March 15, 2008

Math 621 (Number Theory II) / Problem Set 3

1) Let k be an algebraically closed field, and $K = k((t))$ the Laurent power series field over K.

a) Suppose that $\mathrm{char}(k) = 0$. Show that for every $n > 0$, K has up to isomorphism exactly one extension $K_n|K$ of degree n.

b) Is the same true if $\mathrm{char}(k) = p > 0$?

Notation: In the sequel, if not explicitly otherwise stated, K denotes a complete field w.r.t. the discrete valuation v, having a perfect residue field κ_K. If $L|K$ is a finite extension, we denote by $R_L|R_K$ the corresponding extension of (complete) DVR's, and by $\mathcal{D}_{L|K}$ the different of $R_L|R_K$.

2) Let $L|K$ denote arbitrary finite abelian extensions of K.

a) Show that in general there exist $L|K$ which cannot be represented as a compositum $L = L_0L_1L_2$ with $L_0|K$ unramified, $L_1|K$ totally tamely ramified, and $L_2|K$ totally wildly ramified.

b) Try to figure out the most general conditions on K such that the above decomposition is always possible.

c) Suppose that some $L|K$ can be represented in the form $K = L_0L'$ with $L_0|K$ unramified and $L'|K$ totally ramified. Prove or disprove: If $L_1|K$ and $L_2|K$ are maximal totally tamely, respectively totally wildley, ramified sub-extensions of $L|K$, then $L = L_0L_1L_2$.

3) Complete the details of the proof of **Hilbert's Formula**:

Let $L|K$ some finite Galois extension, and $G_0 \supseteq \dots G_i \supseteq \dots \{1\}$ be the higher ramification groups of $\mathrm{Gal}(L|K)$. *Then one has:*

$$v_L(\mathcal{D}_{L|K}) = \textstyle\sum_{i\geq 0}(|G_i| - 1)$$

4) Let $L|K$ be a finite field extension. Prove or disprove:

a) $L|K$ is tamely ramified iff $e(L|K) - 1 = v_L(\mathcal{D}_{L|K})$.

b) $L|K$ is wildly ramified iff $e(L|K) - 1 < v_L(\mathcal{D}_{L|K})$.

5) Prove or disprove: If the residue field κ_K is not perfect, then the assertions from Probelms 1, 2 and 3 above do not necessarily hold.

6) Let G be a finite p-group. Prove or disprove:

a) G has a unique maximal p-elementary abelian quotient $\overline{G}$.

b) The minimal number of generators of G equals the minimal number of generators of $\overline{G}$.

c) The above assertions hold in the same form for all pro-p groups G.

7) Prove the Local Kronecker–Weber Theorem for $\mathbb{Q}_2$.