}$.

b) The resulting map $\overline{s} : \kappa \to R$, $\overline{a} \mapsto a$, is injective and multiplicative, i.e. $\overline{s}(\overline{a}\overline{b}) = \overline{s}(\overline{a})\, \overline{s}(\overline{b})$.

c) $\tilde{\kappa} := \overline{s}(\kappa)$ is the only multiplicatively closed system of representatives for κ.

d) If $\mathrm{char}(K) > 0$, then $\mathrm{char}(K) = p$, and $s : \kappa \to \tilde{\kappa}$ is a field isomorphism.

Language: The system of representatives κ_0 is called the *Teichmüller system of representatives* of K.

4) **Krasner's Lemma**: Let K be a field endowed with a non-trivial absolute value $|\ |$, and $f(X) \in K[X]$ a non-zero separable polynomial over K. Show the following:

a) If $g(X) \in K[X]$ is coefficient-wise "sufficiently close" to $f(X)$, then $f(X)$ and $g(X)$ split in "the same way" over the completion $\widehat{K}$.

b) Give the precise meaning of "sufficiently close" and "the same way".

Isomorphisms of local fields

5) Let K be a field which is not separably closed.

a) Suppose that K is complete w.r.t. non-trivial absolute values $|\ |$ and $|\ |'$. Then $|\ |$ and $|\ |'$ are equivalent.

b) Is the same the case if we replace "complete" by "henselian" (in the case $|\ |$ and $|\ |'$ are n.a.)?

6) In the context of the last Problem above, show the following:

a) If K is complete w.r.t. a non-trivial absolute value $|\ |$, then for each field automorphism σ of K one has $|\ | = \sigma \circ |\ |$.

b) In the case $|\ |$ is non-achimedean, is the same true if we replace "complete" by "henselian"?

7) Using the facts above, prove or disprove:

a) The automorphism groups $\mathrm{Aut}(\mathbb{R})$ and $\mathrm{Aut}(\mathbb{Q}_p)$ consist only of the identity.

b) Is the same true for the Laurent power series field $k((u))$ over some base field k?

c) If L is a finite extensions of $K := \mathbb{R}$ or $K := \mathbb{Q}_p$, then $\mathrm{Aut}(L) = \mathrm{Aut}(L|K)$.

Hint. Use Krasner's Lemma together with the weak approximation Theorem.

8) Let K be complete w.r.t. a non-trivial absolute value $|\ |$. Let $L|K$ be some algebraic extension, endowed with the unique prolongation of $|\ |$ (which we denote again by $|\ |$).

a) Show that L is complete *iff* $L|K$ is finite.

b) If L is countably generated and separable over K, then there exists $x \in \widehat{L}$ such that $K[x]$ is dense in $\widehat{L}$.