Feb 25, 2008

Math 621 (Number Theory II) / Problem Set 2

1) Let K be a locally compact non-discrete topological field. Show that its topology is a V topology.

[**Hint:** Let $U = \{x \in K \mid x^n \to 0\}$. Then U is open, thus U^{-1} is open too. Let S be its complement. Show that S is compact. (Start with some compact neighborhood Σ of 0, and look at $\cup a^n\Sigma$, $a \in \mathbb{Z}$.) Etc.]

2) **Continuity of the roots:** Let K be a complete field with respect to the absolute value $|\ |$. Let $x \in K$ a simple root of $f(X) \in K[X]$, and $\mathcal{U} \subset K$ a neighborhood of x. Show that if $g(X) \in K[X]$ is coefficient-wise close enough to $f(X)$, then $g(X)$ has a simple root $y \in U$.

Witt vectors:

3) Fix a prime number p. For a system of variables $T = (T_m)_m$ define a system $W(T) = \big(W_n(T)\big)_n$ of polynomials from $\mathbb{Z}[T]$ by $W_n(T) = \sum_0^n p^m T_m^{p^{n-m}}$. Set $X = (X_m)_m$ and $Y = (Y_m)_m$. Show the following:

a) There exist systems of polynomials $S = (S_m)_m$ and $P = (P_m)_m$ in $\mathbb{Z}[X, Y]$ such that we have identities:
$$W(S) = W(X) + W(Y) \quad \text{and} \quad W(P) = W(X) \cdot W(Y).$$

b) Let R be any commutative Ring with 1_R. For $a = (a_m)_m$ and $b = (b_m)_m$ from $W(R) := R^{\mathbb{N}}$ define:
$$a + b = (S_0(a, b),\ S_1(a, b), \ldots) \quad \text{and} \quad a \cdot b = (P_0(a, b),\ P_1(a, b), \ldots)$$
Show that the above $+$ and $\cdot$ define a structure of a commutative ring on $W(R)$.

c) What is this ring if p is invertible in R?

Language: $W(R)$ *is called the ring of Witt vectors over* R.

4) In the above context, suppose that k is a perfect field having $\mathrm{char}(k) = p$. Define on $W(k)$ the **Verschiebung** V by $V(a) = (0, a_0, a_1, \ldots)$ and the **Frobenius** F by $F(a) = (a_0^p, a_1^p, \ldots)$. Show the following:

a) V is an endomorphism of $\big(W(k), +\big)$.

b) F is a ring automorphism of $W(k)$.

c) $V \circ F = F \circ V = p\,\mathrm{id}_{W(k)}$.

5) **(Continued)** Show the following:

a) If k is a perfect field with $\mathrm{char}(k) = p$, then $W(k)$ is a complete DVR, $p\,W(k)$ is its maximal ideal, and k is its residue field. Describe the Teichmüller system of representatives $\overline{s} : k \to W(k)$.

b) Let R be a complete DVR such that its maximal ideal is generated by p, and its residue field k is perfect with $\mathrm{char}(k) = p$. Then R is canonically isomorphic to $W(k)$.

c) Moreover, if l is another perfect field, then one has a canonical bijection $\mathrm{Hom}(k, l) \to \mathrm{Hom}\big(W(k), W(l)\big)$.

Higher ramification groups

6) Describe the higher ramification groups at the primes over 2 and 3 in $K|\mathbb{Q}$, where

a) $K = \mathbb{Q}(\zeta_3, \alpha)$, $\alpha^3 = 2$

b) $K = \mathbb{Q}(\zeta_3, \alpha)$, $\alpha^3 = 3$

7) The same question for the n^{th} cyclotomic field $\mathbb{Q}(\zeta_n)|\mathbb{Q}$, where $n = p^e$ for some $e \geq 1$.